

# PROJECTIVE GEOMETRY

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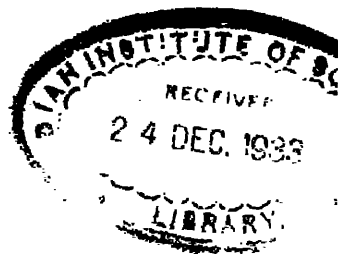
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VOLUME II

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## PREFACE

The present volume is an attempt to carry out the program outlined in the preface to Volume I. Unfortunately, Professor Young was obliged by the pressure of other duties to cease his collaboration at an early stage of the composition of this volume. Much of the work on the first chapters had already been done when this happened, but the form of exposition has been changed so much since then that although Professor Young deserves credit for constructive work, he cannot fairly be held responsible for mistakes or oversights.

Professor Young has kindly read the proof sheets of this volume, as have also Professors A. B. Coble and A. A. Bennett. Most of the drawings were made by Dr. J. W. Alexander. I offer my thanks to all of these gentlemen and also to Messrs. Ginn and Company, who have shown their usual courtesy and efficiency while converting the manuscript into a book.

The second volume has been arranged so that one may pass on a first reading from the end of Chapter VII, Volume I, to the beginning of Volume II. The later chapters of Volume I may well be read in connection with the part of Volume II from Chapter V onward.

I shall pass by the opportunity to discuss any of the pedagogical questions which have been raised in connection with the first volume and which may easily be foreseen for the second. It is to be expected that there will continue to be a general agreement among those who have not made the experiment, that an abstract method of treatment of geometry is unsuited to beginning students.

In this book, however, we are committed to the abstract point of view. We have in mind two principles for the classification of any theorem of geometry: (*a*) the axiomatic basis, or bases, from which it can be derived, or, in other words, the class of spaces in which it can be valid; and (*b*) the group to which it belongs in a given space.



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# PROJECTIVE GEOMETRY

## CHAPTER I

### FOUNDATIONS

**1. Plan of the chapter.** In the first volume of this book we have been concerned with general projective geometry, that is to say, with those theorems which are consequences of Assumptions A, E, P. In many cases we also made use of Assumption  $H_0$ , but most of the theorems which we proved by the aid of this assumption remain true (though trivial) when this assumption is false. The class of spaces to which the geometry of Vol. I applies is very large, and the set of assumptions used is therefore far from categorical.

The main purpose of geometry is, of course, to serve as a theory of that space in which we envisage ourselves and external nature. This purpose can be accomplished only partially by a geometry based on a set of assumptions which is not categorical. We therefore proceed to add the assumptions which are necessary in order to limit attention to the geometry of reals, the geometry in which the number system is the real number system of analysis.

These assumptions are stated in two ways, the one (§ 3) dependent on the theory of the real number system and the other (§§ 7-13) independent of it. We also state the assumptions (§§ 5, 14, 15, 16) necessary for certain other geometries which are of importance because of their relations to the real geometry and to other branches of mathematics. At the end of the chapter we give a summary of the assumptions for the various projective geometries which we are considering.

**2. List of Assumptions A, E, P, and  $H_0$ .** For the sake of having all the assumptions before us in the present chapter, we reprint A, E, P, and  $H_0$ . The assumptions serve to determine a class S of elements called points, and a class of subclasses of S called lines. The phrase

"a point is on a line" or "a line is on a point" means that the point belongs to the line (cf. p. 16, Vol. I).

#### ASSUMPTIONS OF ALIGNMENT:

A 1. *If  $A$  and  $B$  are distinct points, there is at least one line on both  $A$  and  $B$ .*

A 2. *If  $A$  and  $B$  are distinct points, there is not more than one line on both  $A$  and  $B$ .*

A 3. *If  $A, B, C$  are points not all on the same line, and  $D$  and  $E$  ( $D \neq E$ ) are points such that  $B, C, D$  are on a line and  $C, A, E$  are on a line, there is a point  $F$  such that  $A, B, F$  are on a line and also  $D, E, F$  are on a line.*

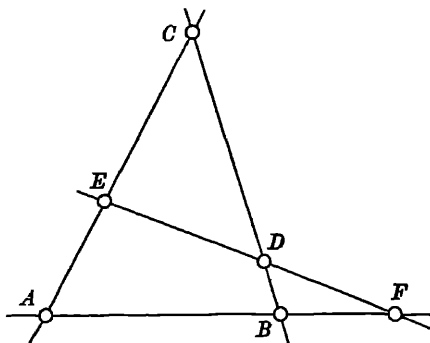


FIG. 1

#### ASSUMPTIONS OF EXTENSION:

E 0. *There are at least three points on every line.*

E 1. *There exists at least one line.*

E 2. *All points are not on the same line.*

E 3. *All points are not on the same plane.\**

E 3'. *If  $S_3$  is a three-space,† every point is on  $S_3$ .*

#### ASSUMPTION OF PROJECTIVITY:

P. *If a projectivity leaves each of three distinct points of a line invariant, it leaves every point of the line invariant.‡*

#### ASSUMPTION $H_0$ :

$H_0$ . *The diagonal points of a complete quadrangle are noncollinear.§*

As was explained when Assumption P was first introduced, this assumption does not appear in the complete list of assumptions for the geometry of reals, but is replaced by certain other assumptions from which it (as well as  $H_0$ ) can be derived as a theorem. The list of assumptions for this geometry will consist of Assumptions A, E, and the new assumptions.

\* Cf. § 7, Vol. I.

† Cf. § 9, Vol. I.

‡ Cf. § 36, Vol. I.

§ Cf. § 18, Vol. I.

**3. Assumption K.** The most summary way of completing the list of assumptions for the geometry of reals is to introduce the following:

K. *A geometric number system (Chap. VI, Vol. I) is isomorphic\* with the real number system of analysis.*

Thus a complete list of assumptions for the geometry of reals is A, E, K.

The use of Assumption K implies a previous knowledge of the real number system.† Its apparent simplicity therefore masks certain real difficulties. What these difficulties are from a geometric point of view will be found on reading §§ 7-13, where K is analyzed into independent statements H, C, R. These sections, however, may be omitted, if desired, on a first reading.

Since a geometric number system in one one-dimensional form is isomorphic with any geometric number system in any one-dimensional form in the same space, it is evident that the principle of duality is valid for all theorems deducible from Assumptions A, E, K.

In order that the results of Vol. I be applicable to the geometry of reals, it must be shown that Assumption P is a logical consequence of Assumptions A, E, K. Since multiplication is commutative in the real number system, this result would follow directly from Theorem 7, Chap. VI, Vol. I. The proof there given is, however, incomplete. It is shown (Theorem 6, loc. cit.) that if P holds, multiplication is commutative; but it is not there proved that if multiplication is commutative, P is satisfied. The needed proof may be made as follows:

**THEOREM 1.** *Assumption P is valid in any space satisfying Assumptions A and E and such that multiplication is commutative in a geometric number system (Chap. VI, Vol. I).*

*Proof.* It is obvious that the number systems determined by any two choices of the fundamental points  $H_0H_1H_\infty$  are isomorphic (cf. Theorems 1 and 3, Chap. VI, Vol. I), so that we may base our argument on an arbitrary choice of these points. We are assuming that multiplication is commutative, and are to prove that any projectivity  $\Pi$

\* This term is defined in § 52, Vol. I.

† The real number system is to be thought of either as defined in terms which rest ultimately on the positive integers (cf. Pierpont, Theory of Functions of Real Variables, pp. 1-94; or Fine, College Algebra, pp. 1-70) or by means of a set of postulates (cf. E. V. Huntington, Transactions of the American Mathematical Society, Vol. VI (1905), p. 17).

which leaves three distinct points of a line fixed is the identity. By definition,  $\Pi$  is the resultant of a sequence of perspectivities

$$[H] \stackrel{S_1}{\underset{\Lambda}{=}} [P] \stackrel{S_2}{\underset{\Lambda}{=}} \dots \stackrel{S_n}{\underset{\Lambda}{=}} [\Pi(H)]$$

where  $[H]$  denotes the points of the given line. By Theorem 5, Chap. III, Vol. I, this chain of perspectivities may be replaced by three perspectivities

$$[H] \stackrel{S}{\underset{\Lambda}{=}} [P] \stackrel{T}{\underset{\Lambda}{=}} [Q] \stackrel{U}{\underset{\Lambda}{=}} [\Pi(H)].$$

Moreover, by Theorem 4, Chap. III, Vol. I, the pencils  $[P]$  and  $[Q]$  may be chosen so that their respective axes pass through two of the given fixed points of  $\Pi$ . Let us denote these points by  $H_x$  and  $H_y$ ,

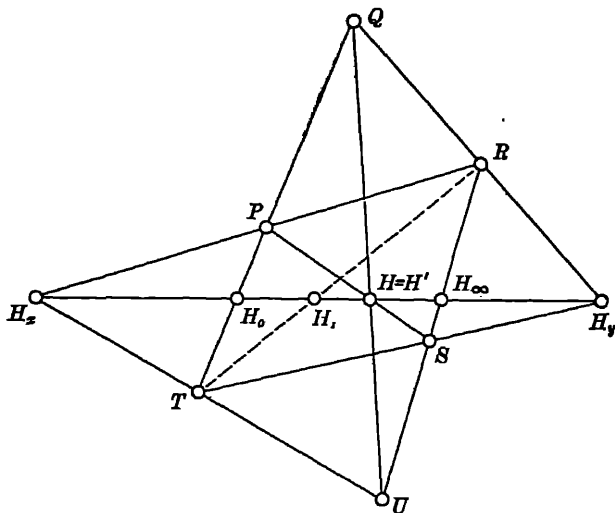


FIG. 2

respectively and let  $H_\infty$  be the third fixed point. By another application of Theorem 4 the pencils  $[P]$  and  $[Q]$  may be chosen so that their common point  $R$  is on the line  $SH_\infty$  (fig. 2).

Now, since  $H_\infty$  is transformed into itself,  $S$ ,  $H_\infty$ , and  $U$  must be collinear. Since  $H_x$  is fixed,  $T$ ,  $H_x$ , and  $U$  must be collinear. Since  $H_y$  is fixed,  $S$ ,  $T$ , and  $H_y$  are collinear. If  $H$  is any point of the line  $H_xH_y$ , it is transformed by the perspectivity with  $S$  as center to a point  $P$  of the line  $H_xR$ ; the perspectivity with  $T$  as center transforms  $P$  to a point  $Q$  of the line  $RH_y$ ; the perspectivity with  $U$  as

center transforms  $Q$  back to a point  $H'$  of the line  $H_x H_y$ . We have to show that  $H' = H$ .

Let  $H_0$  be the trace on the line  $H_x H_y$  of  $PT$ ; let  $H_1$  be the trace of  $RT$ ; and  $H'$  is the trace of  $UQ$ .

The complete quadrangle  $TRSP$  determines  $Q(H_0 H_y H_1, H_\infty H_x H)$ , and hence (Theorem 3, Chap. VI, Vol. I) in the scale  $H_0 H_1 H_\infty$

$$H_y \cdot H_x = H.$$

The complete quadrangle  $TRQU$  determines  $Q(H_0 H_x H_1, H_\infty H_y H')$ , and hence in the scale  $H_0 H_1 H_\infty$

$$H_x \cdot H_y = H'.$$

Since multiplication is commutative,  $H = H'$ , which proves the theorem.

The reader will find no difficulty in using the construction above to prove that the validity of the theorem of Pappus (§ 86, Vol. I) is necessary and sufficient for the commutative law of multiplication and for Assumption P.

**4. Double points of projectivities.** DEFINITION. A projective transformation of a real line into itself is said to be *hyperbolic*, *parabolic*, or *elliptic*,\* according as it has two, one, or no double points.

It was proved in § 58, Vol. I, that the determination of the double points of a projective transformation†

$$(1) \quad \begin{aligned} \rho x'_0 &= ax_0 + bx_1 \\ \rho x'_1 &= cx_0 + dx_1 \end{aligned}$$

depends on the solution of the equation

$$(2) \quad \rho^2 - (a + d)\rho + \Delta = 0,$$

where  $\Delta = ad - bc$ . This equation has two real roots if and only if its discriminant

$$(a + d)^2 - 4\Delta$$

is positive. Hence we have

*If  $\Delta < 0$ , the transformation (1) is hyperbolic. For an elliptic or parabolic projectivity  $\Delta$  is always positive.*

\* These terms are derived from the corresponding types of conic sections (see § 87). In a complex one-dimensional form a somewhat different terminology is used (cf. § 98).

† In this volume we shall generally write homogeneous coordinates in the form  $(x_0, x_1)$ , whereas in Vol. I we used  $(x_1, x_2)$ .

In case the projectivity (1) is an involution,  $a = -d$  (§ 54, Vol. I), and hence  $-4\Delta$  is the discriminant of (2). Hence

*An involution is elliptic or hyperbolic according as  $\Delta$  is positive or negative.*

The intimate connection of these theorems with the theory of linear order is evident on comparison with the first sections of Chap. II. A deduction of the corresponding theorems from the intuitive conceptions of order is to be found in Chap. IV of the *Geometria Proiettiva* of Enriques.

### EXERCISE

A projectivity for which  $\Delta > 0$  is a product of two hyperbolic involutions. A projectivity for which  $\Delta < 0$  is a product of three hyperbolic involutions.

**5. Complex geometry.** Assumption K provides for the solution of many problems of construction which could not be solved in a net of rationality. But even in the real space the fundamental problem of finding the double points of an involution has no general solution.

To see this it is only necessary to set up an involution for which  $\Delta > 0$ . Take any involution of which two pairs of conjugate points  $AA'$  and  $BB'$  form a harmonic set  $H(AA', BB')$ . If the scale  $P_0, P_1, P_2$  is chosen so that  $A = P_0$ ,  $A' = P_2$ ,  $B = P_1$ , then  $B' = P_1$  and the involution is represented by the bilinear equation (§ 54, Vol. I)

$$xx' = -1.$$

The double points of this involution, if existent, would satisfy the equation

$$x^2 = -1,$$

which has no real roots.

An effect of Assumption K is thus to deny the possibility of solving this problem. If, however, we negate Assumption K and replace it by properly chosen other assumptions, we are led to a geometry in which this problem is always soluble, namely, the geometry of the space in which the geometric number system is isomorphic with the complex number system of analysis. Although this geometry does not have the same relation to the space of external nature as the real geometry, it is extremely important because of its relation to other branches of mathematics.

One way of founding this geometry is to replace Assumption K by another assumption of an equally summary character, namely,

J. *A geometric number system is isomorphic with the complex number system of analysis.*

Since this number system obeys the commutative law of multiplication, the corresponding geometry satisfies Assumption P, and all the theorems of Vol. I apply. Thus, a set of postulates for the complex geometry is A, E, J.

The problem of finding the double points of a one-dimensional projectivity is completely solvable in the complex geometry; for any such projectivity may be represented by the bilinear equation (§ 54, Vol. I)

$$cax' + dx' - ax - b = 0,$$

and therefore its double points are given by the roots of

$$cx^2 + (d - a)x - b = 0,$$

which exist in the complex number system.

The analogous result holds good for an  $n$ -dimensional projectivity. In this case the problem reduces to that of finding the roots of an algebraic equation of the  $n$ th degree.

**6. Imaginary elements adjoined to a real space.** In this connection it is desirable to think of another point of view which we may adopt toward the complex space. Suppose we are working in a real geometry on the basis of A, E, K (or of A, E, H, C, R; see below). It is a theorem about the real number system\* that it is contained in a number system (the complex number system) all of whose elements are of the form  $ai + b$  where  $a$  and  $b$  are real and  $i$  satisfies the equation

$$i^2 + 1 = 0.$$

Hence it is a theorem about the real space that it is contained in another space which contains the double points of any given involution.

This may be seen in detail as follows: By the theory of homogeneous coordinates the points of a real projective space S are in a correspondence with the ordered tetrads of real numbers  $(x_0, x_1, x_2, x_3)$ , except  $(0, 0, 0, 0)$ , such that to each tetrad corresponds one point, and to each point a set of tetrads, given by the expression  $(mx_0, mx_1,$

\* This same question is discussed from the point of view of a general space and a general field in Chap. IX, Vol. I.

$mx_3, mx_3)$  where  $x_0, x_1, x_2, x_3$  are fixed and  $m$  takes on all real number values except zero. By the property of the real number system mentioned above, the set of all ordered tetrads of real numbers is contained in the set of all ordered tetrads  $(z_0, z_1, z_2, z_3)$  where  $z_0, z_1, z_2, z_3$  are complex numbers.

Let us define a *complex point* as the class of all ordered tetrads of complex numbers of the form

$$(kz_0, kz_1, kz_2, kz_3)$$

where for a given class  $z_0, z_1, z_2, z_3$  are fixed and not all zero and  $k$  takes on all complex values different from zero. Let the set of these classes satisfying two independent linear equations

$$(3) \quad \begin{aligned} a_0z_0 + a_1z_1 + a_2z_2 + a_3z_3 &= 0, \\ b_0z_0 + b_1z_1 + b_2z_2 + b_3z_3 &= 0 \end{aligned}$$

be called a *complex line*. With these conventions it is easy to see that the set of all complex points and complex lines satisfies the assumptions A, E, P, and thus the complex points constitute a proper projective space. Let us call this space  $S_c$ .

The space  $S_c$  contains the set of all complex points of the form

$$(kx_0, kx_1, kx_2, kx_3)$$

where  $x_0, x_1, x_2, x_3$  are all real. Let us call this subset of complex points  $S_r$ . If any set of complex points of  $S_r$  which satisfy two equations of the form (3) with real coefficients be called a "real line," we have, by reference to the homogeneous coördinate system in  $S$ , that the complex points of  $S_r$  are in such a one-to-one correspondence with the points of  $S$  that to every line in  $S$  corresponds a "real line" in  $S_r$ , and conversely.

Thus,  $S_r$  is a real projective space and is contained in the complex projective space  $S_c$ . Obviously  $S$  may also be regarded as contained in a complex projective space  $S'$  where  $S'$  consists of the points of  $S$  together with the points of  $S_c$  which are not in  $S_r$ , and where each line of  $S'$  consists of the complex points of  $S'$  which satisfy two equations of the form (3) together with the points of  $S$  whose coördinates satisfy the same two equations.

DEFINITION. Points of the real space  $S$  are called *real points*, and points of the extended space  $S'$ , *complex points*. Points in  $S'$  but not in  $S$  are called *imaginary points*.

This discussion of imaginary elements does not require a detailed knowledge or study of the complex number system as such. It is, in fact, a special case of the more general theory in Chap. IX, Vol. I (cf. particularly § 92), which applies to a general projective space. It serves in a large variety of cases where it is sufficient to know merely the *existence* of the complex space  $S'$  containing  $S$  and satisfying Assumptions A, E, P. It is a logically exact way of stating the point of view of the geometers who used imaginary points before the advent of the modern function theory.

There are problems, however, which require a detailed study of the complex space, and this implies, of course, a study of the complex number system and such geometrical subjects as the theory of chains (see §§ 11, 12, below, and later chapters).

There is a very elegant and historically important method of introducing imaginaries in geometry without the use of coördinates, namely, that due to von Staudt.\* It depends essentially on the properties of involutions which are developed in Chap. VIII, Vol. I, and §§ 74-75 of this volume. The reader will find it an excellent exercise to generalize the Von Staudt theory so as to obtain the result stated in Proposition  $K_2$ , Chap. IX, Vol. I.

**7. Harmonic sequence.** We shall now take up a more searching study of the assumptions of the geometry of reals. In Chap. IV, Vol. I, it was proved that every space satisfying Assumptions A, E contains a net of rationality  $R^3$ , and that this net is itself a three-space which satisfies not only Assumptions A and E but also Assumption P (Theorem 20). To this rational subspace, therefore, apply all the theorems in Vol. I which do not depend essentially on Assumption  $H_1$ . For example, every line of  $R^3$  is a linear net of rationality and may be regarded (with the exception of one point chosen as  $\infty$ ) as a commutative number system all of whose numbers are expressible as rational combinations of 0 and 1.

Throughout Vol. I we left the character of this net indeterminate. It might contain only a finite number of points or it might contain an infinite number. We propose now to introduce a new assumption which will fix definitely the structure of a net of rationality.

\* Cf. K. G. C. von Staudt, *Beiträge zur Geometrie der Lage*, Nürnberg (1856 and 1857). J. Lüroth, *Mathematische Annalen*, Vol. VIII (1874), p. 145. Segre, *Memorie della R. Accademia delle scienze di Torino* (2), Vol. XXXVIII (1886).

DEFINITION. Let  $H_0, H_1, H_\infty$  be any three distinct points of a line  $h$ ; let  $S$  and  $T$  be two distinct points collinear with  $H_\infty$  but not on  $h$ ; and let  $K_0$  be a point of intersection of  $SH_0$  and  $TH_1$ . Denote the

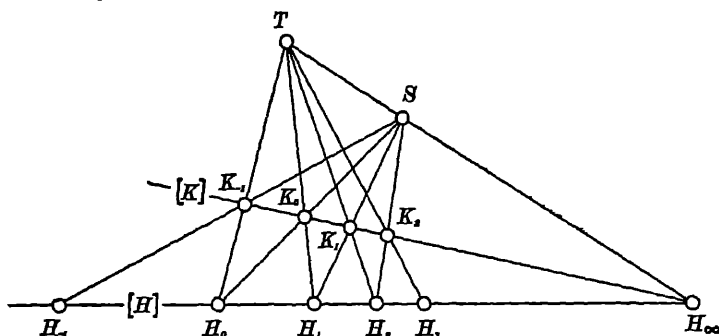


FIG. 3

points of the line  $h$  by  $[H]$  and those of the line  $K_0H_\infty$  by  $[K]$ , and let  $\Pi$  be a projectivity defined by perspectivities as follows:

$$[H] \stackrel{S}{\wedge} [K] \stackrel{T}{\wedge} [\Pi(H)].$$

The set of points

$$H_0, H_1, H_2, \dots, H_i, H_{i+1}, \dots$$

such that  $\Pi(H_i) = H_{i+1}$ , together with the set

$$\dots H_{-i-1}, H_{-i}, \dots, H_{-2}, H_{-1}$$

such that  $\Pi(H_{-i-1}) = H_{-i}$ , is called a *harmonic sequence*. The point  $H_\infty$  is not in the sequence but is called its *limit point*.

The projectivity  $\Pi$  is evidently parabolic and carries  $H_0$  to  $H_1$ .

THEOREM 2. *The middle one of any three consecutive\* points of a harmonic sequence is the harmonic conjugate of the limit point of the sequence with regard to the other two.*

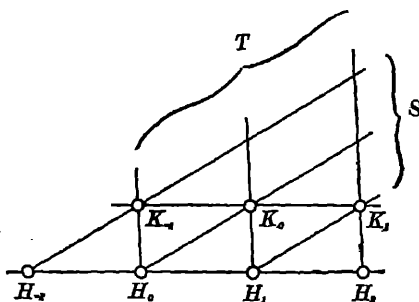


FIG. 4

*Proof.* By construction we have

$$Q(H_\infty H_i H_{i+1}, H_\infty H_{i+2} H_{i+1}).$$

\* This term refers to the subscripts in the notation  $H_j$ .

COROLLARY. *All points of a harmonic sequence belong to the same net of rationality.*

THEOREM 3. *Two harmonic sequences determined by  $H_0, H_1, H_\infty$  and by  $M_0, M_1, M_\infty$  are projective in any projectivity  $\Pi$  by which*

$$H_0 H_1 H_\infty \wedge M_0 M_1 M_\infty.$$

*Proof.* By Theorem 3, Chap. IV, Vol. I, the projectivity  $\Pi$  transforms harmonic sets of points into harmonic sets.

8. **Assumption H.** By reference to fig. 3 it is intuitively evident to most observers that in any picture which can be drawn representing points by dots, and lines by marks drawn with the aid of a straight-edge, no point  $H_i$  which can be accurately marked will ever coincide with  $H_j$  ( $i \neq j$ ). On the other hand, there is nothing in Assumptions A and E to prove that  $H_i \neq H_j$ , because (Introduction, § 2, Vol. I) these assumptions are all satisfied by the miniature spaces discussed in § 72, Chap. VII, Vol. I, and if the number of points on a line is finite, the sequence must surely repeat itself. Thus we are led to make a further assumption.

ASSUMPTION H.\* *If any harmonic sequence exists, not every one contains only a finite number of points.*

The existence of a harmonic sequence determined by any three points follows directly from Assumptions A and E. That any two sequences are projective follows from Theorem 3. Hence Assumption H gives at once

THEOREM 4. *Any three distinct collinear points  $H_0, H_1, H_\infty$  determine a harmonic sequence containing an infinite number of points and having  $H_0$  and  $H_1$  as consecutive points and  $H_\infty$  as the limit point.*

THEOREM 5. *The principle of duality is valid for all theorems deducible from Assumptions A, E, H.*

*Proof.* This principle has been proved in Chap. I, Vol. I, for all theorems deducible from A and E. If  $\eta_0, \eta_1, \eta_\infty$  are any three planes on a line  $l$ , let a line  $l'$  meet them in  $H_0, H_1, H_\infty$  respectively. The projection by  $l$  of the harmonic sequence determined on  $l'$  by  $H_0, H_1, H_\infty$  is the space dual of a harmonic sequence of points. Since the

\* Cf. Gino Fano, *Giornale di Matematiche*, Vol. XXX (1892), p. 106. Obviously Assumption  $H_0$  (Vol. I, p. 45) is a consequence of H. Hence, after introducing Assumption H, we have that a net of rationality satisfies not only A, E, P but also  $H_0$ , and thus every theorem in Vol. I can be applied to a net of rationality.

sequence of points is infinite, so is the sequence of planes. Hence the space dual of Assumption H is true. The principle of duality in a plane or a bundle follows as in § 11, Chap. I, Vol. I.

By reference to the definition of addition in Chap. VI, Vol. I, it is evident on the basis of Assumptions A and E alone that the transformation  $x' = x + a$  is a parabolic projectivity. Denoting it by  $\alpha$ , it is clear that if there is any integer  $n$  such that  $\alpha^n$  is the identity, then  $\alpha^{k+m} = \alpha^m$ ,  $k$  and  $m$  being any integers. Hence, if  $\alpha$  has a finite period, there is only a finite number of points in a harmonic sequence, contrary to Assumption H. Hence

**THEOREM 6.** *A parabolic projectivity never has a finite period. In other words, if of three points determining a harmonic sequence the limit point is taken as  $\infty$  in a scale and two consecutive points as 0 and 1, then the sequence consists of*

$$\begin{array}{ll} 0 & \\ 1 & -1 \\ 1+1=2 & -1-1=-2 \\ 2+1=3 & -2-1=-3 \\ 3+1=4 & -3-1=-4 \\ \vdots & \vdots \end{array}$$

*that is, of zero and all positive and negative integers.*

**COROLLARY 1.** *The net of rationality determined by 0, 1,  $\infty$  consists of zero and all numbers of the form  $\frac{m}{n}$  where  $m$  and  $n$  are positive or negative integers.*

*Proof.* By Theorem 14, Chap. VI, Vol. I, the net of rationality determined by 0, 1,  $\infty$  consists of all numbers obtainable from 0 and 1 by the operations of addition, multiplication, subtraction, and division (excluding division by zero).

**COROLLARY 2.** *The homogeneous coördinates of any point in a linear planar or spatial net of rationality may be taken as integers.*

*Proof.* If  $x_0, x_1, x_2, x_3$  are the homogeneous coördinates of a point in the net, they are defined, according to Chap. VII, Vol. I, in terms of the coördinates in certain linear nets. Hence they may be taken in the form 0 or  $\frac{m_i}{n_1}$  where  $m_i$  and  $n_1$  are integers. If  $m$  is the product of their denominators,  $mx_0, mx_1, mx_2, mx_3$  are integers.

The first of these corollaries enables us to obtain the following simple result with regard to the construction of any point in a net of rationality. Let  $H_{\frac{1}{n}}$  be the harmonic conjugate of  $H_n$  with regard to  $H_1$  and  $H_{-1}$ . The sequence

$$\dots, H_{-\frac{1}{n}}, H_{-\frac{1}{2}}, H_{-1}, H_0, H_1, H_{\frac{1}{2}}, H_{\frac{1}{n}}, \dots$$

is projective (fig. 5) with

$$H_{-3}, H_{-2}, H_{-1}, H_0, H_1, H_2, H_3, \dots$$

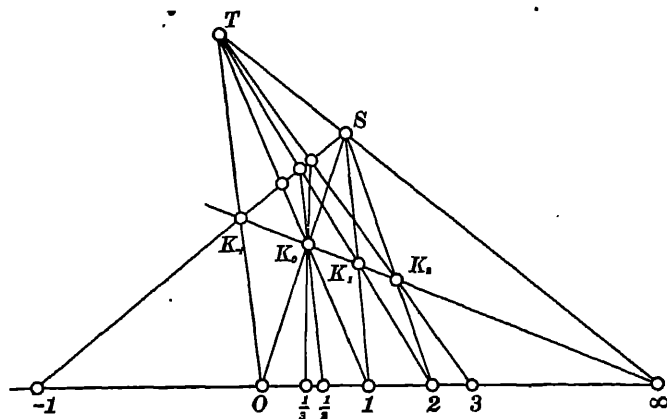


FIG. 5

and therefore must be harmonic. The points  $H_0, H_{\frac{1}{n}}, H_\infty$  determine a harmonic sequence

$$\dots, H_{-\frac{3}{n}}, H_{-\frac{2}{n}}, H_{-\frac{1}{n}}, H_0, H_{\frac{1}{n}}, H_{\frac{2}{n}}, H_{\frac{3}{n}}, \dots$$

By Cor. 1, any point of the net of rationality is contained in a sequence of the last variety for some value of  $n$ .

**9. Order in a net of rationality.** DEFINITION. If  $A$  and  $B$  are points of  $R(H_0H_1H_\infty)$  different from  $H_\infty$ ,  $A$  is said to *precede*  $B$  with respect to the scale  $H_0, H_1, H_\infty$  if and only if the nonhomogeneous coordinate (cf. § 53, Vol. I) of  $A$  is less than the nonhomogeneous coordinate of  $B$ . If  $A$  precedes  $B$ ,  $B$  is said to *follow*  $A$ .

From the corresponding properties of the rational numbers there follow at once the fundamental propositions: With respect to the scale  $H_0, H_1, H_\infty$ , (1) if  $A$  precedes  $B$ ,  $B$  does not precede  $A$ ; (2) if  $A$  precedes  $B$  and  $B$  precedes  $C$ , then  $A$  precedes  $C$ ; (3) if  $A$  and  $B$  are distinct points of  $R(H_0H_1H_\infty)$ , then either  $A$  precedes  $B$  or  $B$  precedes  $A$ .

The use of the properties of numbers in the argument above and in analogous cases does not imply that our treatment of geometry is dependent on analytical foundations. Every theorem which we employ here is a logical consequence of the assumptions A, E, H alone.

The argument which is involved in the present case may be stated as follows: The coördinates relative to a scale  $H_0, H_1, H_\infty$  of the points

$$\dots, H_{-2}, H_{-1}, H_0, H_1, H_2, \dots$$

of a harmonic sequence, when combined according to the rules for addition and multiplication given in Chap. VI, Vol. I, satisfy the conditions which are known to characterize the system of positive and negative integers (including zero). From these conditions (the axioms of the system of positive and negative integers) follow theorems which state the order relations among these integers, and also theorems which state the order relations among the rational numbers, the latter being defined in terms of the integers. But by Theorem 6, Cor. 1, the rational numbers are the coördinates of points in  $R(H_0H_1H_\infty)$ . Hence the points of  $R(H_0H_1H_\infty)$  satisfy the conditions given above.

It would of course be entirely feasible to make the discussion of order in a net of rationality without the use of coördinates.

**\*10. Cuts in a net of rationality.** DEFINITION. Two subsets,  $[A]$  and  $[B]$ , of a net of rationality  $R(H_0H_1H_\infty)$  constitute a *cut*  $(A, B)$  with respect to the scale  $H_0, H_1, H_\infty$  if and only if they satisfy the following conditions: (1) Every point of the net except  $H_\infty$  is in  $[A]$  or  $[B]$ ; (2) with respect to the scale  $H_0, H_1, H_\infty$  every point of  $[A]$  precedes every point of  $[B]$ . If there is a point  $O$  in  $[A]$  or in  $[B]$  such that every point of  $[A]$  distinct from  $O$  precedes it and every point of  $[B]$  distinct from  $O$  follows it, the cut is said to be *closed* and to have  $O$  as its *cut-point*; otherwise the cut is said to be *open*. The class  $[A]$  is said to be the *lower side* and  $[B]$  to be the *upper side* of the cut.

With respect to the scale  $H_0, H_1, H_\infty$  any point  $O (O \neq H_\infty)$  of a net  $R(H_0H_1H_\infty)$  determines two sets of points  $[A]$  and  $[B]$  such that every  $A$  precedes or is identical with  $O$  and  $O$  precedes every  $B$ . These sets of points are therefore a closed cut having  $O$  as cut-point. Not every cut, however, is closed, for consider the set  $[A]$ , including all points whose coördinates in a system of nonhomogeneous coördinates having  $H_\infty$  as the point  $\infty$  are negative or, if positive, such that their squares are less than 2; and the set  $[B]$ , including all points whose

\* An asterisk at the left of a section number indicates that the section may be omitted on a first reading. We have marked in this manner most of the sections which are not essential to an understanding of the discussion of metric geometry in Chaps. III and IV.

coördinates are positive and have their squares greater than 2. Since no rational number can satisfy the equation

$$x^2 = 2,$$

this equation is not satisfied by the coördinates of any point in the net. The sets  $[A]$  and  $[B]$  constitute an open cut.

DEFINITION. With respect to the scale  $H_0, H_1, H_\infty$ , an open cut *precedes* all the points of its upper side and *is preceded by* all points of its lower side. A closed cut *precedes* all the points which its cut-point precedes and *is preceded by* all points by which its cut-point is preceded. A cut  $(A, B)$  *precedes* a cut  $(C, D)$  if and only if there is a point  $B$  preceding a point  $C$ .

THEOREM 7. (1) *If a cut  $(A, B)$  precedes a cut  $(C, D)$ , then  $(C, D)$  does not precede  $(A, B)$ .*

(2) *If a cut  $(A, B)$  is not the same as the cut  $(C, D)$ , then either  $(A, B)$  precedes  $(C, D)$  or  $(C, D)$  precedes  $(A, B)$ , or both cuts are closed and have the same cut-point.*

(3) *If a cut  $(A, B)$  precedes a cut  $(C, D)$  and  $(C, D)$  precedes a cut  $(E, F)$ , then  $(A, B)$  precedes  $(E, F)$ .*

*Proof.* These propositions are direct consequences of the definition above and of the corresponding properties of the relation of precedence between points.

DEFINITION. With respect to the scale  $H_0, H_1, H_\infty$ , a cut  $(A_1, A_2)$  is said to be *between* two cuts  $(B_1, B_2)$  and  $(C_1, C_2)$  in case  $(B_1, B_2)$  precedes  $(A_1, A_2)$  and  $(A_1, A_2)$  precedes  $(C_1, C_2)$  or in case  $(C_1, C_2)$  precedes  $(A_1, A_2)$  and  $(A_1, A_2)$  precedes  $(B_1, B_2)$ . If any one of these cuts is closed, it may be replaced by its corresponding cut-point in this definition. (Thus, for example, any open cut is between any point of its upper side and any point of its lower side.)

An open cut  $(A, B)$  is said to be algebraic if there exists an equation,

$$a_0x^n + a_1x^{n-1} + \dots + a_n = 0,$$

with integral coefficients, and two points  $A_0, B_0$ , such that the coördinates of all points of  $[A]$  between  $A_0$  and  $B_0$  make the left-hand member of this equation greater than zero and all points of  $[B]$  between  $A_0$  and  $B_0$  make it less than zero.\* If it is assumed that this equation has a root between  $A_0$  and  $B_0$ , this is equivalent to assuming that there exists a point corresponding to the cut  $(A, B)$  on the line  $A_0B_0$  but not in the given net.

\* It is perhaps needless to remark that not every algebraic equation with integral coefficients can be associated in this way with a cut. For example,  $x^2 + 1 = 0$ .

For the purposes of geometric constructions it would be sufficient to assume the existence of cut-points for all algebraic open cuts (see Chap. IX, Vol. I). For many purposes, indeed, it would be desirable to make the assumption referred to on p. 97, Chap. IV, Vol. I, and which we here put down for reference as Assumption Q.

**ASSUMPTION Q.** *There is not more than one net of rationality on a line.*

But it is customary in analysis to assume the existence of an irrational number corresponding to every open cut in the system of rationals, and it is convenient in geometry to have a one-to-one correspondence between the points of a line and the system of real numbers. Hence we make the assumption which follows in the next section.

It must not be supposed that in the assumption which follows we are introducing new points in any respect different from those already considered. What we are doing is to postulate that a space is a class of points having certain additional properties. The assumption limits the type of space which we consider; it does not extend the class of points. In this respect our procedure is not parallel to the genetic method of developing the theory of irrational numbers.

### EXERCISE

The points of  $R(H_0H_1H_\infty)$ , together with the open cuts with respect to the scale  $H_0, H_1, H_\infty$ , constitute a set  $[X]$  of things having the following property: If  $[S]$  and  $[T]$  are any two subclasses of  $[X]$  including all  $X$ 's and such that every  $S$  precedes every  $T$ , then there is either an  $S$  or a  $T$  which precedes all other  $T$ 's and is preceded by all other  $S$ 's.

**\*11. Assumption of continuity.** We shall denote the cut-point of a closed cut  $(M, N)$  by  $P_{(M, N)}$ . In the following assumption it is not stated whether the cuts  $(A_1, A_2)$ ,  $(B_1, B_2)$ , and  $(D_1, D_2)$  are open or closed. If one of them is closed, therefore, the corresponding one of the symbols  $P_{(A_1, A_2)}$ ,  $P_{(B_1, B_2)}$ , and  $P_{(D_1, D_2)}$  must be understood in the sense just defined.

**ASSUMPTION C.** *If every net of rationality contains an infinity of points, then on one line  $l$  in one net  $R(H_0H_1H_\infty)$  there is associated with every open cut  $(A, B)$ , with respect to the scale  $H_0, H_1, H_\infty$ , a point  $P_{(A, B)}$  which is on  $l$  and such that the following conditions are satisfied:*

(1) *If two open cuts  $(A, B)$  and  $(C, D)$  are distinct, the points  $P_{(A, B)}$  and  $P_{(C, D)}$  are distinct;*

(2) *If  $(A_1, A_2)$  and  $(B_1, B_2)$  are any two cuts and  $(C, C_2)$  any open cut between two points  $A$  and  $B$  of  $R(H_0H_1H_\infty)$ , and if  $T$  is a projectivity such that*

$$T(H_\infty AB) = H_\infty P_{(A_1, A_2)} P_{(B_1, B_2)},$$

*then  $T(P_{(C, C_2)})$  is a point associated with some cut  $(D_1, D_2)$  between  $(A_1, A_2)$  and  $(B_1, B_2)$ .*

DEFINITION. The set of all points of  $R(H_0H_1H_\infty)$ , together with all points associated with cuts in  $R(H_0H_1H_\infty)$ , with respect to the scale  $H_0, H_1, H_\infty$ , is called the *chain*  $C(H_0H_1H_\infty)$ . The points of  $R(H_0H_1H_\infty)$  are called *rational*, and any other point of the chain is called *irrational* with respect to  $R(H_0H_1H_\infty)$ . A point associated with a cut which follows  $H_0$  is called *positive*, and one associated with a cut which precedes  $H_0$  is called *negative*.

THEOREM 8. *The point  $R_{A, B}$ , associated, by Assumption C, with an open cut  $(A, B)$  of  $R(H_0H_1H_\infty)$ , is not a point of  $R(H_0H_1H_\infty)$ .*

*Proof.* The associated point could not be  $H_\infty$ , because there are projectivities of  $R(H_0H_1H_\infty)$  which leave  $H_\infty$  invariant and change the given cut into different cuts, and therefore, by Assumption C, change the associated point. Now suppose a point  $D$ , distinct from  $H_\infty$  but in  $R(H_0H_1H_\infty)$ , to be associated with some open cut. Since the given cut is open, there must be a point  $A$  between  $D$  and the cut. If  $B$  is a point on the opposite side of the cut from  $D$ ,  $A$  and  $B$  both precede or both follow  $D$  with respect to the scale  $H_0, H_1, H_\infty$ . The transformation which changes every point of  $l$  into its harmonic conjugate with regard to  $H_\infty$  and  $D$  has, when regarded as a transformation of the points of  $R(H_0H_1H_\infty)$  with respect to the scale  $H_0, H_1, H_\infty$ , the equation

$$x' = 2d - x,$$

where  $d$  is the coördinate of  $D$ . It therefore transforms rational points which follow  $D$  into rational points which precede it, and vice versa. Hence  $A$  and  $B$  are transformed into two points,  $A'$  and  $B'$ , which precede  $D$  if  $A$  and  $B$  follow  $D$ , or which follow  $D$  if  $A$  and  $B$  precede  $D$ . By Assumption C (2), the point  $D$  which is associated with an open cut between  $A$  and  $B$  is transformed into a point  $D'$  associated with a cut between  $A'$  and  $B'$ . By Assumption C (1),  $D'$  is distinct from  $D$ , contrary to the hypothesis that  $D$  is a fixed point of the transformation.

THEOREM 9. *The points of  $C(H_0H_1H_\infty)$ , excluding  $H_\infty$ , form, with reference to the scale in which  $H_0 = 0, H_1 = 1, H_\infty = \infty$ , a number system isomorphic with the real number system of analysis.*

*Proof.* The definitions of Chap. VI, Vol. I, give a meaning to the operations of addition and multiplication for all points of the line  $l$ . In that place we derived all the fundamental laws of operation, except

the commutative law of multiplication, on the basis of Assumptions A and E. We have also seen in the present chapter (Theorem 6, Cor. 1) that the coördinates of points in  $R(H_0H_1H_x)$  are the ordinary rational numbers. Hence it remains to show that the geometric laws of combination as applied to the irrational points of  $C(H_0H_1H_x)$  are the same as for the ordinary irrational numbers.

The analytic definition of addition of irrational numbers\* may be stated as follows: If  $a$  and  $b$  are two numbers defined by cuts  $(x_1, y_1)$  and  $(x_2, y_2)$ , then  $a + b$  is the number defined by the cut  $(x_1 + x_2, y_1 + y_2)$ .

To show that our geometric number system satisfies this condition in  $C(H_0H_1H_x)$ , suppose first that  $a$  is a rational point of  $C(H_0H_1H_\infty)$  and  $b$  an irrational point. The projective transformation

$$(4) \quad x' = x + a$$

changes the set of points  $[x_2]$  into the set  $[x_2 + a]$ , which is the same as  $[x_2 + x_1]$ . Similarly, it changes  $[y_2]$  into  $[y_2 + y_1]$ . Hence, it changes the cut  $(x_2, y_2)$  into  $(x_1 + x_2, y_1 + y_2)$ , and hence, by Assumption C (2), changes  $b$  into a point determined by a cut which lies between every pair  $x_1 + x_2$  and  $y_1 + y_2$ . Therefore  $b$  is changed into the point associated with the cut  $(x_1 + x_2, y_1 + y_2)$ . But the transform of  $b$  is  $a + b$ . Hence the geometric sum  $a + b$  is the number defined by the cut  $(x_1 + x_2, y_1 + y_2)$ .

Next, suppose both  $a$  and  $b$  irrational. The transformation (4) changes  $[x_2]$  into the set of irrational points  $[x_2 + a]$ ,  $b$  into  $b + a$ , and  $[y_2]$  into  $[y_2 + a]$ . By the paragraph above, the cut which defines any  $x_2 + a$  precedes the cut which defines any  $y_2 + a$ . Hence, by Assumption C (2), the cut which defines any point  $x_2 + a$  precedes the cut which defines  $b + a$ , and this precedes the cut which defines  $y_2 + a$ . Any point  $x_1 + x_2$  of the lower side of the cut  $(x_1 + x_2, y_1 + y_2)$  precedes the cut defining one of the points  $x_2 + a$ , by the paragraph above, and hence precedes the cut defining  $b + a$ . Similarly, any point of the upper side of this cut follows the cut defining  $b + a$ . Hence  $(x_1 + x_2, y_1 + y_2)$  is the cut defining  $b + a$ . Thus we have identified geometric addition of points in  $C(H_0H_1H_\infty)$  with the addition of ordinary real numbers.

\* Cf. Fine, College Algebra, p. 50; or Veblen and Lennes, Infinitesimal Analysis, Chap. I.

The analytic definition of multiplication of irrational numbers may be stated as follows: If  $a$  and  $b$  are positive numbers defined by the cuts  $(x_1, y_1)$  and  $(x_2, y_2)$ , let  $[x'_1]$  be the set of positive values of  $x_1$ . Then  $ab$  is the number defined by the cut  $(x'_1x_2, y_1y_2)$ . If  $a$  is negative and  $b$  positive,  $ab = -(-a)b$ . If  $a$  is positive and  $b$  negative,  $ab = -(a(-b))$ . If both  $a$  and  $b$  are negative,  $ab = (-a)(-b)$ . If  $a = 0$  or  $b = 0$ ,  $ab = 0$ .

Consider the transformation

$$x' = ax.$$

If  $a$  is positive and rational while  $b$  is positive and irrational, this transforms  $[x_2]$  into  $[ax_2]$ , which is the same as  $[x'_1x_2]$ . It also transforms  $b$  into  $ab$  and  $[y_2]$  into  $[ay_2]$ , which is the same as  $[y_1y_2]$ . Hence, by Assumption C(2),  $ab$  is the number associated with  $(x'_1x_2, y_1y_2)$ .

If both  $a$  and  $b$  are irrational and positive, we again have  $[x_2]$ ,  $b$ , and  $[y_2]$  transformed into  $[ax_2]$ ,  $ab$ , and  $[ay_2]$ , where, as in the analogous case of addition, the cut defining  $ax_2$  precedes the cut defining  $ab$ , which in turn precedes the cut defining  $ay_2$ . Moreover, any  $x'_1x_2$  precedes some  $ax_2$ , and any  $y_1y_2$  follows some  $ay_2$ . Hence, by the same argument as in the case of addition,  $(x'_1x_2, y_1y_2)$  is the cut with which  $ab$  is associated.

The transformation

$$x' = (-1)x$$

changes the cut  $(x_1, x_2)$  defining the irrational number  $a$  into the open cut  $(-x_2, -x_1)$ , which therefore defines an irrational  $a'$ . But since  $x_1 - x_2$  may be any negative rational and  $x_2 - x_1$  may be any positive rational, the sum of  $a$  and  $a'$ , which has been proved to be determined by the cut  $(x_1 - x_2, x_2 - x_1)$ , must be zero. Hence we have that  $(-1)a$  is the irrational  $-a$  such that  $-a + a = 0$ .

The transformation

$$x' = x(-1)$$

is the same as  $x' = (-1)x$  for all rational points. Hence, by Assumption C(2), these transformations are the same for all points of  $C(H_0H_1H_\infty)$ . Hence, for points of  $C(H_0H_1H_\infty)$ ,  $(-1)x = x(-1)$ .

By the associative law of multiplication (which, it is to be remembered, depends only on Assumptions A and E) we have, if  $a$  is negative and  $b$  positive,

$$ab = -(-a)b,$$

where  $(-a)b$  is determined by the analytic (cut) rule. If  $a$  is positive and  $b$  is negative, it follows similarly, with the aid of the relation  $(-1)a = a(-1)$ , that

$$ab = a(-1)(-b) = -(a(-b));$$

and if both  $a$  and  $b$  are negative,

$$ab = (-1)(-a)(-1)(-b) = (-a)(-b).$$

COROLLARY. *With respect to a scale in which  $H_\infty = \infty$ ,  $H_0 = 0$ ,  $H_1 = 1$ , we have  $ab = ba$  whenever  $a$  and  $b$  are in  $C(H_0H_1H_\infty)$ .*

THEOREM 10. *Any projectivity which transforms  $H_0$ ,  $H_1$ , and  $H_\infty$  into points of the chain  $C(H_0H_1H_\infty)$  transforms any point of the chain into a point of the chain.*

*Proof.* We have seen that  $x' = ax$  and  $x' = x + a$ , for rational or irrational values of  $a$ , are projectivities which change  $H_\infty$  into itself and all other points of  $C(H_0H_1H_\infty)$  into points of the chain. The transformation  $x' = 1/x$  is a projectivity which interchanges  $H_\infty$  and  $H_0$  (see § 54, Chap. VI, Vol. I), and by Theorem 9 it changes every point of  $C(H_0H_1H_\infty)$ , except  $H_\infty$  and  $H_0$ , into a point of  $C(H_0H_1H_\infty)$ .

As in the proof of Theorem 11, Chap. VI, Vol. I, it follows that  $H_0$ ,  $H_1$ ,  $H_\infty$  can be transformed into any three points of the chain by a product of transformations of these three types. Moreover, any projectivity is fully determined as a transformation of  $C(H_0H_1H_\infty)$  by the three points  $B_0$ ,  $B_1$ ,  $B_\infty$  into which it transforms  $H_0$ ,  $H_1$ ,  $H_\infty$ . For, suppose there were two such projectivities,  $\Pi$  and  $\Pi'$ , the product  $\Pi^{-1}\Pi'$  would transform  $H_0$ ,  $H_1$ ,  $H_\infty$  into themselves. Hence, by Theorem 16, Chap. IV, Vol. I, it would leave invariant every point of  $R(H_0H_1H_\infty)$ . Hence, by Assumption C(2), it would leave invariant every point of  $C(H_0H_1H_\infty)$ . Hence  $\Pi^{-1}\Pi'$  would be the identity for all points of the chain, and  $\Pi$  would be the same as  $\Pi'$  for all points of the chain. Hence every projectivity changing  $H_0$ ,  $H_1$ ,  $H_\infty$  into points of the chain is expressible as a product of projectivities of the forms  $x' = ax$ ,  $x' = x + a$ ,  $x' = 1/x$ . As all these transform the chain into itself, the theorem follows.

COROLLARY 1. *Any projectivity leaving invariant three points of the chain  $C(H_0H_1H_\infty)$  leaves every point of the chain invariant.*

*Proof.* Let  $\Pi$  be the given projectivity leaving the given points, say  $B_0$ ,  $B_1$ ,  $B_\infty$ , invariant. Let  $P$  be the projectivity such that  $P(B_0B_1B_\infty) = (H_0H_1H_\infty)$ . Then  $\Pi P^{-1}$  leaves  $H_0$ ,  $H_1$ ,  $H_\infty$  invariant and hence

leaves all points of the chain invariant, as shown in the proof of the theorem. Hence  $\Pi$  leaves all points of the chain invariant.

COROLLARY 2. *Any projectivity of the chain  $C(H_0H_1H_\infty)$  into itself is of the form*

$$\begin{aligned} \rho x'_0 &= ax_0 + bx_1, \\ \rho x'_1 &= cx_0 + dx_1, \end{aligned} \quad \begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0,$$

where the coefficients are real numbers.

**\*12. Chains in general.** DEFINITION. If  $(A, B)$  is an open cut in any net of rationality  $R(K_0K_1K_\infty)$  with respect to the scale  $K_0, K_1, K_\infty$ , let  $\Pi$  be a projectivity transforming  $R(K_0K_1K_\infty)$  into  $R(H_0H_1H_\infty)$  and  $K_\infty$  into  $H_\infty$ . This projectivity transforms  $(A, B)$  into a cut  $(C, D)$  in  $R(H_0H_1H_\infty)$  with respect to the scale  $H_0, H_1, H_\infty$ . If  $X$  is the point associated by Assumption C with  $(C, D)$ , the point  $\Pi^{-1}(X) = X'$  is called *the irrational cut-point associated with  $(A, B)$* .

The point  $X'$  is independent of the particular projectivity  $\Pi$ . For let  $\Pi'$  be any projectivity changing  $(A, B)$  into a cut  $(E, F)$  in  $R(H_0H_1H_\infty)$  with respect to the scale  $H_0, H_1, H_\infty$ , and let  $Y$  be the point associated with  $(E, F)$  and  $Y' = \Pi'^{-1}(Y)$ . Then  $\Pi \cdot \Pi'^{-1}$  changes  $(E, F)$  into  $(C, D)$  and hence, by Assumption C (2), must change  $Y$  into  $X$ . This can take place only if  $Y' = X'$ , that is, only if the cut-point  $X'$  associated with  $(A, B)$  is unique.

By projecting any net of rationality into  $R(H_0H_1H_\infty)$  it is shown that the cut-points associated with it satisfy the conditions stated for the points associated with the cuts of  $R(H_0H_1H_\infty)$  in Assumption C. Hence the theorems of the last section also apply to any chain whatever, a chain being defined as follows:

DEFINITION. The totality of points of a net of rationality  $R(ABC)$ , together with all the irrational cut-points defined by open cuts with respect to the scale  $A, B, C$  in  $R(ABC)$ , is called the *chain* defined by  $A, B, C$  and is denoted by  $C(ABC)$ . The irrational cut-points are said to be *irrational with respect to  $R(ABC)$* .

Thus we have

THEOREM 11. (1) *The projective transform of a chain is a chain.*

(2) *Every open cut in any net of rationality defines a unique irrational cut-point collinear with, but not in, the net.*

(3) *If two such cuts with respect to the same scale and in the same net are distinct, their cut-points are distinct.*

(4) *If two open cuts are homologous in a projectivity, their cut-points are homologous in the same projectivity.*

(5) *Any projectivity which transforms three points  $A, B, C$  into three points of the chain  $C(ABC)$  transforms any point of the chain into a point of the chain.*

**THEOREM 12.** *There is one and only one chain containing three distinct points of a line.*

*Proof.* Let  $A, B, C$  be the given points. They belong to the chain  $C(ABC)$  into which  $C(H_0H_1H_\infty)$  is transformed by a projectivity such that  $H_0H_1H_\infty \bar{\wedge} ABC$ . By Theorem 11 (5) any projectivity such that  $ABC \bar{\wedge} BAC$  transforms all points of  $C(ABC)$  into points of  $C(ABC)$ . But by definition such a projectivity transforms  $C(ABC)$  into  $C(BAC)$ ; hence  $C(BAC)$  is contained in  $C(ABC)$ . In like manner  $C(ABC)$  is contained in  $C(BAC)$ . Hence  $C(ABC) = C(BAC) = C(BCA)$ , etc.

Now suppose  $A, B, C$  to be points of some other chain  $C(PQR)$ . By Theorem 11 (5) a projectivity such that  $* PQR \bar{\wedge} QPA$  changes all points of  $C(PQR)$  into points of  $C(PQR)$ . But by definition it changes  $C(PQR)$  into  $C(QPA)$ . Hence  $C(QPA)$  is contained in  $C(PQR)$ . But the same projectivity changes  $C(QPA)$  into  $C(PQR)$ . Hence  $C(PQR) = C(QPA)$ . In like manner  $C(QPA) = C(PBA) = C(CBA) = C(ABC)$ .

**COROLLARY.** *A chain contains the irrational cut-point of every open cut in any net of rationality in the chain.*

**THEOREM 13. THE FUNDAMENTAL THEOREM OF PROJECTIVITY FOR A CHAIN.** *If  $A, B, C, D$  are distinct points of a chain and  $A', B', C'$  any three distinct points of a line, then for any projectivities giving  $(A, B, C, D) \bar{\wedge} (A', B', C', D')$  and  $(A, B, C, D) \bar{\wedge} (A', B', C', D'_1)$  we have  $D' = D'_1$ .*

*Proof.* Let  $\Pi, \Pi_1$  be the two projectivities mentioned in the theorem.  $\Pi_1^{-1}\Pi$  then leaves every point of  $C(ABC)$  fixed; for it leaves every point of  $R(ABC)$  fixed, and hence, by Theorem 11 (4), must leave every irrational cut-point of an open cut in  $R(ABC)$  fixed. But  $\Pi_1^{-1}\Pi$  is then the identical transformation as far as the points of  $C(ABC)$  are concerned. Hence  $D' = D'_1$ .

\* Cf. Theorem 2, Chap. III, Vol. I.

This theorem may also be stated as follows:

*Any projective correspondence between the points of two chains is uniquely determined by three pairs of homologous points.*

Our list of assumptions for the geometry of reals may now be completed by the following assumption of closure.

ASSUMPTION R. *On at least one line, if there is one there is not more than one chain.*

It follows at once, by Theorem 12, that every line is a chain. It also follows, by an argument strictly analogous to the proof of Theorem 5, that the dual propositions of Assumptions C and R are true. Hence we have

THEOREM 14. *The principle of duality is valid for all theorems deducible from Assumptions A, E, H, C, R.*

**\*13. Consistency, categoricalness, and independence of the assumptions.** Let us now apply the logical canons explained in the Introduction (Vol. I) to the foregoing set of assumptions.

THEOREM 15. *Assumptions A, E, H, C, R are consistent if the real number system of analysis is existent.*

*Proof.* Consider the class of all ordered tetrads of real numbers  $(x_0, x_1, x_2, x_3)$ , with the exception of  $(0, 0, 0, 0)$ . Any class of these ordered tetrads such that if one of its members is  $(a_0, a_1, a_2, a_3)$  all its other members are given by the formula  $(ma_0, ma_1, ma_2, ma_3)$ , where  $m$  is any real number not zero, shall be called a point. Any class consisting of all points whose component tetrads satisfy two independent linear homogeneous equations

$$u_0x_0 + u_1x_1 + u_2x_2 + u_3x_3 = 0,$$

$$v_0x_0 + v_1x_1 + v_2x_2 + v_3x_3 = 0$$

shall be called a line. The class of all points and lines so defined satisfy the assumptions A, E, H, C, R (cf. § 4, Vol. I).

THEOREM 16. *Assumptions A, E, H, C, R form a categorical set.*

*Proof.* In Chap. VII, Vol. I, it has been proved that the points of a space satisfying Assumptions A, E, P can be denoted by homogeneous coördinates which are numbers of the geometric number system of Chap. VI, Vol. I. Since P is a logical consequence of A, E, H, C, R (cf. Theorem 13), this result applies here, and by Theorem 9 the

number system in question is isomorphic with the real number system of analysis.

Now if two spaces  $S_1$  and  $S_2$  satisfy  $A, E, H, C, R$ , consider a homogeneous coördinate system in each space and let each point of  $S_1$  correspond to that point of  $S_2$  which has the same coördinates. This correspondence is evidently such that if three points of  $S_1$  are collinear, their correspondents in  $S_2$  are collinear.

It is worthy of remark that the above correspondence may be set up in as many ways as there are collineations of  $S_1$  into itself.

**THEOREM 17.** *Assumptions  $A 1, A 2, A 3, E 0, E 1, E 2, E 3, E 3', H, C, R$  are an independent set.*

*Proof.* The method of proving that a given assumption is not a logical consequence of the other assumptions was explained in the Introduction, p. 6, Vol. I. Suppose there is given a class of objects  $[x]$  and a class of subclasses of  $[x]$ . If we call each  $x$  a point and each element of the class of subclasses a line, then each of our assumptions, when thus interpreted, will be either true or false\* with respect to this interpretation. If all the assumptions but one are true and the one is false, it cannot be a logical consequence of the others; for a logical consequence of true statements must be true. In the sequel we shall call the objects,  $x$ , pseudo-points, and the subclasses of  $[x]$  which play the rôle of lines, pseudo-lines.

**A 1.** The pseudo-points shall be the points of a real projective plane  $\pi$  together with one other point  $O$ . The pseudo-lines shall be the lines of  $\pi$ . **A 1** is false because there is no pseudo-line containing  $O$ . **A 2** is true because it is satisfied by the ordinary projective plane. **A 3** is true because the only sets of points  $A, B, C, D, E$  which satisfy its hypothesis are in  $\pi$ . The only pseudo-plane is  $\pi$ , and there is no pseudo-space. Hence it is evident that **E 0, E 1, E 2, E 3** are true and **E 3'** is vacuously true. Assumptions **H, C, R** are evidently true.

\* If the hypothesis of a statement is not verified, we regard the statement as true. Following the terminology of E. H. Moore (Transactions of the American Mathematical Society, Vol. III, p. 489), we shall describe statements which are true in this sense as "vacuously true" or "vacuous."

It is possible to put any or all of the assumptions into a form such that they are vacuous for the ordinary real space. For example, Professor Moore has pointed out that **A 1** could be replaced by the following proposition, which is vacuous for ordinary space.

**A 1.** Let  $A$  be a point and  $B$  be a point. If there is no line which is on  $A$  and on  $B$ , then  $A = B$ .

A 2. The pseudo-points shall be the points of a real projective three-space  $S_3$  together with one other pseudo-point  $O$ . The pseudo-lines shall be the lines of  $S_3$ , each pseudo-line, however, containing  $O$ . Thus any two pseudo-points are collinear with  $O$ ; a pseudo-plane is an ordinary plane together with  $O$ ; a pseudo-space is  $S_3$  together with  $O$ . Hence it is evident that A 2 is false and A 1, A 3, E 0, E 1, E 2, E 3, E 3' are true. There exist harmonic sequences of pseudo-points, some of which are ordinary harmonic sequences. Hence Assumption H is true. By reference to the definition of a quadrangular set and harmonic conjugate it is clear (because every line contains  $O$ ) that any pseudo-point  $P$  is harmonically conjugate to  $O$  with regard to any two pseudo-points which are collinear with  $P$ . Hence a linear net of rationality contains all the pseudo-points of a pseudo-line. The operations of addition and multiplication are not unique, however, and hence the definition of order does not apply; there are no open cuts, and Assumptions C and R are vacuously true.

A 3. The pseudo-points shall be the points of a real projective space  $S_3$ , with the exception of a single point  $O$ . The pseudo-lines shall be the lines of  $S_3$ , except that in case of those lines which pass through  $O$  the pseudo-lines do not contain  $O$ . Clearly A 3 is false whenever the pseudo-points  $A, B, C, D, E$  are chosen so that the lines  $AB$  and  $DE$  meet in  $O$ . A 1, A 2, E 0, E 1, E 2, E 3, E 3' are obviously true. A harmonic sequence and a net of rationality of pseudo-points can be found identical with an ordinary harmonic sequence and net of rationality on any line not passing through  $O$ . Hence H, C, and R are also true.

E 0. The pseudo-points shall be the vertices of a tetrahedron, and the pseudo-lines the six pairs of pseudo-points. Thus the pseudo-planes are the trios of pseudo-points, and a pseudo-space consists of all four pseudo-points. A 1 and A 2 are obviously true. A 3 is true because we may have  $E=A$  and  $D=B$ . E 1, E 2, E 3, E 3' are true. H, C, R are vacuously true.

E 1. There shall be one pseudo-point and no pseudo-line. E 1 is false and all the other assumptions are vacuously true.

E 2. There shall be three pseudo-points and one pseudo-line containing all three pseudo-points. A 1, A 2, E 0, E 1 are true. A 3, E 3, E 3', H, C, R are vacuously true.

E 3. The pseudo-points and pseudo-lines shall be the points and lines of a real projective plane. A 1, A 2, A 3, E 0, E 1, E 2, H, C, R are true and E 3' is vacuous.

E 3'. The pseudo-points and pseudo-lines shall be the points and lines of a real four-dimensional projective space. E 3' is false and all the other assumptions are true.

H. The pseudo-points and pseudo-lines shall be the points and lines of any modular projective three-space (cf. § 72, Vol. I, and § 16, below). All the assumptions A and E are true, H is false, and C and R are vacuously true.

C. The pseudo-points and pseudo-lines shall be the points and linear nets of rationality of a three-dimensional net of rationality in an ordinary real projective space. All the assumptions are true except C, which is false. R is vacuously true.

R. The pseudo-points and pseudo-lines shall be defined as the points and lines in Theorem 15, the coördinates, however, being elements of the system of ordinary complex numbers. All the assumptions are true except R, which is false.

Assumption C, which is more complicated in its statement than the others, is, however, such that neither of the two statements into which it is separated may be omitted. This result is established in the following theorem:

**THEOREM 18.** *Assumption C(1) is not a consequence of Assumption C(2) and all the other assumptions. Assumption C(2) is not a consequence of C(1) and of the other assumptions even if we add to C(1) the following: If a projectivity transforms  $H_\infty$  into itself and  $H_0$  and  $H_1$  into points of  $R(H_0H_1H_\infty)$ , and transforms an open cut  $(A, B)$  into an open cut  $(C, D)$ , it transforms the point associated with  $(A, B)$  into the point associated with  $(C, D)$ .*

*Proof.\** (1) Any real number  $x$  determines a class  $K_x$  of numbers of the form  $ax + b$  where  $a$  and  $b$  are any rationals.  $K_x$  is the same as  $K_{ax+b}$  for all rational values of  $a$  and  $b$ . Hence, if  $x$  and  $y$  are two irrationals,  $K_x$  and  $K_y$  are either identical or mutually exclusive. Thus the class of all real numbers falls into a set of mutually exclusive

\* This argument makes use of portions of the theory of classes which could not be treated adequately without a long digression. Hence we assume knowledge of the methods and terminology of this branch of mathematics without further explanation.

classes  $[K]$ . With each class  $K$  we associate a particular one of its numbers,\*  $k$ , and thus obtain a set of numbers  $[k]$  such that every real number can be written uniquely in the form  $ak + b$ .

Now consider the number system whose elements are the complex numbers of the form  $ai + b$ , where  $a$  and  $b$  are rational and  $i = \sqrt{-1}$ . If we take as pseudo-points and pseudo-lines the points and lines of a three-space based (as in the proof of Theorem 15) on this number system, it is clear that all the assumptions except C are satisfied. If we also take as the pseudo-points  $H_0, H_1, H_\infty$  those having the coördinates  $(0, 1, 0, 0)$ ,  $(1, 1, 0, 0)$ ,  $(1, 0, 0, 0)$ , the net of rationality  $R(H_0H_1H_\infty)$  consists of  $H_\infty$  and the points whose coördinates are  $(x, 1, 0, 0)$ , where  $x$  is rational. Suppose now that we associate the pseudo-point  $(ai + b, 1, 0, 0)$  with every cut in this net which in the ordinary geometry would determine an irrational point  $(ak + b, 1, 0, 0)$ . Every point is thus associated with an infinity of cuts, contrary to Assumption C(1). Moreover, the cuts with which any point is associated occur between every two pseudo-points and hence between every two cuts of  $R(H_0H_1H_\infty)$ . Therefore Assumption C(2) remains true in this space.

(2) For the second half of the theorem the pseudo-points and pseudo-lines shall be the points and lines of a three-space based on a commutative number system whose elements are the ordinary rational numbers and all open cuts in the rational numbers. The laws of combination shall be such that addition is precisely the same as for the ordinary number system and multiplication is the same between rationals and rationals or rationals and irrationals, but different between irrationals and irrationals. Thus the product of the numbers associated with two open cuts will not, in general, be the number associated with the cut given by the usual rule. Hence the projective transformation  $x' = ax$  will not preserve order relations, and Assumption C(2) must be false. On the other hand, C(1) and the other assumptions are obviously true.

\* We do not show how to set up the correspondence. The assumption that this correspondence exists is a weaker form of the assumption used by Zermelo (*Mathematische Annalen*, Vol. LIX, p. 514) in his proof that any class can be well ordered. Our proof of the second part of the theorem is dependent on the validity of Zermelo's result that the continuum can be well ordered. The whole theorem is therefore subject to the doubts that attach to the Zermelo process because of the lack of explicit methods of setting up the correspondences in question.

The existence of the required new number system can be inferred from Hamel's theorem\* that there exists a well-ordered set of real numbers

$$(5) \quad \alpha_1, \alpha_2, \alpha_3, \dots, \alpha_\omega, \dots$$

such that every real number can be given uniquely by an expression of the form

$$(6) \quad \alpha_0 + \alpha_1 a_1 + \alpha_2 a_2 + \dots + \alpha_n a_n,$$

containing only a finite number of terms, the  $\alpha$ 's all being rational. The ordinary rules of combination for cuts determine a multiplication table for the  $a$ 's; that is, a set of rules of the form

$$(7) \quad a_i a_j = \beta_0 + \beta_1 a_1 + \beta_2 a_2 + \dots + \beta_m a_m,$$

where the  $\beta$ 's are rational. The laws of combination for the number system in general may now be stated as follows: Express the two numbers to be added or multiplied in the form (6); add or multiply by the rules for addition and multiplication of polynomials, reducing the result in the case of multiplication by means of the multiplication table for the  $a$ 's.

Now suppose we denote by

$$(8) \quad a'_1, a'_2, \dots, a'_\omega, \dots$$

the same set of numbers  $[a]$  arranged in a different order of the same type as (5). Such an order would be obtained, for example, by interchanging  $a_1$  and  $a_2$  and leaving the other  $a$ 's unaltered. There is therefore a one-to-one correspondence in which every  $a_i$  corresponds to the  $a'_i$  having the same subscript. Moreover, since the set of all  $a$ 's includes the same elements as the set of all  $a'$ 's, every real number is expressible in the form

$$(9) \quad \alpha_0 + \alpha_1 a'_1 + \alpha_2 a'_2 + \dots + \alpha_n a'_n.$$

A new law of multiplication, which we shall denote by  $\times$ , is now defined by setting up a multiplication table for the  $a'$ 's according to the rule that

$$(10) \quad a'_i \times a'_j = \alpha_0 + \alpha_1 a'_1 + \dots + \alpha_n a'_n$$

whenever

$$(11) \quad a_i a_j = \alpha_0 + \alpha_1 a_1 + \dots + \alpha_n a_n.$$

\* *Mathematische Annalen*, Vol. LX, p. 459.

The product, according to the new law of combination, of two real numbers is obtained by expressing each in the form (9), multiplying according to the rule for polynomials, and reducing by the multiplication table for the  $a$ 's.

Since the set of all expressions of the form

$$\alpha_0 + \alpha_1 a_1 + \alpha_2 a_2 + \dots$$

forms a number system, the set of all expressions of the form

$$\alpha_0 + \alpha_1 a'_1 + \alpha_2 a'_2 + \dots$$

forms a number system isomorphic with the first. For if we let each  $a_i$  correspond to the  $a'_i$  with the same subscript, the sum of any two elements of the first number system corresponds, by definition, to the sum of the corresponding two elements in the second number system. Similarly for the product of a rational by a rational or of a rational by an irrational. The product of two irrationals in the first system corresponds to the product of two irrationals in the second, because the two polynomials in the  $a$ 's are multiplied by the same rules as the two in the  $a$ 's, and are also reduced by corresponding entries in the respective multiplication tables.

We may insure that the two number systems shall be distinct by selecting the  $a$ 's, in the first place, so that  $a_1 = \sqrt{2}$  and  $a_2 = \sqrt{3}$ , and then choosing the  $a$ 's so that  $a'_1 = a_2$ .

**\*14. Foundations of the complex geometry.** Let us add to Assumptions A, E, H, C the following assumption:

ASSUMPTION  $\bar{R}$ . *On some line,  $l$ , not all points belong to the same chain.*

Let  $P_0, P_1, P_2$  be three points of  $l$ . The geometric number system determined by the method of Chap. VI, Vol. I, by the scale  $P_0, P_1, P_2$  is commutative for all the points in the chain  $C(P_0 P_1 P_2)$  but not necessarily for other points. However, it is clear, without assuming the commutativity of multiplication, that

$$x' = x^{-1}, \quad x' = x + a, \quad x' = ax, \quad x' = xa \quad (a = \text{constant})$$

define projectivities. For  $x' = x^{-1}$  this follows from § 54, Vol. I; for  $x' = x + a$  it reduces to Theorem 2, Chap. VI, Vol. I; and for the other two cases, to Theorem 4, Chap. VI, Vol. I.

Let  $J$  be any point of  $l$  not in  $C(P_0 P_1 P_2)$ , and let  $[X]$  be the set of all points in  $C(P_0 P_1 P_2)$ . Then, by Theorem 11 (1), the set of points



$[X+J]$  is a chain. This chain has no point except  $P_2$  in common with  $C(P_0P_1P_2)$ , because, if  $X+J=X' \neq P_2$ , it would follow that  $X'-X=J$ , and thus  $J$  would be a point of  $C(P_0P_1P_2)$ . Let us denote the chain  $[X+J]$  by  $C'$ .

In order to continue this argument we need the following assumption of closure:

ASSUMPTION I. *Through a point  $P$  of any chain  $C$  of the line  $l$ , and any point  $J$  on  $l$  but not in  $C$ , there is not more than one chain of  $l$  which has no other point than  $P$  in common with  $C$ .*

Now let  $P$  be any point of  $l$  not in  $C(P_0P_1P_2)$  or  $C'$ . Such points exist, because, for example, the chain  $C(P_0P_1J)$  does not coincide with  $C(P_0P_1P_2)$  or  $C'$ . The chain  $C(PJP_2)$  has, by Assumption I, a point different from  $P_2$  in common with  $C(P_0P_1P_2)$ . Let  $X_1$  be this point. In case  $X_1 \neq P_0$ , the projectivity

$$(12) \quad X' = X + J(P_1 - X_1^{-1} \cdot X)$$

transforms  $P_0$  into  $J$ ,  $X_1$  into itself, and  $P_2$  into itself. Hence it transforms  $C(P_0P_1P_2) = C(P_0X_1P_2)$  into  $C(JX_1P_2)$ . Hence every point of  $C(JX_1P_2)$ , and in particular  $P$ , is of the form  $X + JX''$ , where  $X$  and  $X''$  are in  $[X]$ . If  $X_1 = P_0$ , the projectivity

$$(13) \quad X' = JX$$

transforms  $C(P_0, P_1, P_2)$  into  $C(P_0JP_2)$ , which contains  $P$ . Hence, in this case  $P$  is of the form  $JX$ . Thus we have

LEMMA 1. *Every point of the line  $l$  is expressible in the form  $A + JB$ , where  $A$  and  $B$  are in  $C(P_0P_1P_2)$ .*

LEMMA 2. *Two points  $A + JB$  and  $A' + JB'$ , where  $A, B, A', B'$  are in  $C(P_0P_1P_2)$ , are identical if and only if  $A = A'$  and  $B = B'$ .*

For if  $B \neq B'$ ,  $A + JB = A' + JB'$  implies  $J = (A' - A)(B - B')^{-1}$ , and thus  $J$  would be in  $C(P_0P_1P_2)$ ; and if  $B = B'$ , it implies directly that  $A = A'$ .

Each of the projectivities  $X' = JX$  and  $X' = XJ$  transforms the chain  $C(P_0P_1P_2)$  into  $C(P_0JP_2)$ . Hence, if  $A$  be any point of  $C(P_0P_1P_2)$ ,

$$(14) \quad AJ = JA',$$

where  $A'$  is also in  $C(P_0P_1P_2)$ .

Each of the projectivities  $X' = (P_1 - J)X$  and  $X' = X(P_1 - J)$  transforms  $C(P_0P_1P_\infty)$  into  $C(P_0(P_1 - J)P_\infty)$ . Hence, if  $A$  be any point of  $C(P_0P_1P_\infty)$ ,

$$A(P_1 - J) = (P_1 - J)A'',$$

where  $A''$  is also in  $C(P_0P_1P_\infty)$ . By the distributive law (Theorem 5, Chap. VI, Vol. I) it follows that

$$A - AJ = A'' - JA''.$$

By (14), this reduces to

$$A - JA' = A'' - JA''.$$

By Lemma 2, it follows that  $A = A'' = A'$ . Hence  $AJ = JA$ . From this we can deduce, by the elementary laws of operation,

$$\begin{aligned} (A + JB)(C + JD) &= A(C + JD) + JB(C + JD) \\ &= AC + AJD + JBC + JBJD \\ &= CA + CJB + JDA + JDJB \\ &= C(A + JB) + JD(A + JB) \\ &= (C + JD)(A + JB). \end{aligned}$$

Hence the geometric number system determined by any scale on  $l$  is commutative. Since chains are transformed into chains by any projective transformation, it follows that the geometric number system determined by any scale on any line in a space satisfying  $A, E, H, C, \bar{R}, I$  satisfies the commutative law of multiplication. Hence, by Theorem 1,

**THEOREM 19.** *Assumption P is satisfied in any space satisfying Assumptions  $A, E, H, C, \bar{R}, I$ .*

Since every point in the geometric number system is expressible in the form  $A + JB$ , we have

$$(15) \quad J^2 = A_0 + JB_0,$$

where  $A_0$  and  $B_0$  are in  $C(P_0P_1P_\infty)$ . Thus  $J$  is one of the double points of the involution

$$(16) \quad XX' - \frac{1}{2}B_0(X + X') - A_0 = 0,$$

which transforms  $C(P_0P_1P_\infty)$  into itself. Any two points of  $C(P_0P_1P_\infty)$  which are conjugate in this involution may be transformed projectively into  $P_0$  and  $P_\infty$  by a transformation which carries  $C(P_0P_1P_\infty)$  into itself. This reduces the involution to

$$(17) \quad XX' = A,$$

where  $A$  must be negative relatively to the scale  $P_0P_1P_2$ , since the double points are not in  $C(P_0P_1P_2)$ . The transformation  $X = \sqrt{-AX'}$  now reduces (17) to

$$XX' = -P_1$$

and thus transforms  $J$  to a point satisfying the equation

$$J^2 = -P.$$

Hence we have

**THEOREM 20.** *The geometric number system in any space satisfying Assumptions A, E, H, C,  $\bar{R}$ , I is isomorphic with the complex number system of analysis, i.e. with the system of numbers  $a + ib$ , where  $i^2 = -1$  and  $a$  and  $b$  are real.*

**\*15. Ordered projective spaces.** There is an important class of projective spaces which may be referred to as the *ordered projective spaces* and which are characterized by the Assumptions S given below. This class of spaces includes the rational and real projective spaces and many others. The set of assumptions, A, E, S, is not categorical, but it may be made so by adding a suitable continuity assumption or by some other assumption of closure.

These assumptions introduce a new class of undefined elements, called *senses*,\* in addition to the points and lines which are the undefined elements of Assumptions A and E. The senses are denoted by symbols of the form  $S(ABC)$ , where  $A, B, C$  denote points.†

S 1. *For any three distinct collinear points  $A, B, C$  there is a sense  $S(ABC)$ .*

S 2. *For any three distinct collinear points there is not more than one sense  $S(ABC)$ .*

S 3.  $S(ABC) = S(BCA)$ .

S 4.  $S(ABC) \neq S(ACB)$ .

S 5. *If  $S(ABC) = S(A'B'C')$  and  $S(A'B'C') = S(A''B''C'')$ , then  $S(ABC) = S(A''B''C'')$ .*

S 6. *If  $S(ABO) = S(BCO)$ , then  $S(ABO) = S(ACO)$ .*

S 7. *If  $OA$  and  $OB$  are distinct lines, and  $S(OAA_1) = S(OAA_2)$  and  $OAA_1A_2 \bar{\wedge} OBB_1B_2$ , then  $S(OBB_1) = S(OBB_2)$ .*

\* Sets of assumptions more or less related to these have been given by A. B. Schweitzer, American Journal of Mathematics, Vol. XXXI, p. 365, and A. N. Whitehead, The Axioms of Projective Geometry, Cambridge Tracts, Cambridge, 1906.

† With respect to the intuitional basis of these assumptions, cf. figs. 6-12, Chap. II.

If  $S(ABC)$  be identified with the sense-class which is discussed below in § 19, Chap. II, it will be seen that S 1 and S 2 are immediately verified and S 3,  $\dots$ , S 7 reduce to Theorems 2-6, Chap. II. This shows that the assumptions S are satisfied by a rational or a real projective space.

These assumptions are capable, as is shown in Chap. II, of serving as a basis for a very complete discussion of geometric order relations. Assumption P is not a consequence of A, E, S alone.

### EXERCISES

1. Prove that Assumption H is a consequence of A, E, and S.
2. Prove that with a proper definition of the symbol  $<$  (less than) the geometric number system in an ordered projective space satisfies the following conditions:

- (1) If  $a$  and  $b$  are distinct numbers,  $a < b$  or  $b < a$ .
- (2) If  $a < b$ , then  $a \neq b$ .
- (3) If  $a < b$  and  $b < c$ , then  $a < c$ .
- (4) If  $a < b$ , there exists a number,  $x$ , such that  $a < x$  and  $x < b$ .
- (5) If  $0 < a$ , then  $b < a + b$  for every  $b$ .
- (6) If  $0 < a$  and  $0 < b$ , then  $0 < a \cdot b$ .

(Cf. E. V. Huntington, Transactions of the American Mathematical Society, Vol. VI (1905), p. 17.)

3. Introduce an assumption of continuity, and with this assumption and A, E, S prove Assumption P.
4. Prove that P is not a consequence of A, E, S alone.

**\* 16. Modular projective spaces.** We have seen (§ 7) that, in any space satisfying Assumptions A and E, any two harmonic sequences are projective. Hence, if one harmonic sequence contains an infinity of points, every such sequence contains an infinity of points, and by § 8 these points are in one-to-one reciprocal correspondence with the ordinary rational numbers. On the other hand, if one harmonic sequence contains a finite number of points, every other harmonic sequence in the same space contains the same finite number of points. Hence the spaces satisfying Assumptions A and E fall into two classes—those satisfying Assumption H and those satisfying the following:

**ASSUMPTION  $\bar{H}$ .** *If any harmonic sequence exists, at least one contains only a finite number of points.*

The spaces satisfying  $\bar{H}$  may be called *modular*, and those satisfying  $H$  *nonmodular*.

It follows, just as in Theorem 5, that the principle of duality is true for any modular space.

Let  $\Pi$  be any parabolic projectivity on a line, and let  $H_\infty$  be its invariant point. If  $H_0$  be any other point of the line, the points

$$\dots \Pi^{-2}(H_0), \Pi^{-1}(H_0), H_0, \Pi(H_0), \Pi^2(H_0) \dots$$

form a harmonic sequence, by definition. If this is to contain only a finite number of points, there must be some positive integer  $n$  such that  $\Pi^m(H_0) = \Pi^n(H_0)$ , where  $m$  is zero or a positive integer less than  $n$ . If  $n - m = k$ , we have

$$\Pi^k(\Pi^m(H_0)) = \Pi^m(H_0),$$

and hence

$$\Pi^k = 1.$$

Hence all the points of the harmonic sequence are contained in the set

$$H_0, \Pi(H_0), \dots, \Pi^{k-1}(H_0).$$

In case  $k$  is not a prime number, that is, if there exist two positive integers,  $k_1, k_2$  different from unity such that  $k = k_1 \cdot k_2$ , let us consider the parabolic projectivity  $\Pi^{k_1}$ . The points

$$H_0, \Pi^{k_1}(H_0), \Pi^{2k_1}(H_0), \dots, \Pi^{(k_1-1)k_2}(H_0)$$

satisfy the definition of a harmonic sequence. Since any two harmonic sequences contain the same number of points, it follows that the given sequence could not have contained more than  $k_1$  points. In case  $k_1$  breaks up into two factors, the same argument shows that the given harmonic sequence could not contain a number of points larger than either factor. This process can be repeated only a finite number of times and can stop only when we arrive at a prime number. Hence we have

**THEOREM 21.** *The number of points in a harmonic sequence is prime. The points of a harmonic sequence may be denoted by*

$$H_0, \Pi(H_0), \dots, \Pi^{p-1}(H_0),$$

*where  $\Pi$  is a parabolic projectivity. The period,  $p$ , of any parabolic projectivity is a prime number.*

With reference to a scale in which  $H_0 = 0$ ,  $\Pi(H_0) = 1$ , and the limit point of the harmonic sequence is  $\infty$ ,  $\Pi$  has the equation

$$x' = x + 1.$$

Hence the coördinates of the points in the harmonic sequence are

$$0, 1, 2, \dots, p-1,$$

respectively, where 2 represents  $1+1$ , 3 represents  $2+1$ , etc. Since  $\Pi^p = 1$ , we must have that  $p = 0$ ,  $p+1 = 1$ ,  $np+k=k$ , etc. In other words, the coördinates of the points in a harmonic sequence are elements of the field obtained by reducing the integers modulo  $p$ , as explained in § 72, Vol. I.

By Theorem 14, Chap. VI, Vol. I, the net of rationality determined by the points whose coördinates are 0, 1,  $\infty$  consists of the point  $\infty$  and all points whose coördinates are obtainable from 0 and 1 by the operations of addition, subtraction, multiplication, and division (except division by zero). Since all numbers of this sort are contained in the set

$$0, 1, \dots, p-1,$$

we have

**THEOREM 22.** *The number of points in a net of rationality in a modular space is  $p+1$ ,  $p$  being a prime number constant for the space in question.*

Obviously, if Assumption Q (§ 10) be added to the set A, E,  $\overline{H}$ , the number of points on any line must be  $p+1$ ,  $p$  being prime. A space satisfying A, E,  $\overline{H}$  shall be called a *rational modular space*. The problem of finding the double points of a projectivity in a rational modular space of one or more dimensions leads to the consideration of modular spaces bearing a relation to the rational ones analogous to the relation which the complex geometry bears to the real geometry. The existence of such spaces follows from the considerations in Chap. IX, Vol. I (Propositions  $K_2$  and  $K_n$ ). The geometric number systems for such spaces may be finite\* (Galois fields) or infinite.†

\* E. H. Moore, The Subgroups of the Generalized Finite Modular Group, Decennial publications of The University of Chicago, Vol. IX (1903), pp. 141-190; L. E. Dickson, Linear Groups, Chap. I.

† L. E. Dickson, Transactions of the American Mathematical Society, Vol. VIII (1907), p. 389. See also the article by E. Steinitz referred to in § 92, Vol. I.

**17. Recapitulation.** The various groupings of assumptions which we have considered thus far may be resumed as follows: A space satisfying Assumptions

A, E	is a general projective space;
A, E, P	is a proper projective space;
A, E, H	is a nonmodular projective space;
A, E, $\bar{H}$	is a modular projective space;
A, E, S	is an ordered projective space;
A, E, $\bar{H}$ , Q	is a rational modular projective space;
A, E, H, Q	is a rational nonmodular projective space;
A, E, H, C, R } or A, E, K	is a real projective space;
A, E, H, C, $\bar{R}$ , I } or A, E, J	
	is a complex projective space.

The first six sets of assumptions are not, and the remaining ones are, categorical. The set of theorems deducible from any one of these sets of assumptions is called a projective geometry, and the various geometries may be distinguished by the adjectives applied above to the corresponding spaces.

## CHAPTER II

### ELEMENTARY THEOREMS ON ORDER

**18. Direct and opposite projectivities on a line.** In § 9 a point  $A$  was said to precede a point  $B$  relative to a scale  $P_0, P_1, P_2$  if the coördinate of  $A$  in this scale was less than the coördinate of  $B$ . Supposing the coördinate of  $A$  to be  $a$  and that of  $B$  to be  $b$ , the projectivity changing  $P_0$  to  $A$  and  $P_1$  to  $B$  and leaving  $P_2$  fixed has the equation

$$(1) \quad x' = (b - a)x + a.$$

In this transformation the coefficient of  $x$  is positive if and only if  $A$  precedes  $B$ . But the transformations of the form

$$(2) \quad x' = \alpha x + \beta,$$

where  $\alpha$  is positive, evidently form a group. This group is a subgroup of the group of all projectivities leaving  $P_2$  invariant, for the latter group contains all transformations (2) for which  $\alpha \neq 0$ .

The group of transformations (2) for which  $\alpha$  is positive is, by what we have just seen, such that whenever a pair of points  $A$  and  $B$  are transformed to  $A'$  and  $B'$  respectively,  $A$  precedes  $B$  if and only if  $A'$  precedes  $B'$ . The discussion of order relative to a scale could therefore be based on the theory of this group.

The order relations defined by means of this group have all, however, a special relation to the point  $P_2$ , and they can all be derived by specialization from a more general relation defined by means of a more extensive group. We shall therefore enter first into the discussion of this larger group, and afterwards (§ 23) show how to derive the relations of "precede" and "follow" from the general notion of "sense." The definitions for the general case, like those for the special one, will be seen to depend simply on the distinction between positive and negative numbers.

A projective transformation of a line may be written in the form

$$(3) \quad \begin{aligned} x'_0 &= a_{00}x_0 + a_{01}x_1, \\ x'_1 &= a_{10}x_0 + a_{11}x_1, \end{aligned} \quad \Delta = \begin{vmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{vmatrix} \neq 0,$$

where the  $a_{ij}$ 's are numbers of the geometric number system.

Under Assumptions A, E, H, C, R (or A, E, K) the  $a_{ij}$ 's are real. If attention be restricted to a single net of rationality satisfying Assumption H, the  $a_{ij}$ 's may be taken (Theorem 6, Cor. 2, Chap. I) as integers. The discussion which follows is valid on either hypothesis.\*

DEFINITION. The projectivities of the form (3) for which  $\Delta > 0$  are called *direct*, and those for which  $\Delta < 0$  are called *opposite*.

Since the determinant of the product of two transformations (3) is the product of the determinants, the direct projectivities form a subgroup of the projective group. The same transformation (3) cannot be both direct and opposite, for two transformations (3) are identical only if the coefficients of one are obtainable from those of the other by multiplying them all by the same constant  $\rho$ ; but this merely changes  $\Delta$  into  $\rho^2\Delta$ .

In form, the definition is dependent on the choice of the coördinate system which is used in equations (3). Actually, however, the definition is independent of the coördinate system, for if a given projectivity has a positive  $\Delta$  with respect to one scale, it has a positive  $\Delta$  with respect to every scale. This may be proved as follows:

Let the fundamental points of the scale to which the coördinates in (3) refer be  $P_0, P_1, P_\infty$ , and let  $Q_0, Q_1, Q_\infty$  be the fundamental points of any other scale. By § 56, Vol. I, the coördinates  $y_0, y_1$  of any point  $R$  with respect to any scale  $Q_0, Q_1, Q_\infty$  are such that  $y_1/y_0 = R(Q_\infty Q_0, Q_1 R)$ . Suppose that, relative to the scale  $P_0, P_1, P_\infty$ , the projectivity which transforms  $Q_0, Q_1, Q_\infty$  to  $P_0, P_1, P_\infty$  respectively has the equations

$$(4) \quad \begin{aligned} y_0 &= b_{00}x_0 + b_{01}x_1, \\ y_1 &= b_{10}x_0 + b_{11}x_1, \end{aligned} \quad \begin{vmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{vmatrix} = D \neq 0.$$

Thus any point  $R$  whose coördinates relative to the scale  $P_0, P_1, P_\infty$  are  $(x_0, x_1)$  is transformed by this projectivity to a point  $R'$  whose coördinates relative to the scale  $P_0, P_1, P_\infty$  are  $(y_0, y_1)$ .

Since cross ratios are unaltered by projective transformations,

$$R(Q_\infty Q_0, Q_1 R) = R(P_\infty P_0, P_1 R') = \frac{y_1}{y_0}.$$

Hence it follows that if  $x_0$  and  $x_1$  are the coördinates of any point  $R$  relative to the scale  $P_0, P_1, P_\infty$ , the corresponding values of  $y_0$  and  $y_1$  given

\* It is, in fact, valid in any space satisfying Assumptions A, E, S, P. The purely ordinal theorems are indeed valid in any ordered projective space (§ 16), but those regarding involutions, conic sections, etc., necessarily involve Assumption P also. Cf. the fine print at the end of § 19.

by (4) are the coördinates of  $R$  relative to the scale  $Q_0, Q_1, Q_\infty$ . Let us indicate (4) by  $(y_0, y_1) = T(x_0, x_1)$ , and (3) by  $(x'_0, x'_1) = S(x_0, x_1)$ .

Now a direct transformation (3) carries a point whose coördinates relative to the scale  $P_0, P_1, P_\infty$  are  $(x_0, x_1)$  into one whose coördinates relative to the same scale are  $(x'_0, x'_1)$ , where  $(x'_0, x'_1) = S(x_0, x_1)$ . The coördinates of these two points relative to the scale  $Q_0, Q_1, Q_\infty$  are  $(y_0, y_1) = T(x_0, x_1)$  and  $(y'_0, y'_1) = T(x'_0, x'_1)$  respectively. Hence, by substitution,

$$(y'_0, y'_1) = T(S(x_0, x_1)) = T(S(T^{-1}(y_0, y_1))),$$

or

$$(y'_0, y'_1) = TST^{-1}(y_0, y_1),$$

where  $T^{-1}$  indicates, as usual, the inverse of  $T$ . The determinant of the transformation  $TST^{-1}$  is

$$\Delta' = D\Delta \frac{K^2}{D},$$

where  $K$  is real (or rational), and  $\Delta'$  therefore has the same sign as  $\Delta$ . Thus the definition of a direct projectivity is independent of the choice of the coördinate system.

This result can be put in another form which is important in the sequel:

DEFINITION. Two figures are said to be *conjugate under* or *equivalent with respect to* a group of transformations if and only if there exists a transformation of the group carrying one of the figures into the other.

THEOREM 1. *If two sets of points are conjugate under the group of direct projectivities on a line, so are also the two sets of points into which they are transformed by any projectivity of the line.*

*Proof.* Let  $S$  be a direct projectivity changing a set of points  $[A]$  into a set of points  $[B]$ , and let  $T$  be any other projectivity on the line, and let  $T(A) = A'$  and  $T(B) = B'$ . Since  $T^{-1}(A') = A$ ,  $S(A) = B$ , and  $T(B) = B'$ , it follows that  $TST^{-1}(A') = B'$ . But the discussion above shows that  $TST^{-1}$  is a direct projectivity. Hence  $[A']$  and  $[B']$  are conjugate under the group of direct projectivities, as was to be proved.

According to the definition in § 75, Vol. I (see also § 89, below), the group of direct projectivities is a self-conjugate subgroup of the group of all projectivities on a line. Since this is the only relation between the two groups which we have employed in the proof of the theorem above, this theorem can be generalized to any case in which we have one group of transformations appearing as a self-conjugate subgroup of another.

## EXERCISES

1. Within the field of all real numbers the positive numbers may be defined as those numbers different from zero which possess square roots. Generalize this definition to other fields, and thus generalize the definitions of direct projectivities. In each case determine how far the theorems on sense and order in the following sections can be generalized (cf. § 72, Vol. I).

2. The group of projectivities which transform a net of rationality into itself has a self-conjugate subgroup consisting of those transformations which are products of pairs of involutions having their double points in the net of rationality. This group contains all projectivities for which the determinant is the square of a rational number.

\*3. Work out a definition and theory of the group of direct projectivities independent of the use of coordinates. This may be done by the aid of theorems in Chap. VIII, Vol. I (cf. §§ 69 and 70, below).

19. The two sense-classes on a line. DEFINITION. Let  $A_0, B_0, C_0$  be any three distinct points of a line. The class of all ordered\* triads of points  $ABC$  on the line, such that the projectivities

$$A_0B_0C_0 \overline{\wedge} ABC$$

are direct, is called a *sense-class* and is denoted by  $S(A_0B_0C_0)$ . Two ordered triads in the same sense-class are said to *have the same sense* or to *be in the same sense*. Two collinear ordered triads not in the same sense-class are said to *have opposite senses* or to *be in opposite senses*.

One sense-class chosen arbitrarily may be referred to by a particular name, as *right-handed*, *clockwise*, *positive*, etc.†

The term "sense," standing by itself, might have been defined as follows: "The senses are any set of objects in one-to-one and reciprocal correspondence with the sense classes." This is analogous to the definition of a vector given in § 42. When there is question only of one line, any two objects whatever may serve as the two senses—for example, the signs + and −. This agrees with the definition of sense as "the sign of a certain determinant." When dealing with more than one line, it is no longer correct to say that there are two senses; there are, in fact, two senses for each line.

\* "Order," here, is a logical rather than a geometrical term, just as in the definition of "throw" (§ 23, Vol. I). It is a device for distinguishing the elements of a set. For example, when we say that  $ABC$  cannot be transformed into  $ACB$  by any transformation of a given group, it is a way of saying that the group contains no transformation changing  $A$  into  $A$ ,  $B$  into  $C$ , and  $C$  into  $B$ .

† A partial list of references on the notion of sense in one and more dimensions would include: Möbius, *Barycentrische Calcul*, note in § 140; Ganas, *Werke*, Vol. VIII, p. 248; von Staudt, *Beiträge zur Geometrie der Lage*, §§ 3, 14; Study, *Archiv der Mathematik und Physik*, Vol. XXI (1913), p. 193; *Encyclopädie der Math. Wiss.* III AB 7, p. 618.

When one adopts, as we do, the symbol  $S(ABC)$  to stand for a sense-class, there is no occasion for attaching a separate meaning to the word "sense." It may be regarded as an incomplete symbol,\* like the  $\frac{d}{dx}$  in the  $\frac{dy}{dx}$  of the calculus.

**THEOREM 2.** *If the ordered triad  $ABC$  is in the sense-class  $S(A_0B_0C_0)$ , then  $S(ABC) = S(A_0B_0C_0)$ . If  $S(ABC) = S(A'B'C')$  and  $S(A'B'C') = S(A''B''C'')$ , then  $S(ABC) = S(A''B''C'')$ .*

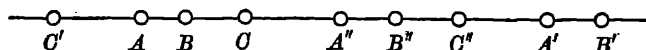


FIG. 6

*Proof.* Both statements are consequences of the fact that the direct projectivities form a group.

**THEOREM 3.** *If  $S(ABC) \neq S(A'B'C')$  and  $S(A'B'C') \neq S(A''B''C'')$ , then  $S(ABC) = S(A''B''C'')$ .*

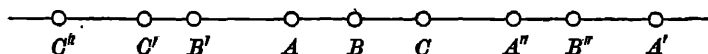


FIG. 7

*Proof.* If  $S(ABC) \neq S(A'B'C')$ , the projectivity  $ABC \overline{\wedge} A'B'C'$  is opposite. Hence the theorem follows from the fact that the product of two opposite projectivities is direct.

**COROLLARY.** *There are two and only two sense-classes on a line.*

**THEOREM 4.** *If  $A, B, C$  are distinct collinear points,  $S(ABC) = S(BCA)$  and  $S(ABC) \neq S(ACB)$ .†*

*Proof.* Let  $A, B, C$  be taken as  $(1, 1)$ ,  $(1, 0)$ ,  $(0, 1)$  respectively. Then

$$x'_0 = x_1,$$

$$x'_1 = x_0$$

is an opposite projectivity interchanging  $B$  and  $C$  and leaving  $A$  invariant. Hence  $S(ABC) \neq S(ACB)$ . In like manner, we can prove that  $S(ACB) \neq S(BCA)$ . It follows, by Theorem 3, that  $S(ABC) = S(BCA)$ .

\* The term "incomplete symbol" appears in Whitehead and Russell's *Principia Mathematica*, Vol. I, Chap. III, of the Introduction, together with a discussion of its logical significance.

† This may be expressed by the phrase "Sense is preserved by even and altered by odd permutations." A *transposition* is a permutation in which two and only two elements are interchanged, and an *even (odd)* permutation is the resultant of an even (odd) number of transpositions. Cf. Burnside, *Theory of Groups of Finite Order*, Chap. I.

THEOREM 5. If  $S(ABD) = S(BCD)$ , then  $S(ABD) = S(ACD)$ .

*Proof.* Choose the coördinates so that  $D = (0, 1)$ ,  $A = (1, 0)$ ,  $B = (1, 1)$ . The transformation of  $ABD$  to  $BCD$  may be written in the form

$$x'_0 = x_0,$$

$$x'_1 = x_0 + ax_1,$$

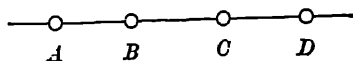


FIG. 8

because  $(0, 1)$  is invariant and  $(1, 0)$  goes to  $(1, 1)$ . This transformation will be direct if and only if  $a > 0$ . The point  $C$ , being the transform of  $(1, 1)$ , is  $(1, 1 + a)$ . The transformation carrying  $ABD$  to  $ACD$  is

$$x' = x_0,$$

$$x'_1 = (1 + a)x_1,$$

which is direct because  $(1 + a) > 0$ .

As an immediate consequence of Theorem 1 we have

THEOREM 6. If  $S(ABC) = S(A_1B_1C_1)$  and  $ABCA_1B_1C_1 \wedge A'B'C' A'_1B'_1C'_1$ , then

$$S(A'B'C') = S(A'_1B'_1C'_1).$$

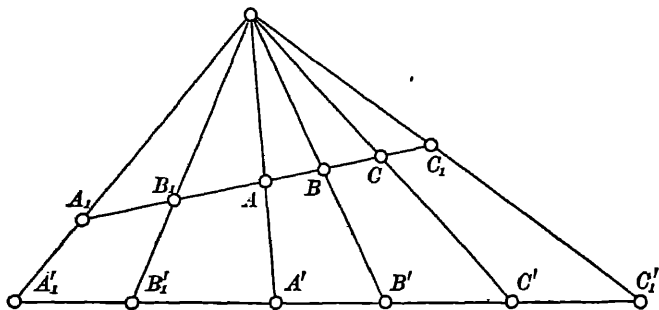


FIG. 9

Theorems 2-6 contain the propositions given in § 15, Chap. I, as Assumptions S. Theorem 6 is slightly more general than S7 but is directly deducible from it. The developments of the following sections will be based entirely on these propositions, and hence belong to the theory of any ordered projective space, except where reference is made to figures whose existence depends on Assumption P. Theorems of the latter sort hold in any space satisfying A, E, P, S.

These propositions have the advantage, as assumptions, of corresponding to some of our simplest intuitions with regard to the linear order relations. The reader may verify this by constructing the figures to which they correspond (cf. figs. 6-9). Each proposition will be found to correspond to a number of visually distinct figures.

**20. Sense in any one-dimensional form.** DEFINITION. If 1, 2, 3, 1', 2', 3' are elements of the same one-dimensional form, and  $A, B, C, A', B', C'$  are collinear points such that

$$1231'2'3' \overline{\wedge} ABCA'B'C',$$

then the ordered triad 123 is said to have the same sense as 1'2'3' if and only if  $S(ABC) = S(A'B'C')$ . The set of all ordered triads having the same sense as 123 is called a *sense-class* and denoted by  $S(123)$ .

In view of Theorem 6 this definition is independent of the choice of the points  $A, B, C, A', B', C'$ . It is an immediate corollary of the definition that the plane and space duals of Theorems 2-6 all hold good (cf. figs. 10 and 13).

By the definition of a point conic there is a one-to-one correspondence between the points  $[P]$  of the conic and the lines joining them to a fixed point  $P_0$  of the conic. We now define any statement in

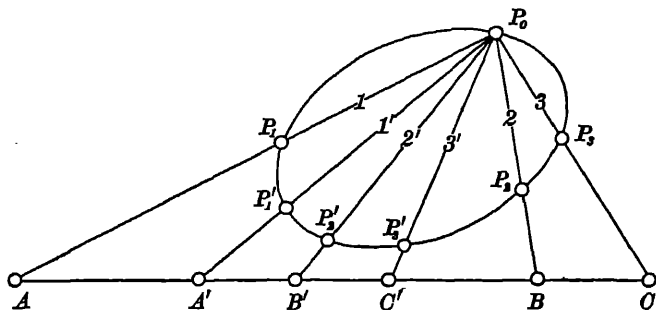


FIG. 10

terms of order relations among the points of the conic  $[P]$  to mean that the same statement holds for the corresponding lines  $[P_0P]$ . By Theorem 6, above, together with Theorem 2, Chap. V, Vol. I, it follows that this definition is independent of the choice of the point  $P_0$ . The definitions of the order relations in the line conic, the cone of lines, and the cone of planes are made dually.\*

The propositions with regard to sense are perhaps even more evident intuitively when stated with regard to a conic or a flat pencil than with regard to the points of a line (cf. figs. 10 and 11).

\* These definitions are in reality special cases of the definition given above for any one-dimensional form, since the cones and conic sections are one-dimensional forms of the second degree (§ 41, Vol. I) and since the notion of projectivity between one-dimensional forms of the first and second degrees has been defined in § 76, Vol. I. However, at present we do not need to avail ourselves of the theorems in Chap. VIII, Vol. I, on which the latter definition is based.

**21. Separation of point pairs.** DEFINITION. Two points  $A$  and  $B$  of a line are said to *separate* two points  $C$  and  $D$  of the same line if and only if  $S(ABC) \neq S(ABD)$ . This is indicated by the symbol  $AB \parallel CD$ .

**THEOREM 7.** (1) *The relation  $AB \parallel CD$  implies the relations  $CD \parallel AB$  and  $AB \parallel DC$ , and excludes the relation  $AC \parallel BD$ .* (2) *Given any four distinct points of a line, we have either  $AB \parallel CD$  or  $AC \parallel BD$  or  $AD \parallel BC$ .* (3) *From the relations  $AB \parallel CD$  and  $AD \parallel BE$  follows the relation  $AD \parallel CE$ .* (4) *If  $AB \parallel CD$  and  $ABCD \nparallel A'B'C'D'$ , then  $A'B' \parallel C'D'$ .*\*

*Proof.* (1) If  $AB \parallel CD$ , we have

$$(5) \quad S(ABC) \neq S(ABD),$$

which, by the definition of separation, implies  $AB \parallel DC$ . By Theorems 2-6 we obtain successively, from (5),

$$S(ABC) = S(ADE),$$

$$S(ABC) = S(ADC),$$

$$S(ACB) = S(DAB),$$

$$S(ACB) = S(DCB),$$

$$S(ABC) = S(CDB),$$

$$S(CDA) \neq S(CDB),$$

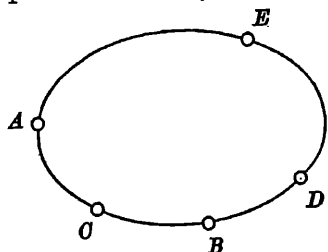


FIG. 11

the last of which implies  $CD \parallel AB$ . The relation  $AC \parallel BD$  is excluded because it means  $S(ACB) \neq S(ACD)$ , which contradicts the second of the equations above.

(2) By the corollary of Theorem 3 we have either  $S(ABC) \neq S(ABD)$  (in which case  $AB \parallel CD$ ) or  $S(ABC) = S(ABD)$ . In the latter case either  $S(ABC) \neq S(ADC)$  or  $S(ABC) = S(ADC)$ . The first of these alternatives is equivalent to  $S(ACB) \neq S(ACD)$  and yields  $AC \parallel BD$ ; the second implies  $S(ADC) = S(ABC) = S(ABD) \neq S(ADB)$ , and thus yields  $AD \parallel BC$ .

(3) The hypotheses give  $S(ABC) \neq S(ABD)$  and  $S(ADB) \neq S(ADE)$ . The first of these gives  $S(BCA) = S(DBA)$ , which, by Theorem 5, implies  $S(DBA) = S(DCA)$ , and thus  $S(ADB) = S(ADC)$ . Hence, by the second hypothesis,  $S(ADC) \neq S(ADE)$ , and therefore  $AD \parallel CE$ .

(4) This is a direct consequence of Theorem 6.

\* The properties expressed in this theorem are sufficient to define abstractly the relation of separation. Cf. Vailati, *Revue de Mathématiques*, Vol. V, pp. 76, 183; also Padoa, *Revue de Mathématiques*, Vol. VI, p. 35.

THEOREM 8. *If  $A$  and  $B$  are harmonically conjugate with regard to  $C$  and  $D$ , they separate  $C$  and  $D$ .*

*Proof.* By Theorem 7 (2) we have either  $AB \parallel CD$  or  $AC \parallel BD$  or  $AD \parallel BC$ . We also have  $ABCD \overline{\wedge} BACD$ . Hence  $AC \parallel BD$  would imply  $BC \parallel AD$ , contrary to Theorem 7 (1); and  $AD \parallel BC$  would imply  $BD \parallel AC$ , contrary to Theorem 7 (1). Hence we must have  $AB \parallel CD$ .

THEOREM 9. *An involution in which two pairs separate one another has no double points.*

*Proof.* Suppose that the given involution had the double points  $M, N$ , and that the two pairs which separate one another are  $A, A'$  and  $B, B'$  respectively. Since the involution would be determined by the projectivity  $MNA \overline{\wedge} MNA'$ , in which, by Theorem 8,

$$S(MNA) \neq S(MNA'),$$

it would follow, by Theorem 6, that every ordered triad was carried into an ordered triad in the opposite sense. Since the involution carries  $AA'B$  to  $A'AB'$ , we should have

$$S(AA'B) \neq S(A'AB');$$

and hence

$$S(AA'B) = S(AA'B'),$$

contrary to hypothesis.

This theorem can also be stated in the following form:

COROLLARY 1. *An involution with double points is such that no two pairs separate one another.*

COROLLARY 2. *If an involution is direct, each pair separates every other pair. If an involution is opposite, no pair separates any other pair.*

**22. Segments and intervals.** DEFINITION. Let  $A, B, C$  be any three distinct points of a line. The set of all points  $X$  such that

$$S(AXC) = S(ABC)$$

is called a *segment* and is denoted by  $\overline{ABC}$ . The points  $A$  and  $C$  are called the *ends* of the segment. The segment  $\overline{ABC}$ , together with its ends, is called the *interval*  $ABC$ . The points of  $\overline{ABC}$  are said to be *interior* to the interval  $ABC$ , and  $A$  and  $C$  are called its *ends*.

COROLLARY 1. *A segment does not contain its ends.*

COROLLARY 2. *If  $D$  is in  $\overline{ABC}$ , then*

$$\overline{ABC} = \overline{ADC}.$$

COROLLARY 3. If  $D$  is in  $\overline{ABC}$ , then  $B$  and  $D$  are not separated by  $A$  and  $C$ .

THEOREM 10. If  $A$  and  $B$  are any two distinct points of a line, there are two and only two segments, and also two and only two intervals, of which  $A$  and  $B$  are ends.

*Proof.* Let  $C$  and  $D$  be two points which separate  $A$  and  $B$  harmonically. If  $X$  is any point of the line distinct from  $A$  and  $B$ , either

$$S(\overline{AXB}) = S(\overline{ACB})$$

or

$$S(\overline{AXB}) = S(\overline{ADB}).$$

In one case  $X$  is in  $\overline{ACB}$ , and in the other case in  $\overline{ADB}$ .

DEFINITION. Either of the two segments (or of the two intervals) whose ends are two points  $A, B$  may be referred to as a *segment*  $\overline{AB}$  (or an *interval*  $AB$ ). The two segments or intervals  $AB$  are said to be *complementary* to one another.

COROLLARY. If  $A, B, C$  are any three distinct points of a line, the line consists of the three segments complementary to  $\overline{ABC}$ ,  $\overline{BCA}$ ,  $\overline{CAB}$ , together with the points  $A, B$ , and  $C$ .

*Proof.* Any point  $X$  distinct from  $A, B, C$  satisfies one of the relations  $AC \parallel BX$  or  $AB \parallel CX$  or  $AX \parallel BC$ .

THEOREM 11. If  $A_1, A_2, \dots, A_n$  is any set of  $n$  ( $n > 1$ ) distinct points of a line, the remaining points of the line constitute  $n$  segments, each of which has two of the points  $A_1, A_2, \dots, A_n$  as end points and no two of which have a point in common.

*Proof.* The theorem is true for  $n = 2$ , by Theorem 10. Suppose it true for  $n = k$ . If  $k + 1$  points are given, the point  $A_{k+1}$  is, by the theorem for the case  $n = k$ , on one of the  $k$  segments determined by the other  $k$  points, say on the segment whose ends are  $A_i$  and  $A_j$ . By the corollary to Theorem 10, this segment consists of  $\overline{A_{k+1}A_i}$ , together with two segments whose ends are respectively  $\overline{A_{k+1}A_i}$  and  $\overline{A_{k+1}A_j}$ . Hence the theorem is valid for  $n = k + 1$  if valid for  $n = k$ . Hence the theorem is established by mathematical induction.

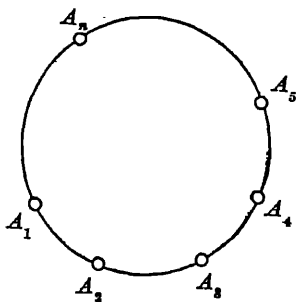


FIG. 12

DEFINITION. A finite set of collinear points,  $A_i$  ( $i = 1, \dots, n$ ), is in the *geometrical order*  $\{A_1 A_2 A_3 A_4 \dots A_n\}$  if no two of its points are

separated by any of the pairs  $A_1A_2, A_2A_3, \dots, A_nA_1$ . As an obvious consequence of Theorem 11 we now have

**THEOREM 12.** *To any set  $[A]$  of  $n$  points of a line the notation  $A_1, A_2, \dots, A_n$  may be assigned so that they are in the order  $\{A_1A_2 \dots A_n\}$ . A set of points in the order  $\{A_1A_2 \dots A_n\}$  is also in the orders  $\{A_2A_3 \dots A_nA_1\}$  and  $\{A_nA_{n-1} \dots A_2A_1\}$ .*

### EXERCISES

1. If  $AB \parallel CD$  and  $AC \parallel BE$ , then  $CD \parallel BE$ .
2. The relations  $AB \parallel CD, AB \parallel CE, AB \parallel DE$  are not possible simultaneously.
3. Any two points  $A, B$  are in the orders  $\{AB\}$  and  $\{BA\}$ . Any three collinear points are in the orders  $\{ABC\}, \{ACB\}, \{CAB\}$ .

**23. Linear regions.** The set of all points on a line, the set of all points on a line with the exception of a single one, and the segment are examples (cf. Ex. 1 below) of what we shall define as linear regions on account of their analogy with the planar and spatial regions considered later.

**DEFINITION.** A *region* on a line is a set of collinear points such that (1) any two points of the set are joined by an interval consisting entirely of points of the set and (2) every point is interior to at least one segment consisting entirely of points of the set. A region is said to be *convex* if it satisfies also the condition that (3) there is at least one point of the line which is not in the set.

**DEFINITION.** An ordered pair of distinct points  $AB$  of a convex region  $R$  is said to be *in the same sense* as an ordered pair  $A'B'$  of  $R$  if and only if  $S(ABA_\infty) = S(A'B'A_\infty)$ , where  $A_\infty$  is a point of the line not in  $R$ . The set of all ordered pairs of  $R$  in the same sense as  $AB$  is denoted by  $S(AB)$  and is called a *sense-class*. The segment complementary to  $AA_\infty B$  is called the *segment  $\overline{AB}$* . The corresponding interval is called the *interval  $AB$* . A set of points of  $R$  is said to be in the *order  $\{A_1A_2 \dots A_n\}$*  if they are in the order  $\{A_1A_2 \dots A_nA_\infty\}$ . If  $C$  is separated from  $A_\infty$  by  $A$  and  $B$ ,  $C$  is *between  $A$  and  $B$*  with respect to  $R$ . If  $S(AB) = S(CD)$ , then  $C$  is said to *precede  $D$* , and  $D$  to *follow  $C$* , in the sense  $AB$ .

If there is a point  $B_\infty$ , other than  $A_\infty$ , which is not in the convex region  $R$ , the sense  $S(ABA_\infty)$  is the same as the sense  $S(ABB_\infty)$ , and the segment  $\overline{AA_\infty B}$  is the same as the segment  $\overline{AB_\infty B}$ . Hence

THEOREM 13. *For a given convex region  $R$  the above definition has the same meaning if any other point collinear with  $R$  but not in  $R$  be substituted for  $A_{\infty}$ .*

COROLLARY 1. *If  $S(AB) = S(A'B')$  and  $S(A'B') = S(A''B'')$ , then  $S(AB) = S(A''B'')$ .*

COROLLARY 2. *If  $S(AB) \neq S(A'B')$  and  $S(A'B') \neq S(A''B'')$ , then  $S(AB) = S(A''B'')$ .*

COROLLARY 3.  $S(AB) \neq S(BA)$ .

COROLLARY 4. *If  $S(AB) = S(BC)$ , then  $S(AB) = S(AC)$ .*

These corollaries are direct translations of Theorems 2-5 into our present terminology. Theorem 7 translates into the following statements in terms of betweenness:

THEOREM 14. (1) *If  $C$  is between  $A$  and  $B$ , then  $B$  is not between  $A$  and  $C$ .* (2) *If three points  $A, B, C$  are distinct,  $C$  is between  $A$  and  $B$  or  $B$  is between  $A$  and  $C$  or  $A$  is between  $C$  and  $B$ .* (3) *If  $C$  is between  $A$  and  $B$  and  $A$  is between  $B$  and  $E$ , then  $C$  is between  $B$  and  $E$ .*

Theorem 7 translates into the following statements in terms of "precede" and "follows."

THEOREM 15. (1) *If  $C$  precedes  $B$  in the sense  $AC$ , then  $B$  does not precede  $C$  in this sense.* (2) *In the sense  $AC$ , either  $B$  precedes  $C$  or  $C$  precedes  $B$ .* (3) *If, in the sense  $AB$ ,  $A$  precedes  $C$  and  $E$  precedes  $A$ , then  $E$  precedes  $C$ .*

DEFINITION. If  $A$  and  $B$  are any two points of a convex region  $R$ , the set consisting of all points which follow  $A$  in the sense  $AB$  is called the *ray*  $AB$ . The point  $A$  is called the *origin* of the ray. The ray consisting of all points which precede  $A$  in the sense  $AB$  is said to be *opposite* to the ray  $AB$ . The set of all points which precede  $A$  in the sense  $AB$  is sometimes called the *prolongation* of the segment  $AB$  beyond  $A$ .

### EXERCISES

1. A convex region on a line is either a segment or the set of all points on the line with the exception of one point.\*

2. If three points of a convex region are in the order  $\{ABC\}$ , they are in the order  $\{CBA\}$  but not in the order  $\{ACB\}$  or  $\{CAB\}$ .

3. In a convex region, if  $A$  is between  $B$  and  $C$ , it is between  $C$  and  $B$ .

4. Between any two points there is an infinity of points.

\* This exercise requires the use of an assumption of continuity ( $G$  and  $R$ , or  $K$ ).

5. If  $B$  is on  $\overline{AC}$  and  $C$  is on  $\overline{BD}$ , then  $C$  is on  $\overline{AD}$  and  $B$  is on  $\overline{AD}$ .

6. The relations  $B$  is on  $\overline{AC}$ ,  $B$  is on  $\overline{AD}$ ,  $B$  is on  $\overline{CD}$  are not possible simultaneously.

7. If  $B$  and  $C$  are on  $\overline{AD}$ , then  $B$  is on  $\overline{AC}$  or on  $\overline{CD}$ .

8. Choosing a system of nonhomogeneous coordinates in which  $A_\infty$  is  $\infty$ , show that the sense  $AB$  is the same as the sense  $A'B'$  if and only if  $B - A$  is of the same sign as  $B' - A'$ ; also that two point pairs have the same sense if and only if they are conjugate under the group

$$x' = ax + b,$$

where  $a > 0$ .

**24. Algebraic criteria of sense.** If  $A = (a_0, a_1)$ ,  $B = (b_0, b_1)$ , and  $C = (c_0, c_1)$  are any three distinct points of the line, the transformation

$$(6) \quad \begin{aligned} x'_0 &= \rho_0 a_0 x_0 + \rho_1 b_0 x_1, \\ x'_1 &= \rho_0 a_1 x_0 + \rho_1 b_1 x_1 \end{aligned}$$

changes  $(1, 0)$ ,  $(0, 1)$ , and  $(1, 1)$  into  $A$ ,  $B$ , and  $C$  respectively if and only if  $\rho_0$  and  $\rho_1$  satisfy the equations

$$\begin{aligned} c_0 &= \rho_0 a_0 + \rho_1 b_0, \\ c_1 &= \rho_0 a_1 + \rho_1 b_1, \end{aligned}$$

that is, if

$$\frac{\rho_0}{\rho_1} = \frac{\begin{vmatrix} c_0 & b_0 \\ c_1 & b_1 \end{vmatrix}}{\begin{vmatrix} a_0 & c_0 \\ a_1 & c_1 \end{vmatrix}}.$$

With this choice of  $\rho_0/\rho_1$  the determinant of the transformation (6) is of the same sign as

$$S = \begin{vmatrix} a_0 & b_0 \\ a_1 & b_1 \end{vmatrix} \cdot \begin{vmatrix} b_0 & c_0 \\ b_1 & c_1 \end{vmatrix} \cdot \begin{vmatrix} c_0 & a_0 \\ c_1 & a_1 \end{vmatrix}.$$

By definition the projectivity is direct if and only if  $S$  is positive. Now if  $A' = (a'_0, a'_1)$ ,  $B' = (b'_0, b'_1)$ ,  $C' = (c'_0, c'_1)$  are any three points of the line, and

$$S' = \begin{vmatrix} a'_0 & b'_0 \\ a'_1 & b'_1 \end{vmatrix} \cdot \begin{vmatrix} b'_0 & c'_0 \\ b'_1 & c'_1 \end{vmatrix} \cdot \begin{vmatrix} c'_0 & a'_0 \\ c'_1 & a'_1 \end{vmatrix},$$

two cases are possible. If  $S'$  is of the same sign as  $S$ , the projectivities in which

$$(7) \quad (1, 0)(0, 1)(1, 1) \overline{\wedge} ABC,$$

$$(8) \quad (1, 0)(0, 1)(1, 1) \overline{\wedge} A'B'C'$$

are both direct or both opposite, and hence the projectivity in which

$$(9) \quad ABC \overline{\wedge} A'B'C'$$

is direct. If  $S'$  is opposite in sign to  $S$ , one of the projectivities (7) and (8) is direct and the other opposite, and hence (9) is opposite. Hence

**THEOREM 16.** *Let  $A = (a_0, a_1)$ ,  $B = (b_0, b_1)$ ,  $C = (c_0, c_1)$ ,  $A' = (a'_0, a'_1)$ ,  $B' = (b'_0, b'_1)$ ,  $C' = (c'_0, c'_1)$  be collinear points. Then  $S(ABC) = S(A'B'C')$  if and only if the expressions*

$$\begin{vmatrix} a_0 & b_0 \\ a_1 & b_1 \end{vmatrix} \cdot \begin{vmatrix} b_0 & c_0 \\ b_1 & c_1 \end{vmatrix} \cdot \begin{vmatrix} c_0 & a_0 \\ c_1 & a_1 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} a'_0 & b'_0 \\ a'_1 & b'_1 \end{vmatrix} \cdot \begin{vmatrix} b'_0 & c'_0 \\ b'_1 & c'_1 \end{vmatrix} \cdot \begin{vmatrix} c'_0 & a'_0 \\ c'_1 & a'_1 \end{vmatrix}$$

have the same sign.

**COROLLARY 1.** *Three points given by the finite nonhomogeneous coordinates  $a, b, c$  are conjugate under the group of all direct projectivities to three points given by the finite nonhomogeneous coordinates  $a', b', c'$ , respectively, if and only if  $(a-b)(b-c)(c-a)$  and  $(a'-b')(b'-c')(c'-a')$  have the same sign.*

*Proof.* Set  $a = a_1/a_0$ ,  $b = b_1/b_0$ ,  $c = c_1/c_0$ , and apply the theorem.

**COROLLARY 2.** *Two points given by the finite nonhomogeneous coordinates  $a$  and  $b$  are conjugate under the group of all direct projectivities leaving the point  $\infty$  of the nonhomogeneous coordinate system invariant to the two points given by the finite nonhomogeneous coordinates  $a'$  and  $b'$  respectively if and only if  $a-b$  and  $a'-b'$  have the same sign.*

*Proof.* Set  $a = a_1/a_0$ ,  $b = b_1/b_0$ ,  $c_0 = 0$ ,  $c_1 = 1$ , and apply the theorem.

**THEOREM 17.**  *$A, B$  separate  $C, D$  if and only if the cross ratio  $R(AB, CD)$  is negative.*

*Proof.* By the last theorem,  $A, B$  separate  $C, D$  if and only if

$$\begin{vmatrix} a_0 & b_0 \\ a_1 & b_1 \end{vmatrix} \cdot \begin{vmatrix} b_0 & c_0 \\ b_1 & c_1 \end{vmatrix} \cdot \begin{vmatrix} c_0 & a_0 \\ c_1 & a_1 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} a_0 & b_0 \\ a_1 & b_1 \end{vmatrix} \cdot \begin{vmatrix} b_0 & d_0 \\ b_1 & d_1 \end{vmatrix} \cdot \begin{vmatrix} d_0 & a_0 \\ d_1 & a_1 \end{vmatrix}$$

are opposite in sign. But the quotient of these two expressions has the same sign as  $R(AB, CD)$  (cf. p. 165, Chap. VI, Vol. I).

With the aid of this theorem the proof of Theorem 7 can be made much more simply than in § 21.

**25. Pairs of lines and of planes.** **THEOREM 18.** *The points of space not on either of two planes  $\alpha$  and  $\beta$  fall into two classes such that two points  $O_1, O_2$  of the same class are not separated by the points in which the line  $O_1O_2$  meets the planes  $\alpha$  and  $\beta$ , while two points  $O, P$  of different classes are separated by the points in which the line  $OP$  meets  $\alpha$  and  $\beta$ .*

*Proof.* By the space dual of Theorem 10 the planes of the pencil  $\alpha\beta$  are separated by  $\alpha$  and  $\beta$  into two segments. Let  $[O]$  be the set

of points on the planes of one of these segments but not on the line  $\alpha\beta$ , and let  $[P]$  be the set of the points on the planes of the other segment but not on the line  $\alpha\beta$ .

The two planes  $\omega$  and  $\pi$  of the pencil  $\alpha\beta$  which are on any two points  $O$  and  $P$  are separated by  $\alpha$  and  $\beta$ . Hence, by Theorem 7 and § 20, the points in which the line  $OP$  meets  $\alpha$  and  $\beta$  are separated by  $O$  and  $P$ . In like manner, any two points  $O_1, O_2$  de-

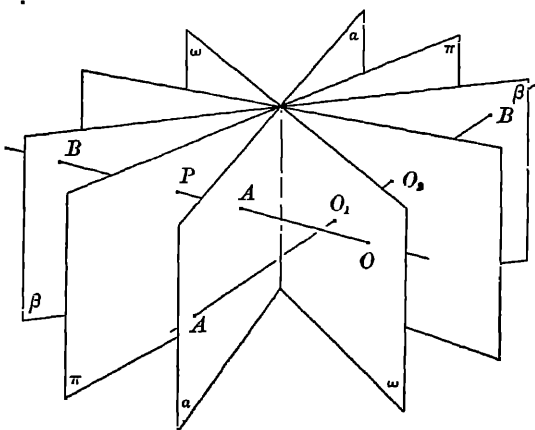


FIG. 13

termine with the line  $\alpha\beta$  a pair of planes (or a single plane) not separated by  $\alpha$  and  $\beta$ , and hence the line  $O_1O_2$  meets  $\alpha$  and  $\beta$  in points (or a single point) not separated by  $O_1$  and  $O_2$ . By the same reasoning, any line  $P_1P_2$  meets  $\alpha$  and  $\beta$  in points (or a point) not separated by  $P_1$  and  $P_2$ .

**COROLLARY 1.** *If  $l$  and  $m$  are two coplanar lines, the points of the plane which are not on  $l$  or  $m$  fall into two classes such that two points  $O_1, O_2$  of the same class are not separated by the points in which the line  $O_1O_2$  meets  $l$  and  $m$ , while two points  $O, P$  of different classes are separated by the points in which  $OP$  meets  $l$  and  $m$ .*

**COROLLARY 2.** *There is only one pair of classes  $[O]$  and  $[P]$  satisfying the conditions of the above theorem (or its first corollary) determined by a given pair of planes (or lines).*

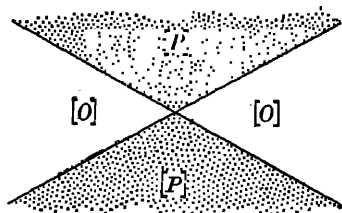


FIG. 14

**DEFINITION.** Two points in different classes (according to Corollary 1) relative to two coplanar lines are said to be *separated* by the two lines; otherwise they are said not to be separated by the lines. Two points in different classes (according to Theorem 18) relative to two planes are said to be *separated* by the two planes; otherwise they are said not to be separated by the planes.

## EXERCISES

1. If  $l_1$  and  $l_2$  are two coplanar lines and  $O$  any point of their common plane, all triads of points in a fixed sense-class  $S_1$  on  $l_1$  are projected from  $O$  into triads in a fixed sense-class  $S_2$  on  $l_2$  (Theorem 6). If  $P$  is any other point of the plane, it is separated from  $O$  by  $l_1$  and  $l_2$  if and only if triads in the sense  $S_1$  are not projected from  $P$  into triads in the sense  $S_2$ .

This problem can be stated also in terms of the sense of pairs of points in the region obtained on  $l_1$  or  $l_2$  respectively by leaving out the common point. The theorem in this form is generalized in § 30. In the form stated in Ex. 1 it has the following generalization.

2. If  $l_1$  and  $l_2$  are two noncoplanar lines, and  $o$  is any line not intersecting them, all triads in a fixed sense  $S_1$  on  $l_1$  are axially projected from  $o$  into triads in a fixed sense  $S_2$  on  $l_2$  (Theorem 6). The lines not intersecting  $l_1$  and  $l_2$  fall into two classes: those by which triads in the sense  $S_1$  are projected into triads in the sense  $S_2$ , and those by which triads in the sense  $S_1$  are projected into triads in the sense opposite to  $S_2$ .

3. Obtain the definition of separation of two coplanar lines by two points as the plane dual of the definition of separation of two points by two coplanar lines. Prove that if two coplanar lines separate two points, then the points separate the lines. State and prove the corresponding result for pairs of points and of planes.

## 26. The triangle and the tetrahedron.

THEOREM 19. If a line  $l$  not passing through any vertex of a triangle  $ABC$  meets the sides  $BC$ ,  $CA$ ,  $AB$  in  $A_1$ ,  $B_1$ ,  $C_1$  respectively, then any other line  $m$  which meets the segments  $\overline{BA_1C}$ ,  $\overline{CB_1A}$  also meets the segment  $\overline{AC_1B}$ .

*Proof.* Suppose first that  $m$  passes through  $A_1$ ; then

$$ACB_1B_2 \stackrel{A}{\wedge} (ABC_1C_2),$$

and hence, if  $B_1$  and  $B_2$  do not separate  $A$  and  $C$ ,  $C_1$  and  $C_2$  do not separate  $A$  and  $B$ . Similarly, the theorem is true if  $m$  passes through  $B_1$ .

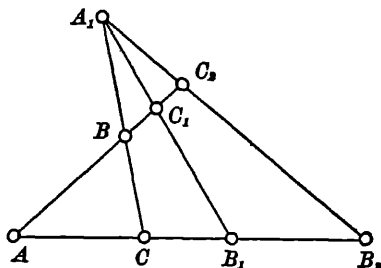


FIG. 15

If  $m$  does not pass through  $A_1$  or  $B_1$ , let  $m'$  be a line joining  $A_1$  to the point in which  $m$  meets  $CA$ . By the argument above we have first that  $m'$  meets all three segments  $\overline{BA_1C}$ ,  $\overline{CB_1A}$ , and  $\overline{AC_1B}$ , and then that  $m$  meets them.

Let us denote the segment  $\overline{AC_1B}$  by  $\gamma$ ,  $\overline{BA_1C}$  by  $\alpha$ , and  $\overline{CB_1A}$  by  $\beta$ , and the segments complementary to  $\alpha$ ,  $\beta$ ,  $\gamma$  by  $\bar{\alpha}$ ,  $\bar{\beta}$ ,  $\bar{\gamma}$  respectively. The

above theorem then gives the information that every line which meets two of the segments  $\alpha, \beta, \gamma$  meets the third. Any line which meets  $\alpha$  and  $\bar{\beta}$  meets  $\bar{\gamma}$ , for, as it does not pass through  $A$  or  $B$ , it meets either  $\gamma$  or  $\bar{\gamma}$ ; but if it met  $\gamma$ , and by hypothesis meets  $\alpha$ , it would meet  $\beta$ . Hence the theorem gives that  $\alpha, \bar{\beta}, \bar{\gamma}$  are such that any line meeting two of these segments meets the third. By a repetition of this argument it follows that every line of the plane which does not pass through a vertex of the triangle meets all three segments of one of the trios  $\alpha\beta\gamma, \bar{\alpha}\bar{\beta}\bar{\gamma}, \alpha\bar{\beta}\gamma, \alpha\beta\bar{\gamma}$ , and no line whatever meets all three segments in any of the trios  $\alpha\beta\gamma, \bar{\alpha}\bar{\beta}\bar{\gamma}, \alpha\bar{\beta}\gamma, \alpha\beta\bar{\gamma}$ .

The lines of the plane, exclusive of those through the vertices, therefore fall into four classes:

- (1) those which meet  $\alpha, \beta, \gamma$ ,
- (2) those which meet  $\bar{\alpha}, \bar{\beta}, \bar{\gamma}$ ,
- (3) those which meet  $\alpha, \bar{\beta}, \bar{\gamma}$ ,
- (4) those which meet  $\bar{\alpha}, \beta, \gamma$ .

No two lines  $l_1, l_2$  of the same class are separated by any pair of the lines joining the point  $l_1, l_2$  to the vertices of the triangle, while any two lines  $l_1, m_1$  of different classes are separated by two of the lines joining the point  $l_1, m_1$  to the vertices. This result is perhaps more intuitively striking when put into the dual form, as follows:

**THEOREM 20.** *The points of a plane not on the sides of a triangle fall into four classes such that no two points  $L_1, L_2$  of the same class are separated by any pair of the points in which the line  $L_1L_2$  meets the sides of the triangle, while*

*any two points  $L_1, M_1$  of different classes are separated by two of the points in which the line  $L_1M_1$  meets the sides of the triangle.*

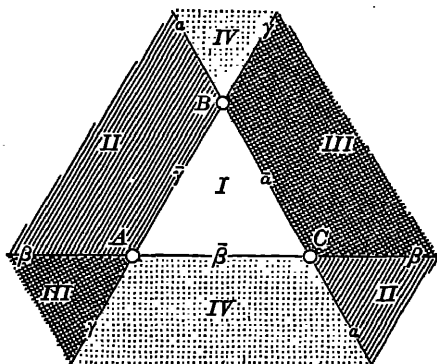


FIG. 16

**DEFINITION.** Any one of the four classes of points in Theorem 20 is called a *triangular region*. The vertices of the triangle are also called *vertices of the triangular region*.

The property of the triangle stated in Theorem 19 can also serve as a basis for a discussion of the ordinal theorems on the tetrahedron and for those of the  $(n+1)$ -point in  $n$ -space. Suppose we have a tetrahedron whose vertices are  $A_1, A_2, A_3, A_4$ . Let us denote its faces by  $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ , the face  $\alpha_1$  being opposite to the vertex  $A_1$ , etc.; let us denote the edges by  $a_{12}, a_{13}, a_{14}, a_{23}, a_{24}, a_{34}$ , the edge  $a_{ij}$  being the line  $A_i A_j$ . Each edge  $a_{ij}$  is separated by the vertices  $A_i, A_j$  into two segments, which we shall denote by  $\sigma_{ij}$  and  $\bar{\sigma}_{ij}$ . Let  $\pi$  be a plane not passing through any vertex; the six segments which it meets may be denoted by  $\sigma_{12}, \sigma_{13}, \dots, \sigma_{42}$ , and the complementary segments by  $\bar{\sigma}_{12}, \bar{\sigma}_{13}, \dots, \bar{\sigma}_{42}$ . Then as a corollary of Theorem 19 we have that any plane which meets three noncoplanar segments of the set  $\sigma_{12}, \sigma_{13}, \dots, \sigma_{42}$  meets all the rest of them, and, moreover, no plane meets all the segments  $\bar{\sigma}_{12}, \bar{\sigma}_{13}, \dots, \bar{\sigma}_{42}$ . If we observe that any plane not passing through a vertex must meet the edges  $a_{12},$

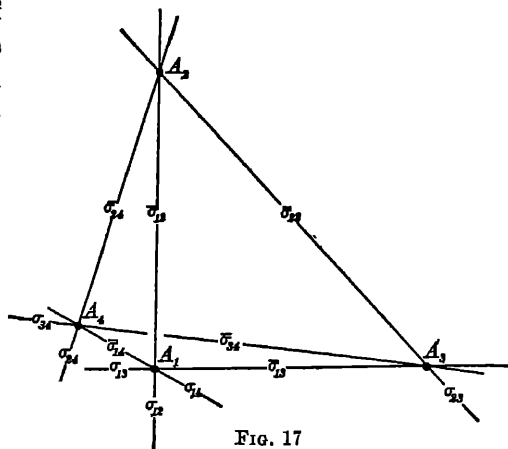


FIG. 17

$a_{13}, a_{14}$  in three distinct points, it becomes clear that the planes not passing through any vertex fall into eight classes such that two planes of the same class are not separated by a pair of vertices, whereas two planes of different classes are separated by a pair of vertices. Under duality we have

**THEOREM 21.** *The points not upon the faces of a tetrahedron fall into eight classes such that two points of the same class are not separated by the points in which the line joining them meets the faces, whereas two points of different classes are separated by two of the points in which their line meets the faces of the tetrahedron.*

**DEFINITION.** Any one of the eight classes of points in Theorem 21 is called a *tetrahedral region*. The vertices of the tetrahedron are also called vertices of any one of the tetrahedral regions.

It would be easy to complete the discussion of the triangle and the tetrahedron at this point — for example, to define the term “boundary” and to prove that the boundary of any one of the classes of points in Theorem 20 is composed of  $A, B, C$  and three segments having the property that no line meets them all. We shall defer this discussion, however, to a later chapter, where the results will appear as special cases of more general theorems.

### 27. Algebraic criteria of separation. Cross ratios of points in space.

The classes of points determined (Theorems 18–21) by a pair of intersecting lines, a triangle, a pair of planes or by a tetrahedron can be discussed by means of some very elementary algebraic considerations. As these are similar in the plane and in space, let us carry out the work only for the three-dimensional cases.

Suppose that the homogeneous coördinates of four noncoplanar points  $A_1, A_2, A_3, A_4$  are given by the columns of the matrix,

$$(10) \quad \begin{pmatrix} a_{01} & a_{02} & a_{03} & a_{04} \\ a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix},$$

and let  $(x_0, x_1, x_2, x_3)$  be the homogeneous coördinates of any other point  $X$ . Let us indicate by  $|x, a_2, a_3, a_4|$  the determinant of the matrix obtained by substituting  $x_0, x_1, x_2, x_3$  respectively for the elements of the first column in the matrix above; by  $|a_1, x, a_3, a_4|$  the determinant obtained by performing the same operation on the second column, etc. The expressions  $|y, a_2, a_3, a_4|$  etc. have similar meanings in terms of the coördinates of a point  $(y_0, y_1, y_2, y_3) = Y$ . The following expressions are formed analogously to the cross ratios of four points on a line (cf. § 58, Vol. I):

$$(11) \quad \begin{aligned} k_{14} &= \frac{|x, a_2, a_3, a_4|}{|a_1, a_2, a_3, x|} \div \frac{|y, a_2, a_3, a_4|}{|a_1, a_2, a_3, y|}, \\ k_{24} &= \frac{|a_1, x, a_3, a_4|}{|a_1, a_2, a_3, x|} \div \frac{|a_1, y, a_3, a_4|}{|a_1, a_2, a_3, y|}, \\ k_{34} &= \frac{|a_1, a_2, x, a_4|}{|a_1, a_2, a_3, x|} \div \frac{|a_1, a_2, y, a_4|}{|a_1, a_2, a_3, y|}. \end{aligned}$$

Clearly there are twelve numbers  $k_{ij}$  which could be defined analogously to these; and if the notation  $A_1, A_2, A_3, A_4, X, Y$  be permuted among the six points, 720 such expressions are defined. Each number  $k_{ij}$

is an absolute invariant of the six points, for it is unaltered if the coördinates of any point be multiplied by a constant or if all six points be subjected to the same linear transformation.

If  $Y$  be not upon any of the planes determined by the points  $A_1, A_2, A_3, A_4$ , there exists a projectivity which carries  $Y$  into  $(1, 1, 1, 1)$  and the points  $A_1, A_2, A_3, A_4$  into the points represented by the columns of

$$(12) \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let  $(X_0, X_1, X_2, X_3)$  be the point into which  $(x_0, x_1, x_2, x_3)$  is carried by this projectivity. By substituting in (11) we see that

$$k_{14} = \frac{X_0}{X_3}, \quad k_{24} = \frac{X_1}{X_3}, \quad k_{34} = \frac{X_2}{X_3}.$$

From this it follows that  $|x, a_2, a_3, a_4|, |a_1, x, a_3, a_4|$ , etc. could be taken as the homogeneous coördinates with respect to the tetrahedron of reference whose vertices are  $A_1, A_2, A_3, A_4$ .

The line  $(X_0, X_1, X_2, X_3) - \lambda(1, 1, 1, 1)$

meets the planes determined by the four points represented by (12) in four points given by the values  $\lambda = X_0, \lambda = X_1, \lambda = X_2, \lambda = X_3$ . The cross ratios of pairs of these points with  $(X_0, X_1, X_2, X_3)$  and  $(1, 1, 1, 1)$  are  $X_0/X_3, X_1/X_3$ , and  $X_2/X_3$ . Hence  $k_{14}, k_{24}, k_{34}$  are cross ratios of  $X$  and  $Y$  with pairs of points in which the line joining them meets the faces of the tetrahedron  $A_1A_2A_3A_4$ .

By Theorem 17, the points  $X$  and  $Y$  are separated by the planes  $A_2A_3A_4$  and  $A_2A_3A_1$  if and only if  $k_{14}$  is negative. They will be separated by  $A_1A_3A_4$  and  $A_1A_3A_2$  if and only if  $k_{24}$  is negative, and by  $A_1A_2A_4$  and  $A_1A_2A_3$  if and only if  $k_{34}$  is negative. Hence, by Theorem 21, we have

**THEOREM 22.** *The points  $X$  and  $Y$  will be in the same class with respect to the tetrahedron  $A_1A_2A_3A_4$  if and only if  $k_{14}, k_{24}, k_{34}$  are all positive.*

**COROLLARY.** *The eight regions determined by the tetrahedron  $A_1A_2A_3A_4$  are those for which the algebraic signs of  $k_{14}, k_{24}, k_{34}$  appear in the following combinations:  $(+, +, +), (+, +, -), (+, -, +), (-, +, +), (-, -, -), (-, -, +), (-, +, -), (+, -, -)$ .*

Recalling that  $|x, \alpha_2, \alpha_3, \alpha_4| = 0$  is the equation of the plane  $A_2A_3A_4$  (cf. § 70, Vol. I), we see that if

$$\begin{aligned} \alpha(x) &\equiv \alpha_0x_0 + \alpha_1x_1 + \alpha_2x_2 + \alpha_3x_3 = 0 \\ \text{and} \quad \beta(x) &\equiv \beta_0x_0 + \beta_1x_1 + \beta_2x_2 + \beta_3x_3 = 0 \end{aligned}$$

are the equations of two planes, the formula given above for the cross ratio of two points  $X$  and  $Y$  with the points of intersection of the line  $XY$  with these planes becomes

$$(13) \quad \frac{\alpha(x)}{\alpha(y)} + \frac{\beta(x)}{\beta(y)}.$$

Thus two points are in the same one of the two classes determined by the planes  $\alpha(x)$  and  $\beta(x)$  if and only if this expression is positive.

This result assumes an even simpler form when specialized somewhat with respect to a system of nonhomogeneous coördinates. Suppose that  $x_0 = 0$  be chosen as the singular plane in a system of nonhomogeneous coördinates; then the same point is represented nonhomogeneously by  $(x, y, z)$  or homogeneously by  $(1, x, y, z)$ , and the plane represented above by  $\alpha(x) = 0$  has the equation

$$\alpha_1x + \alpha_2y + \alpha_3z + \alpha_0 = 0.$$

If  $\beta(x) = 0$  be the plane  $x_0 = 0$ , the expression for the cross ratio written above becomes

$$\frac{\alpha(x)}{\alpha(y)} + \frac{x_0}{y_0},$$

which reduces in nonhomogeneous coördinates, when  $(x_0, x_1, x_2, x_3)$  and  $(y_0, y_1, y_2, y_3)$  are replaced by  $(1, x', y', z')$  and  $(1, x'', y'', z'')$ , to

$$(14) \quad \frac{\alpha_1x' + \alpha_2y' + \alpha_3z' + \alpha_0}{\alpha_1x'' + \alpha_2y'' + \alpha_3z'' + \alpha_0}.$$

Hence two points  $(x', y', z')$  and  $(x'', y'', z'')$  are separated by the singular plane, and  $\alpha_1x + \alpha_2y + \alpha_3z + \alpha_0 = 0$  if and only if the numerator and denominator of (14) are of opposite sign. For reference we shall state this as a theorem in the following form:

**THEOREM 23.** *The two classes of points determined, according to Theorem 18, by the singular plane of a nonhomogeneous coördinate system and a plane  $ax + by + cz + d = 0$  are respectively the points  $(x, y, z)$  for which  $ax + by + cz + d$  is positive and the points for which it is negative.*

## EXERCISES

1. Carry out the discussion analogous to the above in the two-dimensional case. Generalize to  $n$  dimensions.

2. How many of the 720 numbers analogous to  $k_{14}$  are distinct?

**28. Euclidean spaces.** DEFINITION. The set of all points of a projective space\* of  $n$  dimensions, with the exception of those on a single  $(n-1)$ -space  $S^\infty$  contained in the  $n$ -space, is called a *Euclidean space of  $n$  dimensions*. Thus, in particular, the set of all but one of the points of a projective line is called a *Euclidean line*, and the set of all the points of a projective plane, except those on a single line, is called a *Euclidean plane*.

DEFINITION. The projective  $(n-1)$ -space  $S^\infty$  is called the *singular  $(n-1)$ -space* or the  *$(n-1)$ -space at infinity* or the *ideal  $(n-1)$ -space associated with the Euclidean space*. Any figure in  $S^\infty$  is said to be *ideal* or to be *at infinity*, whereas any figure in the Euclidean  $n$ -space is said to be *ordinary*.

The ordinary points of any line in a Euclidean plane or space form a Euclidean line and thus satisfy the definition (§ 23) of a linear convex region. The definitions and theorems of that section may therefore be applied at once in discussing Euclidean spaces. Thus, if  $A$  and  $B$  are any two ordinary points, we shall speak of "the segment  $AB$ ," "the ray  $AB$ ," etc.

The first corollary of Theorem 18 yields a very simple and important theorem if the line  $m$  be taken as the line at infinity, namely:

**THEOREM 24.** *The points of a Euclidean plane which are not on a line  $l$  fall into two classes such that the segment joining two points of the same class does not meet  $l$  and the segment joining two points of different classes does meet  $l$ .*

**COROLLARY.** *If  $\alpha$  is any ray whose origin is a point of  $l$ , all points of  $\alpha$  are either on  $l$  or on the same side of  $l$ .*

In like manner Theorem 18 yields

**THEOREM 25.** *The points of a Euclidean three-space which are not on a plane  $\pi$  fall into two classes such that the segment joining two points of the same class does not meet  $\pi$  and the segment joining two points of different classes does meet  $\pi$ .*

\* We shall refer to a line, plane, or  $n$ -space in the sense of Chap. I, Vol. I, as a projective line, plane, or  $n$ -space whenever there is possibility of confusion with other types of spaces.

DEFINITION. The two classes of points determined by a line  $l$  in a Euclidean plane, according to Theorem 24, are called the two *sides* of  $l$ . The two classes of points determined by a plane  $\pi$  in a Euclidean three-space, according to Theorem 25, are called the two *sides* of  $\pi$ .

The two sides of  $\pi$  are characterized algebraically in Theorem 23.

DEFINITION. An ordered pair of rays  $h, k$  having a common origin is called an *angle* and is denoted by  $\angle hk$ . If the rays are  $AB$  and  $AC$ , the angle may also be denoted by  $\angle BAC$ . If the rays are opposite, the angle is called a *straight* angle; if the rays coincide, it is called a *zero* angle. The rays  $h, k$  are called the *sides* of  $\angle hk$ , and their common origin the *vertex* of  $\angle hk$ .

### EXERCISES

1. The points of a Euclidean plane not on the sides or vertex of a nonzero angle  $\angle hk$  fall into two classes such that the segment joining two points of different classes contains one point of  $h$  or  $k$ . In case  $\angle hk$  is not a straight angle, one of these two classes consists of every point which is between a point of  $h$  and a point of  $k$ .

2. Generalize Theorem 25 to  $n$  dimensions.

**29. Assumptions for a Euclidean space.** A Euclidean space can be characterized completely by means of a set of assumptions stated in terms of order relations. Such a set of assumptions is given below. It is a simple exercise, which we shall leave to the reader, to verify that these assumptions are all satisfied by a Euclidean space as defined in the last section.

The reverse process is also of considerable interest. This consists (1) in deriving the elementary theorems of alignment and order from Assumptions I–VIII below, and (2) in defining ideal elements and showing that these, together with the elements of the Euclidean space, form a projective space. For the details of (1) and an outline of (2) the reader may consult the article by the writer, in the Transactions of the American Mathematical Society, Vol. V (1904), pp. 343–384, and also a note by R. L. Moore, in the same journal, Vol. XIII (1912), p. 74. On (2) one may consult the article by R. Bonola, *Giornale di Matematiche*, Vol. XXXVIII (1900), p. 105, and also that by F. W. Owens, Transactions of the American Mathematical Society, Vol. XI (1910), p. 141. Compare also the Introduction to Vol. I.

This set of assumptions refers to an undefined class of elements called points and an undefined relation among points indicated by saying "the points  $A, B, C$  are in the order  $\{ABC\}$ ."

The assumptions are as follows:

- I. If points  $A, B, C$  are in the order  $\{ABC\}$ , they are distinct.  
 II. If points  $A, B, C$  are in the order  $\{ABC\}$ , they are not in the order  $\{BCA\}$ .

DEFINITION. If  $A$  and  $B$  are distinct points, the segment  $\overline{AB}$  consists of all points  $X$  in the order  $\{AXB\}$ ; all points of the segment  $\overline{AB}$  are said to be *between*  $A$  and  $B$ ; the segment together with  $A$  and  $B$  is called the *interval*  $AB$ ; the *line*  $AB$  consists of  $A$  and  $B$  and all points  $X$  in one of the orders  $\{ABX\}$ ,  $\{AXB\}$ ,  $\{XAB\}$ ; and the *ray*  $AB$  consists of  $B$  and all points  $X$  in one of the orders  $\{AXB\}$  and  $\{ABX\}$ .

III. If points  $C$  and  $D$  ( $C \neq D$ ) are on the line  $AB$ , then  $A$  is on the line  $CD$ .

IV. If three distinct points  $A, B$ , and  $C$  do not lie on the same line, and  $D$  and  $E$  are two points in the orders  $\{BCD\}$  and  $\{CEA\}$ , then a point  $F$  exists in the order  $\{AFB\}$  and such that  $D, E$ , and  $F$  lie on the same line.

V. If  $A$  and  $B$  are two distinct points, there exists a point  $C$  such that  $A, B$ , and  $C$  are in the order  $\{ABC\}$ .

VI. There exist three distinct points  $A, B, C$  not in any of the orders  $\{ABC\}$ ,  $\{BCA\}$ ,  $\{CAB\}$ .

DEFINITION. If  $A, B, C$  are three noncollinear points, the set of all points collinear with pairs of points on the intervals  $AB, BC, CA$  is called the *plane*  $ABC$ .

VII. If  $A, B, C$  are three noncollinear points, there exists a point  $D$  not in the same plane with  $A, B$ , and  $C$ .

VIII. Two planes which have one point in common have two distinct points in common.

IX. If  $A$  is any point and a any line not containing  $A$ , there is not more than one line through  $A$  coplanar with  $a$  and not meeting  $a$ .

XVII. If there exists an infinitude of points, there exists a certain pair of points  $A, C$  such that if  $[\sigma]$  is any infinite set of segments of the line  $AC$ , having the property that each point of the interval  $AC$  is a point of a segment  $\sigma$ , then there is a finite subset,  $\sigma_1, \sigma_2, \dots, \sigma_n$ , with the same property.\*

\* The proposition here stated about the interval  $AC$  is commonly known as the Heine-Borel theorem. The continuity assumption is more usually stated in the form of the "Dedekind Cut Axiom." Cf. R. Dedekind, *Stetigkeit und irrationalen Zahlen*, Braunschweig, 1872.

Assumptions I to VIII are sufficient to define a three-space which is capable of being extended by means of ideal elements into a projective space satisfying A, E, S. This space will not, in general, satisfy Assumption P. If the continuity assumption, XVII, be added, the corresponding projective space is real and hence properly projective. Assumption IX is the assumption with regard to parallel lines. Assumption VIII limits the number of dimensions to three.

**30. Sense in a Euclidean plane.** Suppose that  $l_{\infty}$  is the line at infinity of a Euclidean plane. Every collineation transforming the Euclidean plane into itself effects a projectivity on  $l_{\infty}$  which is either direct or opposite (§ 18). Since the direct projectivities on  $l_{\infty}$  form a group, the planar collineations which effect these transformations on  $l_{\infty}$  also form a group.

**DEFINITION.** A collineation of a Euclidean plane which effects a direct projectivity on the line at infinity of this plane is said to be a *direct collineation* of the Euclidean plane. Any other collineation of the Euclidean plane is said to be *opposite*. Let  $A, B, C$  be three noncollinear points; the class of all ordered triads  $A'B'C'$  such that the collineation carrying  $A, B$ , and  $C$  to  $A', B'$ , and  $C'$  respectively is direct, is called a *sense-class* and is denoted by  $S(ABC)$ . Two ordered triads of noncollinear points in the same sense-class are said to *have the same sense* or to *be in the same sense*. Otherwise they are said to *have opposite senses* or to *be in opposite senses*.

Since the direct projectivities form a group, it follows that if a triad  $A'B'C'$  is in  $S(ABC)$ , then  $S(ABC) = S(A'B'C')$ .

**THEOREM 26.** *There are two and only two sense-classes in a Euclidean plane. If  $A, B$ , and  $C$  are noncollinear points,  $S(ABC) = S(BCA) \neq S(ACB)$ .*

*Proof.* Let  $A, B, C$  be three noncollinear points. If  $A', B', C'$  are any three noncollinear points such that the projectivity carrying  $A, B, C$  to  $A', B', C'$  respectively is direct,  $S(ABC)$  contains the triad  $A'B'C'$ . Because the direct projectivities form a group,  $S(ABC) = S(A'B'C')$ . The triads to which  $ABC$  is carried by collineations which are not direct all form a sense-class, because the product of two opposite collineations is direct. Thus there are two and only two sense-classes.

Suppose we denote the lines  $BC, CA, AB$  by  $a, b, c$  respectively and let  $A', B', C'$  be the points of intersection of  $a, b, c$  respectively

with  $l_\infty$ . The projectivity carrying  $ABC$  to  $BCA$  evidently carries  $a, b$ , and  $c$  to  $b, c$ , and  $a$  respectively, and thus carries  $A'B'C'$  to  $B'C'A'$ , and thus is direct (§ 19). Hence

$$S(ABC) = S(BCA).$$

The projectivity carrying  $ABC$  to  $ACB$  carries  $A'B'C'$  to  $A'C'B'$ , and hence is not direct; and hence

$$S(ABC) \neq S(ACB).$$

**THEOREM 27.** *Two points  $C$  and  $D$  are on opposite sides of a line  $AB$  if and only if*

$$S(ABC) \neq S(ABD).$$

This theorem can be derived as a consequence of Ex. 1, § 25. It can also be derived from the following algebraic considerations.

Let us choose a system of nonhomogeneous coördinates in such a way that the singular line of the coördinate system is the same as the singular line of the Euclidean plane. The group of all projective collineations transforming the Euclidean plane into itself then reduces (§ 67, Vol I) to

$$(15) \quad \begin{aligned} x' &= a_1x + b_1y + c_1, \\ y' &= a_2x + b_2y + c_2, \end{aligned} \quad \Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0.$$

If we change to the homogeneous coördinates for which  $x = x_1/x_0$  and  $y = x_2/x_0$ , the line at infinity has the equation  $x_0 = 0$ , and the equations (15) reduce to

$$(16) \quad \begin{aligned} x'_0 &= x_0, \\ x'_1 &= c_1x_0 + a_1x_1 + b_1x_2, \\ x'_2 &= c_2x_0 + a_2x_1 + b_2x_2. \end{aligned}$$

On the line at infinity this effects the transformation

$$\begin{aligned} x'_1 &= a_1x_1 + b_1x_2, \\ x'_2 &= a_2x_1 + b_2x_2, \end{aligned}$$

which is direct if and only if  $\Delta > 0$ .

Let the nonhomogeneous coördinates of three points  $A, B, C$  be  $(a_1, a_2), (b_1, b_2), (c_1, c_2)$  respectively. The determinant

$$(17) \quad S = \begin{vmatrix} a_1 & a_2 & 1 \\ b_1 & b_2 & 1 \\ c_1 & c_2 & 1 \end{vmatrix}$$

is multiplied by  $\Delta$  whenever the points  $A, B, C$  are subjected to the transformation (15). This is verified by a direct substitution. Hence

the algebraic sign of  $S$  is left invariant by all direct collineations and changed by all others. Hence we have

**THEOREM 28.** *An ordered triad of points  $(a_1, a_2), (b_1, b_2), (c_1, c_2)$  has the same sense as an ordered triad  $(a'_1, a'_2), (b'_1, b'_2), (c'_1, c'_2)$  if and only if the determinants*

$$\begin{vmatrix} a_1 & a_2 & 1 \\ b_1 & b_2 & 1 \\ c_1 & c_2 & 1 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} a'_1 & a'_2 & 1 \\ b'_1 & b'_2 & 1 \\ c'_1 & c'_2 & 1 \end{vmatrix}$$

*have the same sign.*

Theorem 27 now follows as a corollary of Theorem 23, § 27.

### EXERCISES

1. If  $\angle ABC = \angle A'BC'$ ,  $S(ABC) = S(A'BC')$ .
2. Let  $\angle hk$  be said to have the same sense as  $\angle h'k'$  if  $S(ABC) = S(A'B'C')$ , where  $B$  is the vertex of  $\angle hk$ ,  $A$  a point of  $h$ ,  $C$  a point of  $k$ , and  $A', B', C'$  points analogously defined for  $\angle h'k'$ . Define positive and negative angles and develop a theory of the order relations of rays through a point.
3. Let  $\rho$  and  $\sigma$  be two planes of a projective space which meet in a line  $l_\infty$ ; let us denote the two Euclidean planes obtained by leaving  $l_\infty$  out of  $\rho$  and  $\sigma$  by  $\rho_1$  and  $\sigma_1$  respectively; and let  $S_\rho$  be an arbitrary sense-class in  $\rho_1$ . All ordered point triads of  $S_\rho$  are projected from a point  $Q$  not on  $\rho$  or  $\sigma$  into triads of a fixed sense-class  $S_\sigma$  in  $\sigma_1$ . Any other point  $P$  not on  $\rho$  or  $\sigma$  is separated from  $O$  by  $\rho$  and  $\sigma$  if and only if triads in the sense-class  $S_\rho$  are not projected from  $P$  into triads of  $S_\sigma$ .

**\*31. Sense in Euclidean spaces.** The definition given above of direct transformations in a Euclidean plane, based on the concept of direct transformations on the singular line, cannot be generalized to three dimensions. This is because the plane at infinity is projective and, as will be proved in the next section, does not admit of a distinction between direct and opposite projectivities. Nevertheless, the algebraic criterion  $\Delta > 0$  does generalize and is made the basis of the definition which follows.

With reference to a nonhomogeneous coordinate system, of which the singular  $(n-1)$ -space is the  $(n-1)$ -space at infinity, the equations of any projective collineation of a Euclidean  $n$ -space take the form\*

$$(18) \quad x'_i = b_i + \sum_{j=1}^n a_{ij}x_j, \quad (i=1, \dots, n)$$

where the determinant  $|a_{ij}|$  is different from zero. The resultant of

\* The reader may, if he wishes, limit attention to the case  $n=3$ . We have not actually developed the theory of coordinate systems in  $n$  dimensions, but as there is no essential difference in this theory between the three-dimensional case and the  $n$ -dimensional, we do not intend to write out the details.

two transformations of this form has a determinant which is the product of the determinants of the two transformations. Since the coefficients appear nonhomogeneously in (18), it is clear that a self-conjugate subgroup of the group of all transformations (18) is defined by the condition  $|a_{ij}| > 0$ . It follows by the same reasoning as used in § 18 that this subgroup is independent of the choice of the frame of reference, so long as the singular  $(n-1)$ -space coincides with the singular  $(n-1)$ -space of the corresponding Euclidean  $n$ -space.

DEFINITION. The group of all transformations (18) for which the determinant  $|a_{ij}| > 0$  is called the group of *direct* collineations. In a Euclidean  $n$ -space let  $A_1, A_2, \dots, A_{n+1}$  be  $n+1$  linearly independent points; the class of all ordered  $(n+1)$ -ads\*  $A'_1 A'_2 \dots A'_{n+1}$  such that the collineation transforming  $A_1, A_2, \dots, A_{n+1}$  into  $A'_1, A'_2, \dots, A'_{n+1}$  respectively is direct is called a *sense-class* and is denoted by  $S(A_1 A_2 \dots A_{n+1})$ .

THEOREM 29. *There are two and only two sense-classes in a Euclidean  $n$ -space. The sense-class of an ordered  $n$ -ad is unaltered by even permutations and altered by odd permutations.*

*Proof.* The argument for the three-dimensional case is typical of the general case. Let the coördinates of four points  $A, B, C, D$  be  $(a_1, a_2, a_3), (b_1, b_2, b_3), (c_1, c_3, c_3), (d_1, d_3, d_3)$  respectively. The determinant

$$(19) \quad \begin{vmatrix} a_1 & a_2 & a_3 & 1 \\ b_1 & b_2 & b_3 & 1 \\ c_1 & c_2 & c_3 & 1 \\ d_1 & d_2 & d_3 & 1 \end{vmatrix}$$

is multiplied by  $|a_{ij}|$  whenever the points are simultaneously subjected to a transformation (18). Hence the algebraic sign of (19) is left invariant by all direct collineations.

Since an odd permutation of the rows of (19) would change the sign of (19), no such permutation can be effected by a direct collineation. The remaining statements in the theorem now follow directly from the theorem that any ordered tetrad of points can be transformed by a transformation of the form (18) into any other ordered tetrad.

\*32. *Sense in a projective space.* Let us consider the group of all linear transformations

$$(20) \quad x'_i = \sum_{j=0}^n a_{ij} x_j, \quad (i = 0, \dots, n)$$

for which the determinant  $|a_{ij}|$  is different from zero.

\* An  $n$ -ad is a set of  $n$  objects (cf. § 19).

If  $(x_0, \dots, x_n)$  is a set of homogeneous coördinates, the equations (20) continue to represent the same transformation when all the  $a_{ij}$ 's are multiplied by the same constant  $\rho$ ; and two sets of equations like (20) represent the same transformation only if the coefficients of one are proportional to those of the other.

If each  $a_{ij}$  be multiplied by  $\rho$ ,  $|a_{ij}|$  is multiplied by  $\rho^{n+1}$ . Hence, if  $|a_{ij}|$  is negative and  $n$  is even, we may multiply each  $a_{ij}$  by  $-1$  and thus obtain an equivalent expression of the form (20) for which  $|a_{ij}|$  is positive. If, however,  $n$  is odd,  $\rho^{n+1} = k < 0$  has no real root. Hence, if  $n$  is odd, a transformation (20) for which  $|a_{ij}|$  is negative is not equivalent to one for which  $|a_{ij}|$  is positive. Hence the condition  $|a_{ij}| > 0$  determines a subset of the transformations (20) if and only if  $n$  is odd. This subset of transformations forms a group for the reason given in § 18 for the case  $n = 1$ .

DEFINITION. If  $n$  is odd, the group of transformations (20) for which  $|a_{ij}| > 0$  is called the group of *direct* collineations in  $n$ -space.

This definition of the group of direct collineations is independent of the choice of the frame of reference, as follows by an argument precisely like that used to prove the corresponding proposition in § 18.

In a space of three dimensions, let us inquire into what sets of five points the set  $(1, 0, 0, 0)$ ,  $(0, 1, 0, 0)$ ,  $(0, 0, 1, 0)$ ,  $(0, 0, 0, 1)$ ,  $(1, 1, 1, 1)$  can be transformed by direct collineations. If the initial points are to be transformed respectively into the points whose coördinates are the columns of the matrix

$$(21) \quad \begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} & a_{04} \\ a_{10} & a_{11} & a_{12} & a_{13} & a_{14} \\ a_{20} & a_{21} & a_{22} & a_{23} & a_{24} \\ a_{30} & a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix},$$

the collineation must take the form

$$(22) \quad \begin{aligned} x'_0 &= \rho_0 a_{00} x_0 + \rho_1 a_{01} x_1 + \rho_2 a_{02} x_2 + \rho_3 a_{03} x_3, \\ x'_1 &= \rho_0 a_{10} x_0 + \rho_1 a_{11} x_1 + \rho_2 a_{12} x_2 + \rho_3 a_{13} x_3, \\ x'_2 &= \rho_0 a_{20} x_0 + \rho_1 a_{21} x_1 + \rho_2 a_{22} x_2 + \rho_3 a_{23} x_3, \\ x'_3 &= \rho_0 a_{30} x_0 + \rho_1 a_{31} x_1 + \rho_2 a_{32} x_2 + \rho_3 a_{33} x_3, \end{aligned}$$

where the  $\rho$ 's satisfy the equations

$$(23) \quad \begin{aligned} \rho_0 a_{00} + \rho_1 a_{01} + \rho_2 a_{02} + \rho_3 a_{03} &= a_{04}, \\ \rho_0 a_{10} + \rho_1 a_{11} + \rho_2 a_{12} + \rho_3 a_{13} &= a_{14}, \\ \rho_0 a_{20} + \rho_1 a_{21} + \rho_2 a_{22} + \rho_3 a_{23} &= a_{24}, \\ \rho_0 a_{30} + \rho_1 a_{31} + \rho_2 a_{32} + \rho_3 a_{33} &= a_{34}. \end{aligned}$$

Substituting the values of  $\rho_i$  determined from these equations in the determinant of the transformation (22), we see that the value of this determinant is

$$(24) \quad \frac{(a_{04}a_{11}a_{22}a_{33})(a_{00}a_{14}a_{22}a_{33})(a_{00}a_{11}a_{24}a_{33})(a_{00}a_{11}a_{22}a_{34})}{(a_{00}a_{11}a_{22}a_{33})^3},$$

where the expressions in parentheses are abbreviations for determinants formed from the matrix (21) having these expressions as their main diagonals. The number (24) has the same sign as

$$(25) \quad (a_{04}a_{11}a_{22}a_{33})(a_{00}a_{14}a_{22}a_{33})(a_{00}a_{11}a_{24}a_{33})(a_{00}a_{11}a_{22}a_{34})(a_{00}a_{11}a_{22}a_{33}),$$

which is entirely analogous to the expression found in Theorem 16. The initial set of points is transformable into the points whose coördinates are the columns of (21) by a direct transformation if and only if (25) is positive.

This result may be stated in the form of a theorem as follows:

**THEOREM 30.** *If a set of five points whose homogeneous coördinates are the columns of the matrix (21) be such that the product of the four-rowed determinants obtained by omitting columns of this matrix is positive, it can be transformed by a direct collineation into any other set of points having the same property, but not into a set for which the analogous product is zero or negative.*

**COROLLARY.** *Any even permutation but no odd permutation of the vertices of a complete five-point can be effected by a direct collineation.*

**DEFINITION.** Let  $A, B, C, D, E$  be five points no four of which are coplanar. The class of all ordered pentads obtainable from the pentad  $A, B, C, D, E$  by direct collineations is called a *sense-class* and is denoted by  $S(ABCDE)$ .

Theorem 30 and its corollary now give at once the following:

**THEOREM 31.** *There are two and only two sense-classes in a real projective three-space. The sense-class of a set of five points is unaltered by even permutations and altered by odd permutations.*

If an analogous definition of sense-class had been made in the plane, we should have had that all planar collineations are direct, and hence that there is only one sense-class in the plane. This remark, together with Theorem 31, expresses in part what is meant by the proposition:

*The real projective plane is one-sided and the real projective three-space is two-sided.*

Although we have grounded this discussion upon propositions regarding certain groups of collineations, the notion of sense is connected with a much more extensive group. We shall return to this study, which will give a deeper insight into the notions of sense and of one- and two-sidedness, in a later chapter.

**33. Intuitional description of the projective plane.** We may assist our intuitive conception\* of the one-sidedness of the real projective plane by a further consideration of the regions into which a plane is separated by a triangle. These are represented in fig. 16. Since any triangular region is projectively transformable into any other, it follows that any triangular region may be represented like Region I in fig. 16. In fig. 18 the four regions are thus represented, together with a portion of the relations among them.

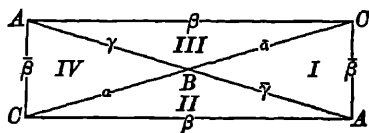


FIG. 18

The representation is more complete if the two segments labeled  $\beta$  are superposed in such a way that the end labeled  $A$  of one coincides with the end labeled  $A$  of the other. This is represented in fig. 19 and may be realized in a model by cutting out a rectangular strip of paper, giving it a half twist, and pasting together the two ends.

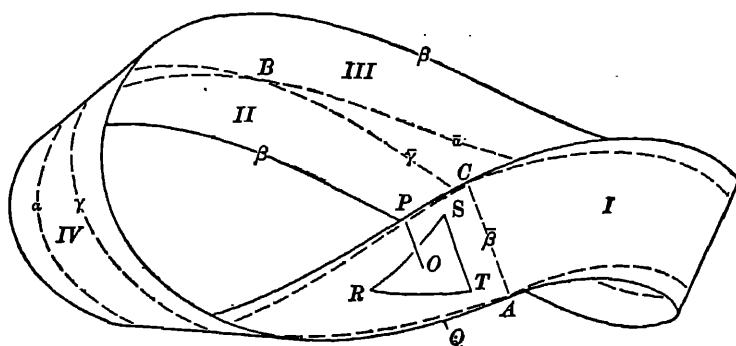


FIG. 19

To complete the model it would be necessary to bring the two edges labeled  $\beta$  in fig. 18 into coincidence. This, however, is not possible in a finite three-dimensional figure without letting the surface cut itself.†

The twisted strip as an example of a one-sided surface is due to Möbius.‡ It has only *one* boundary  $A\beta C\beta A$ . An imaginary man  $OP$  on the surface (fig. 19) could walk, without crossing the boundary, along a path which is the

\* It would not be difficult to give a rigorous treatment of the propositions in this section, but it is thought better to postpone this to a later chapter.

† Plaster models showing this surface are manufactured by Martin Schilling of Leipzig.

‡ Gesammelte Werke, Vol. II, p. 519.

image of a straight line in the projective plane, till he arrived at the antipodal position  $OQ$ . If a small triangle  $RST$  were to be moved with the man without being lifted from the surface or being allowed to touch the man, it would be found, when the man arrived at the position  $OQ$ , that the triangle could be superposed upon itself,  $R$  coinciding with itself, but  $S$  and  $T$  interchanged. In other words, the boundary of the triangular region containing  $O$  would coincide with itself with sense reversed.

It is not essential that the triangular region  $RST$  be small, but merely that the figure  $ORST$  move continuously so that the triangle  $RST$  remains a triangle and the point  $O$  is never on one of its sides. The possibility of making this transformation of the figure  $ORST$  into  $ORTS$  is not affected by joining the two  $\beta$ -edges together, because none of the paths need meet the boundary of the strip. Therefore a corresponding continuous deformation can be made in the projective plane.

If we think of the figure  $ORST$  in the projective plane, the four points enter symmetrically. Thus, since  $S$  and  $T$  can be interchanged by continuously moving the complete quadrangle, any two vertices can be interchanged by such a motion, and hence any permutation of the four vertices can be effected by such a motion. This is intimately associated with the fact that all projectivities in the plane are direct (§ 32), as will be proved in a later chapter, where the notion of continuous deformation of a complete quadrangle in a projective plane is given a precise formulation.

The triangle  $RST$  may be replaced by any small circuit containing  $O$ , and it still remains true that  $O$  and the circuit may be continuously deformed till  $O$  coincides with itself and the circuit coincides with itself reversed. For example, the circuit may be taken as a conic section, and the projective plane imaged as the plane of elementary geometry plus "a line at infinity" (see the introduction to Vol. I, §§ 3, 4, 5, and also § 28 above). The ellipse I (fig. 20) may be deformed into the parabola II, this into the hyperbola III, this into the parabola IV, and this into the ellipse V. The reader can easily verify that the sense indicated by the arrow on I goes continuously to that indicated on V. The figures may be regarded as the projections from a variable center of an ellipse in a plane at right angles to the plane of the paper.

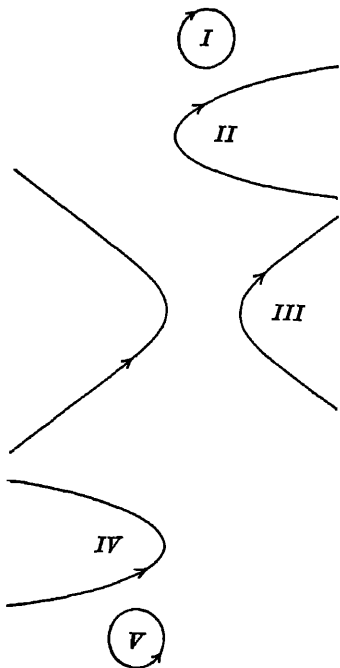


FIG. 20

This deformation of an ellipse and also the corresponding one of the quadrangle  $ORST$  depend on internal properties of the surface; i.e. they are independent of the situation of the surface in a three-dimensional space. They are sharply to be distinguished from the property expressed by saying that the man  $OP$  comes back to the position  $OQ$ , for the latter is a property of the space in which the surface lies.\* In fact, the closely related proposition, that if the man  $OP$  walk along a straight line in a projective plane till he comes back to the position  $OQ$ , the triangle  $RST$  comes back to  $R'TS'$ , implies that if a tetrahedron (e.g.  $PQRS$ ) be deformed into coincidence with itself so that two vertices are interchanged, the other two vertices will also be interchanged. And the last statement is a manifestation of the theorem (§ 32) that although the projective plane is one-sided, the projective three-space is two-sided.

A sort of model of the projective three-space may be obtained by generalizing the discussion of the plane given above. Any one of the eight regions determined by a tetrahedron is projectively equivalent to any other. Hence we pass from fig. 17 to fig. 21, which represents in full only the relations among the segments, triangular regions, and tetrahedral regions having  $A_1$  as an end, or vertex. Each of the triangles having  $A_2, A_3, A_4$  as vertices is represented by two triangles in fig. 21. Thus, in

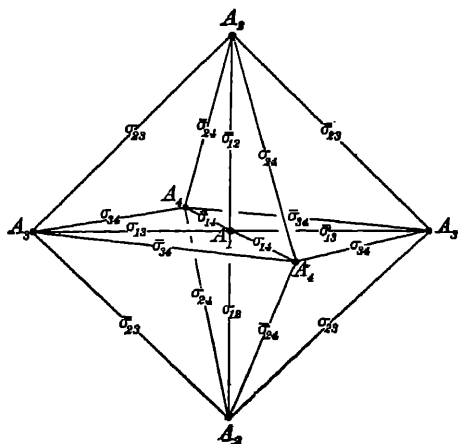


FIG. 21

order to represent the projective space completely we should have to bring each of the triangular regions  $A_2A_3A_4$  into coincidence with the one which is symmetrical with it with respect to  $A_1$ . In other words, fig. 21 would represent a projective three-space completely if each point on the octahedral surface formed by the triangular regions  $A_2A_3A_4$  were brought into coincidence with the opposite point.

### EXERCISE

Show that the octahedron in fig. 21 may be distorted into a cube so that the projective three-space is represented by a cube in which each point coincides with its symmetric point with respect to the center of the cube.

\* E. Steinitz, Sitzungsberichte der Berliner Mathematischen Gesellschaft, Vol. VII (1908), p. 35.

## CHAPTER III

### THE AFFINE GROUP IN THE PLANE

#### 34. The geometry corresponding to a given group of transformations.

The theorems which we have hitherto considered, whether in general projective geometry or in the particular geometry of reals, state properties of figures which are unchanged when the figures are subjected to collineations. For example, we have had no theorems about individual triangles, because any two triangles are equivalent under the general projective group, and thus are not to be distinguished from one another. On the other hand, there does not, in general, exist a collineation carrying a given pair of coplanar triangles into another given pair of coplanar triangles; and thus we have the theorem of Desargues, and other theorems, stating projective properties of pairs of triangles. We have thus considered only very general properties of figures, and so have dealt hardly at all with the familiar relations, such as perpendicularity, parallelism, congruence of angles and segments, which make up the bulk of elementary Euclidean geometry. These properties are not invariant under the general projective group, but only under certain subgroups. We shall therefore approach their study by a consideration of the properties of these subgroups.

There are, in general, at least two groups of transformations to consider in connection with a given geometrical relation: (1) a group by means of which the relation may be defined, and (2) a group under which the relation is left invariant. These two groups may or may not be the same.\*

We have already had one example of a definition of a geometrical relation by means of a group of transformations. In § 19 two collinear triads of points are defined as being in the same sense-class if they are conjugate under the group of direct projectivities on the line. The relation between pairs of triads which is thus defined is invariant under the group of all projectivities (§ 18).

\*The group (1) will always be a self-conjugate subgroup of (2), as follows directly from the definition of a self-conjugate subgroup. See § 39, below, where the rôle of self-conjugate subgroups is explained and illustrated.

The system of definitions and theorems which express properties invariant under a given group of transformations may be called, in agreement with the point of view expounded in Klein's Erlangen Programm,\* a *geometry*. Obviously, all the theorems of the geometry corresponding to a given group continue to be theorems in the geometry corresponding to any subgroup of the given group; and the more restricted the group, the more figures will be distinct relatively to it, and the more theorems will appear in the geometry. The extreme case is the group corresponding to the identity, the geometry of which is too large to be of consequence.

For our purposes we restrict attention to groups of projective collineations,† and in order to get a more exact classification of theorems we narrow the Kleinian definition by assigning to the geometry corresponding to a given group only the theory of those properties which, while invariant under this group, are *not invariant under any other group of projective collineations containing it*. This will render the question definite as to whether a given theorem belongs to a given geometry.

Perhaps the simplest example of a subgroup of the projective group in a plane is the set of all projective collineations which leave a line of the plane invariant. The present chapter is concerned chiefly with the geometry belonging to this group.

The chapter is based entirely on Assumptions A, E, P,  $H_0$ . In fact, the theorems of §§ 36, 38, 39, 40, 42, 45, 46, 48 depend only on A, E,  $H_0$ . The class of theorems which depend on assumptions with regard to order relations has already been touched on in §§ 28–30.

**35. Euclidean plane and the affine group.** Let  $l_\infty$  be an arbitrary but fixed line of a projective plane  $\pi$ . In accordance with the definition in § 28 we shall refer to  $l_\infty$  as the *line at infinity*. The points of  $l_\infty$  shall be called *ideal‡ points* or *points at infinity*, whereas the remaining points and lines of  $\pi$  shall be called *ordinary* points and lines. The set of all ordinary points is a *Euclidean plane*. In the rest of this chapter the term “point,” when unmodified, will refer to an ordinary point.

\* Cf. F. Klein, Vergleichende Betrachtungen über neuere geometrische Forschungen, Erlangen 1872; also in Mathematische Annalen, Vol. XLIII (1893), p. 63.

† From some points of view it would have been desirable to include also all projective groups containing correlations.

‡ There is some divergence in the literature with respect to the use of this word and the word “improper.” On the latter term see § 85, Vol. I.

DEFINITION. Any projective collineation transforming a Euclidean plane into itself is said to be *affine*; the group of all such collineations is called the *affine group*, and the corresponding geometry the *affine geometry*.

THEOREM 1. *There is one and only one affine collineation transforming three vertices  $A, B, C$  of a triangle to three vertices  $A', B', C'$  respectively of a triangle.*

*Proof.* Since  $l_\infty$  is transformed into itself, this is a corollary of Theorem 18, § 35, Vol. I.

With respect to any system of nonhomogeneous coördinates of which  $l_\infty$  is the singular line, any affine collineation may be written in the form (§ 67, Vol. I)

$$(1) \quad \begin{aligned} x' &= a_1x + b_1y + c_1, \\ y' &= a_2x + b_2y + c_2, \end{aligned}$$

where

$$\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0.$$

**36. Parallel lines.** DEFINITION. Two ordinary lines not meeting in an ordinary point are said to be *parallel* to each other, and the pair of lines is said to be *parallel*. A line is also said to be *parallel* to itself.

Hence, in a Euclidean plane we have the following theorem as a consequence of the theorems in Chap. I, Vol. I:

THEOREM 2. *In a Euclidean plane, two points determine one and only one line; two lines meet in a point or are parallel; two lines parallel to a third line are parallel to each other; through a given point there is one and only one line parallel to a given line  $l$ .*

DEFINITION. A simple quadrangle  $ABCD$  such that the side  $AB$  is parallel to  $CD$  and  $BC$  to  $DA$  is called a *parallelogram*.

DEFINITION. The lines  $AC$  and  $BD$  are called the *diagonals* of the simple quadrangle  $ABCD$ .

In terms of parallelism, most projective theorems lead to a considerable number of special cases. Moreover, since the affine geometry is not self-dual, theorems which are dual in projective geometry may have essentially different affine special cases. A few affine theorems which are obtainable by direct specialization are given in the following list of exercises, and a larger number in the next section.

## EXERCISES

1. If the sides of two triangles are parallel by pairs, the lines joining corresponding vertices meet in a point or are parallel.

2. If in two projective flat pencils three pairs of corresponding lines are parallel, then each line is parallel to its homologous line.

3. With respect to any system of nonhomogeneous coördinates in which  $l_\infty$  is the singular line, the equation of a line parallel to  $ax + by + c = 0$  is  $ax + by + c' = 0$ .

4. A homology (or an elation) whose center and axis are ordinary transforms  $l_\infty$  into a line parallel to the axis.

5. If the number of points on a projective line is  $p + 1$ , the number of points in a Euclidean plane is  $p^2$ , the number of triangles in a Euclidean plane is  $p^2(p - 1)^2(p + 1)/6$ , and the latter is also the number of projective collineations transforming a Euclidean plane into itself.

37. **Ellipse, hyperbola, parabola.** DEFINITION. A conic meeting  $l_\infty$  in two distinct points is called a *hyperbola*, one meeting it in only one point a *parabola*, and one meeting it in no point an *ellipse*. The

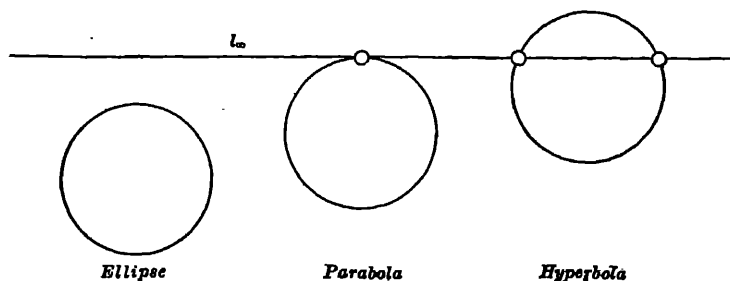


FIG. 22

pole of  $l_\infty$  is called the *center* of the conic. Any line through the center is called a *diameter*. The tangents to a hyperbola at its points of intersection with  $l_\infty$  are called its *asymptotes*. A conic having an ordinary point as center is called a *central conic*.

## EXERCISES

1. An ellipse or a hyperbola is a central conic, but a parabola is not.
2. The center of a parabola is its point of contact with  $l_\infty$ .
3. No two tangents to a parabola are parallel.
4. The asymptotes of a hyperbola meet at its center.
5. Two conjugate diameters (cf. § 44, Vol. I) of a hyperbola are harmonically conjugate with respect to the asymptotes.
6. If a simple hexagon be inscribed in a conic in such a way that two of its pairs of opposite sides are parallel, the third pair of opposite sides is parallel.

7. If a parallelogram be inscribed in a conic, the tangents at a pair of opposite vertices are parallel.

8. If the vertices of a triangle are on a conic and two of the tangents at the vertices are parallel to the respectively opposite sides, the third tangent is parallel to the third side.

9. If a parallelogram be circumscribed to a conic, its diagonals meet in the center and are conjugate diameters.

10. If a parallelogram be inscribed in a conic, any pair of adjacent sides are parallel to conjugate diameters. Its diagonals meet at the center of the conic.

11. Let  $P$  and  $P'$  be two points which are conjugate with respect to a conic, let  $p$  be the diameter parallel to  $PP'$ , and let  $Q$  and  $Q'$  be points of intersection with the conic of the diameter conjugate to  $p$ . The lines  $PQ$  and  $P'Q'$  meet on the conic.

12. If a parallelogram  $OAPB$  is such that the sides  $OA$  and  $OB$  are conjugate diameters of a hyperbola and the diagonal  $OP$  is an asymptote, then the other diagonal  $AB$  is parallel to the other asymptote.

13. If two lines  $OA$  and  $OB$  are conjugate diameters of a conic which they meet in  $A$  and  $B$ , then any two parallel lines through  $A$  and  $B$  respectively meet the conic in two points  $A'$  and  $B'$  such that  $OA'$  and  $OB'$  are conjugate diameters.

14. Any two parabolas are conjugate under a collineation transforming  $l_\infty$  into itself.\*

15. Any two hyperbolas are conjugate under a collineation transforming  $l_\infty$  into itself.\*

16. Derive the equation of a parabola referred to a nonhomogeneous coordinate system with a tangent and a diameter as axes.

17. Derive the equation of a hyperbola referred to a nonhomogeneous coordinate system with the asymptotes as axes.

18. Derive the equation of an ellipse or a hyperbola referred to a nonhomogeneous coordinate system with a pair of conjugate diameters as axes.

38. The group of translations. DEFINITION. Any elation having  $l_\infty$  as an axis is called a *translation*. If  $l$  is any ordinary line through the center of a translation, the translation is said to be *parallel* to  $l$ .

COROLLARY. A translation carries every proper line into a parallel line and leaves invariant every line of a certain system of parallel lines.

THEOREM 3. There is one and only one translation carrying a point  $A$  to a point  $B$ .

*Proof.* Any translation carrying  $A$  to  $B$  must be an elation with  $l_\infty$  as axis and the point of intersection of the line  $AB$  with  $l_\infty$  as center. Hence the theorem follows from Theorem 9, Chap. III, Vol. I.

\* On the corresponding theorem for ellipses, see § 76, Ex. 7.

**THEOREM 4.** *An ordered point pair  $AB$  can be carried by a translation to an ordered point pair  $A'B'$  such that  $A'$  is not on the line  $AB$ , if and only if  $ABB'A'$  is a parallelogram.*

*Proof.* Let  $L_\infty$  and  $M_\infty$  be the points at infinity on the lines  $AA'$  and  $AB$  respectively. The translation carrying  $A$  to  $A'$  must carry the line  $AM_\infty$  to  $A'M_\infty$  and leave the line  $BL_\infty$  invariant. Hence the point  $B$ , which is the intersection of  $AM_\infty$  with  $BL_\infty$ , is carried to  $B'$ , which is the intersection of  $A'M_\infty$  with  $BL_\infty$ . Hence the points  $A'$  and  $B'$  to which  $A$  and  $B$  respectively are carried by a translation are such

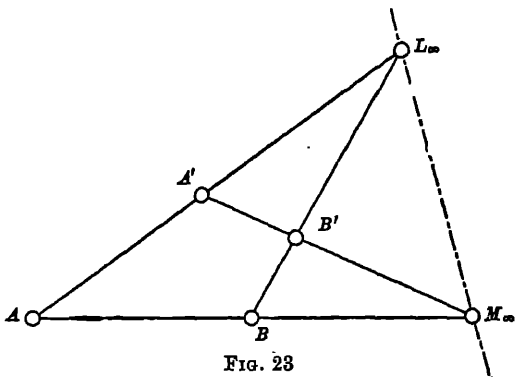


FIG. 23

that  $ABB'A'$  is a parallelogram. Since there is one and only one translation carrying  $A$  to  $A'$ , the same reasoning shows that whenever  $ABB'A'$  is a parallelogram there exists a translation carrying  $A$  and  $B$  to  $A'$  and  $B'$  respectively.

**THEOREM 5.** *An ordered point pair  $AB$  is carried by a translation to an ordered point pair  $A'B'$ , where  $A'$  is on the line  $AB$ , if and only if  $Q(L_\infty AA', L_\infty B'B)$ ,  $L_\infty$  being the point at infinity of  $AB$ .*

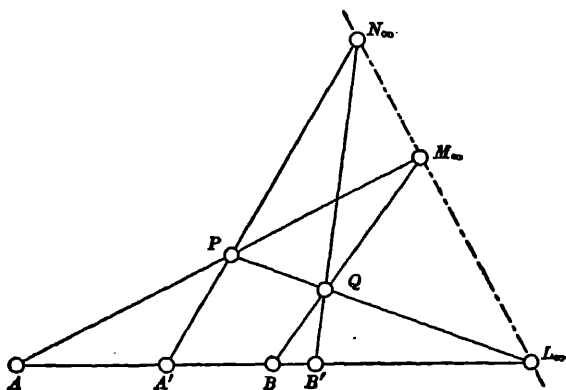


FIG. 24

*Proof.* Let  $P$  be any point not on the line  $AB$ , and let  $M_\infty$  and  $N_\infty$  respectively be the points of intersection of  $PA$  and  $PA'$  with  $l_\infty$ . Let  $Q$  be the point of intersection of  $BM_\infty$  with  $PL_\infty$ . Then, by the last theorem, the translation carrying

$A$  to  $B$  carries  $P$  to  $Q$ , and hence carries  $A'$  to the point of intersection of  $QN_x$  with  $AB$ . Hence  $N_x$ ,  $Q$ , and  $B'$  are collinear, and hence we have  $Q(L_xAA', L_xB'B)$ .

**THEOREM 6.** *If  $A, B, C$  are any three points, the resultant of the translations carrying  $A$  to  $B$  and  $B$  to  $C$  is the translation carrying  $A$  to  $C$ .*

*Proof.* Let  $A_x, B_x, C_x$  be the points of intersection of the lines  $BC, CA, AB$  respectively with  $l_x$ . Suppose first that the three points  $A_x, B_x, C_x$  are all distinct. The translation carrying  $A$  to  $B$  changes the line  $AB_x$  into the line  $BB_x$ , and the translation carrying  $B$  to  $C$  changes the line  $BB_x$  into  $CB_x$ . Hence the line  $AB_x$  is invariant under the resultant of these two translations.

Consider now any other line through  $B_x$ , and let it meet  $AA_x$  in  $A'$  and  $BC$  in  $C'$ ; also let  $B'$  be the point of intersection of  $A'C_x$  with  $BC$  (fig. 25). We then have that the translation carrying  $A$  to  $B$  carries  $A'$  to  $B'$  (Theorem 4), and on account of  $Q(A_xBB', A_xC'C)$  (Theorem 5) the translation carrying  $B$  to  $C$  carries  $B'$

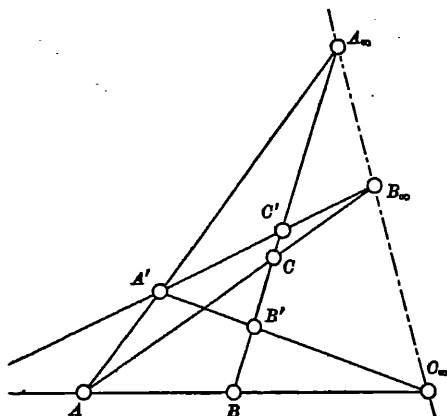


FIG. 25

to  $C'$ . Hence the resultant of the two translations carries  $A'$  to  $C'$  and thus leaves the line  $A'B_x$  invariant; that is, it leaves all the lines through  $B_x$  invariant. Since it obviously leaves all points on  $l_x$  invariant, it is a translation (Cor. 3, Theorem 9, Chap. III, Vol. I).

If two of the three points  $A_x, B_x, C_x$  coincide, they all coincide, and in this case the theorem is obvious.

By definition, the identity is a translation. Hence we have

**COROLLARY.** *The set of all translations form a group.*

**THEOREM 7.** *The group of translations is commutative.*

*Proof.* Given two translations  $T_1$  and  $T_2$  and let  $A$  be any point,  $T_1(A) = A'$  and  $T_2(A') = B'$ . If  $B' = A$ ,  $T_2$  is the inverse of  $T_1$ , and hence  $T_1$  and  $T_2$  are obviously commutative. If  $B' \neq A$  and  $B'$  is not

on the line  $AA'$ , let  $B$  (fig. 23) be the point of intersection of the line through  $A$  parallel to  $A'B'$  with the line through  $B'$  parallel to  $AA'$ , then  $ABB'A'$  is a parallelogram, and it is obvious that  $T_1(B) = B'$  and  $T_2(A) = B$ . Hence  $T_1T_2(A) = B'$ . But, by the definition of  $A'$  and  $B'$ ,  $T_2T_1(A) = B'$ . Hence, in this case also,  $T_1$  and  $T_2$  are commutative.

In case  $B'$  is on the line  $AA'$ , let  $P$  and  $Q$  (fig. 24) be two points such that  $A'B'QP$  is a parallelogram, let  $B$  be the point of intersection of  $AA'$  with the line through  $Q$  parallel to  $AP$ , and let  $L_\infty, M_\infty, N_\infty$  be the points at infinity of  $PQ, PA$ , and  $PA'$  respectively. Then, since  $T_2(A') = B'$ , it is obvious that  $T_2(P) = Q$ , and hence that  $T_2(A) = B$ . Moreover, on account of  $Q(L_\infty AB, L_\infty B'A')$ ,  $T_1(A) = A'$  implies that  $T_1(B) = B'$ . Hence  $T_1T_2(A) = B'$ , and thus, in this case also,  $T_1$  and  $T_2$  are commutative.

**THEOREM 8.** *If  $OX$  and  $OY$  are two nonparallel lines and  $T$  is any translation, there is a unique pair of translations  $T_1, T_2$  such that  $T_1$  is parallel to  $OX$ ,  $T_2$  parallel to  $OY$ , and  $T_1T_2 = T$ .*

*Proof.* In case  $T$  is parallel to  $OX$  or  $OY$  the theorem is trivial. If  $T$  is parallel to neither of them, let  $P = T(O)$  and let  $X_1$  and  $Y_1$  be the points in which the lines through  $P$  parallel to  $OY$  and  $OX$  respectively meet  $OX$  and  $OY$  respectively. Then  $OX_1PY_1$  is a parallelogram, and if  $T_1$  be the translation carrying  $O$  to  $X_1$ , and  $T_2$  the translation carrying  $O$  to  $Y_1$ , it follows, by Theorems 4 and 6, that  $T_1T_2 = T$ .

On the other hand, if  $T'_1$  is any translation parallel to  $OX$ , and  $T'_2$  any translation parallel to  $OY$ , and  $T'_1(O) = X'_1$  and  $T'_2(O) = Y'_1$ , the product  $T'_1T'_2$  carries  $O$  to a point  $P'$  such that  $OX'_1P'Y'_1$  is a parallelogram. But  $P' = P$  if and only if  $X'_1 = X_1$  and  $Y'_1 = Y_1$ . Hence  $T$  determines  $T_1$  and  $T_2$  uniquely.

**THEOREM 9.** *With respect to a nonhomogeneous coordinate system in which  $l_\infty$  is the singular line a translation parallel to the  $x$ -axis has the equations*

$$(2) \quad \begin{aligned} x' &= x + a, \\ y' &= y. \end{aligned}$$

*Proof.* The point into which  $(0, 0)$  is transformed by a given translation parallel to the  $x$ -axis may be denoted by  $(a, 0)$ . By Theorem 5 and § 48, Vol. I, it then follows that any point  $(x, 0)$  of the  $x$ -axis

is transformed into  $(x + a, 0)$ . Since lines parallel to the  $y$ -axis are transformed into lines parallel to the  $y$ -axis, and since lines parallel to the  $x$ -axis are invariant, it follows that the given translation takes the given form (2).

Conversely, any transformation of the type (2) leaves all lines parallel to the  $x$ -axis invariant and transforms any other line into a line parallel to itself. Hence it is a translation parallel to the  $x$ -axis.

**THEOREM 10.** *With respect to a nonhomogeneous coordinate system in which  $l_\infty$  is the singular line, any translation can be expressed in the form*

$$(3) \quad \begin{aligned} x' &= x + a, \\ y' &= y + b. \end{aligned}$$

*Proof.* By Theorem 8 any translation is the product of a translation parallel to the  $x$ -axis by one parallel to the  $y$ -axis. Hence it is the product of a transformation of the form

$$\begin{aligned} x' &= x + a, \\ y' &= y, \end{aligned}$$

by a transformation of the form

$$\begin{aligned} x' &= x, \\ y' &= y + b. \end{aligned}$$

### EXERCISE

Investigate the subgroups of the group of translations.

**39. Self-conjugate subgroups. Congruence.** **DEFINITION.** Any subgroup  $G'$  of a group  $G$  is said to be *self-conjugate* or *invariant*\* under  $G$  if and only if  $\Sigma T \Sigma^{-1}$  is an operation of  $G'$  whenever  $\Sigma$  is an operation of  $G$  and  $T$  of  $G'$ .

The geometric significance of this notion is as follows: Suppose that two figures  $F_1$  and  $F_2$  are conjugate under  $G'$ , and  $T$  is a transformation of  $G'$  such that  $F_2 = T(F_1)$ . If  $F_1$  and  $F_2$  are changed into  $F'_1$  and  $F'_2$  by any transformation  $\Sigma$  of  $G$ , then  $\Sigma^{-1}(F'_1) = F_1$ . Hence†  $T \Sigma^{-1}(F'_1) = F_2$ ,

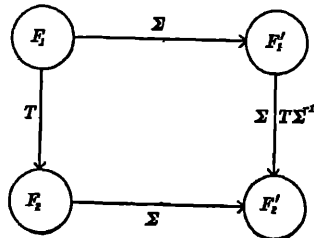


FIG. 26

\* These terms have already been defined in § 75, Vol. I.

† These relations may be illustrated by the accompanying diagram (probably due to S. Lie).

and  $\Sigma T \Sigma^{-1}(F'_1) = F'_2$ . Therefore, if  $G'$  is self-conjugate under  $G$ , the figures  $F'_1$  and  $F'_2$  are conjugate under  $G'$ . Hence *the property of being conjugate under the self-conjugate subgroup  $G'$  is a property left invariant by the group  $G$* . Thus the theory of figures conjugate under  $G'$  belongs to the geometry corresponding to  $G$ , provided that  $G$  is not a self-conjugate subgroup of any other group of projective collineations.

**THEOREM 11.** *The group of translations is self-conjugate under the affine group.*

*Proof.* Let  $T$  be an arbitrary translation and  $\Sigma$  an arbitrary affine transformation. We have to show that  $\Sigma T \Sigma^{-1}$  is a translation. If  $P$  be any point of  $l_\infty$ ,  $\Sigma(P)$  is also on  $l_\infty$ . Therefore, since  $T$  leaves all points of  $l_\infty$  invariant, so does  $\Sigma T \Sigma^{-1}$ . The system of lines through the center of  $T$  is a system of parallel lines;  $\Sigma$  transforms this system of parallel lines into a system of parallel lines; and hence the latter system of parallel lines is invariant under  $\Sigma T \Sigma^{-1}$ . Hence (cf. Cor. 3, Theorem 9, Chap. III, Vol. I)  $\Sigma T \Sigma^{-1}$  is a translation.

**COROLLARY 1.** *The group of translations is self-conjugate under any subgroup of the affine group which contains it.*

**COROLLARY 2.** *For any affine collineation  $\Sigma$ , and any translation  $T$ , there exists a translation  $T'$  such that  $\Sigma T = T' \Sigma$  and a translation  $T''$  such that  $T \Sigma = \Sigma T''$ .*

*Proof.* Let  $\Sigma T \Sigma^{-1} = T'$  and  $\Sigma^{-1} T \Sigma = T''$ . By the theorem,  $T'$  and  $T''$  are translations. But

$$\Sigma T \Sigma^{-1} = T' \quad \text{and} \quad \Sigma^{-1} T \Sigma = T''$$

imply  $\Sigma T = T' \Sigma$  and  $T \Sigma = \Sigma T''$  respectively.

**DEFINITION.** Two figures are said to be *congruent* if they are conjugate under the group of translations.

This definition will presently be extended by giving other conditions under which two figures are said to be congruent.\* In view of Theorem 11, the theory of congruence as thus far defined belongs to the affine geometry.

\* A complete definition would be of the form, "Two figures are said to be congruent if and only if . . ."

**40. Congruence of parallel point pairs.** The figure consisting of two distinct points  $A, B$  may be looked at in two ways with respect to congruence. We consider either the two ordered\* point pairs  $AB$  and  $BA$  or the point pair  $AB$  without regard to order. In the second case  $AB$  and  $BA$  mean the same thing and  $AB$  is congruent to  $BA$  because the identity belongs to the group of translations. On the other hand, the ordered pair  $AB$  is not conjugate to the ordered pair  $BA$  under the group of translations, because the translation carrying  $A$  to  $B$  does not carry  $B$  to  $A$  (this is under Assumptions  $A, E, H_0$ ).

**THEOREM 12.** *If  $ABDC$  is a parallelogram, the ordered point pair  $AB$  is congruent to the ordered point pair  $CD$ . If the condition  $Q(P_\infty AC, P_\infty DB)$  is satisfied where  $P_\infty$  is an ideal point, the ordered point pair  $AB$  is congruent to the ordered point pair  $CD$ .*

*Proof.* This is a corollary of Theorems 4 and 5.

**COROLLARY 1.** *Let  $A$  and  $B$  be any two distinct points and  $O$  the harmonic conjugate of the point at infinity of the line  $AB$  with respect to  $A$  and  $B$ . Then the pair  $AO$  is congruent to the pair  $OB$ .*

**DEFINITION.** The point  $O$  in the last corollary is called the *mid-point* of the pair  $AB$ . In case  $B = A$ ,  $A$  is called the *mid-point* of the pair  $AB$ .

**COROLLARY 2.** *The line joining the mid-points of the pairs of vertices  $AB$  and  $AC$  of a triangle  $ABC$  is parallel to the line  $BC$ .*

*Proof.* Let  $B_\infty$  and  $C_\infty$  be the points at infinity of the lines  $AB$  and  $AC$  respectively, and let  $B_1$  and  $C_1$  be the mid-points of the pairs  $AB$  and  $AC$  respectively. Then, by the definition of "mid-point,"

$$AB_1BB_\infty \overline{\wedge} AC_1CC_\infty.$$

Hence the lines  $B_1C_1$ ,  $BC$ , and  $B_\infty C_\infty$  concur, which means that  $B_1C_1$  and  $BC$  are parallel.

**DEFINITION.** The line joining a vertex, say  $A$ , of a triangle  $ABC$  to the mid-point of  $BC$  is called a *median* of the triangle.

**THEOREM 13.** *The three medians of a triangle meet in a point.*

\* Cf. footnote on page 40.

*Proof.* Let the triangle be  $ABC$ ; let  $A_\infty, B_\infty, C_\infty$  be the points at infinity of the sides  $BC, CA, AB$  respectively; and let  $A_1, B_1, C_1$  be the points of intersection of the pairs of lines  $BB_\infty$  and  $CC_\infty, CC_\infty$  and  $AA_\infty$ ,

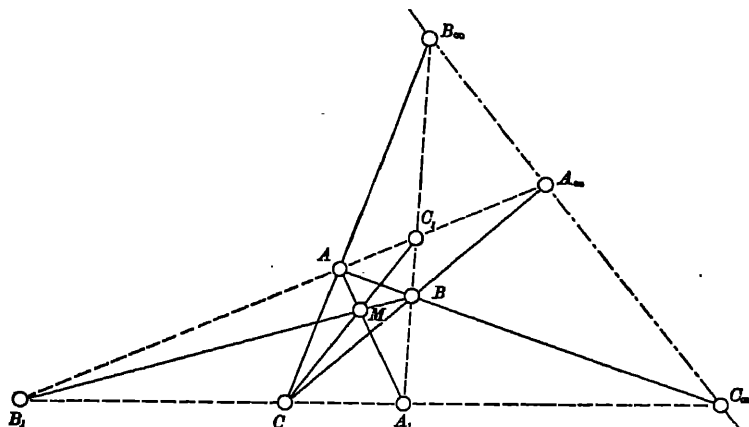


FIG. 27

$AA_\infty$  and  $BB_\infty$  respectively (fig. 27). Then, by well-known theorems on harmonic sets (§ 31, Vol. I), the medians of the triangle  $ABC$  are  $AA_1$ ,  $BB_1$ , and  $CC_1$ , and these three lines concur.

### EXERCISES

1. The diagonals of a parallelogram bisect one another; that is, if  $ABCD$  is a parallelogram, the mid-points of the pairs  $AC$  and  $BD$  coincide.

2. Let  $a$  and  $b$  be two parallel lines. The mid-points of all the pairs  $AB$  where  $A$  is on  $a$  and  $B$  on  $b$  are on a line parallel to  $a$  and  $b$ .

3. If the sides  $AB, BC, CA$  of a triangle  $ABC$  are respectively parallel to the sides  $A'B', B'C', C'A'$  of a triangle  $A'B'C'$ , and the ordered point pair  $AB$  is congruent to the ordered point pair  $A'B'$ , then the two triangles are congruent.

4. The mid-points of the pairs of opposite vertices of a complete quadrilateral are collinear. Let us call this line the *diameter* of the quadrilateral.

5. A line through a diagonal point  $O$  of a complete quadrangle, parallel to the opposite side of the diagonal triangle, is met by either pair of opposite sides of the quadrangle which do not pass through  $O$  in a pair of points having  $O$  as mid-point.

41. **Metric properties of conics.** The following list of exercises contains a number of theorems on conics which involve the congruence of parallel point pairs and can be derived by aid of the theorems in the last sections.

## EXERCISES

1. The mid-points of a system of pairs of points of a conic  $AA'$ ,  $BB'$ ,  $CC'$ , etc. are collinear if the lines  $AA'$ ,  $BB'$ ,  $CC'$  are parallel. The line containing the mid-points is a diameter conjugate to the diameter parallel to  $AA'$ .

2. Let  $A$  and  $B$  be two points of a parabola. If the line joining the mid-point  $C$  of the pair  $AB$  to the pole  $P$  of the line  $AB$  meets the conic in  $O$ , then  $O$  is the mid-point of the pair  $CP$ .

3. If a line meets a hyperbola in a pair of points  $H_1H_2$ , and its asymptotes in a pair  $A_1A_2$ , the two pairs have the same mid-point. The pair  $H_1A_1$  is congruent to the pair  $H_2A_2$ .

4. The point of contact of a tangent to a hyperbola is the mid-point of the pair in which the tangent meets the asymptotes.

5. Let  $A_1$  and  $A_2$  be each a fixed and  $X$  a variable point of a hyperbola, and let  $X_1$  and  $X_2$  be the points in which the lines  $XA_1$  and  $XA_2$  meet one of the asymptotes. The point pairs  $X_1X_2$  determined by different values of  $X$  are all congruent.

6. The centers of all conics inscribed in\* a simple quadrilateral  $ABCD$  are on the line joining the mid-points of the point pairs  $CA$  and  $BD$ .

7. The centers of all conics which pass through the vertices of a complete quadrangle  $ABCD$  are on a conic  $C^2$ , which contains the six mid-points of the pairs of vertices of the quadrangle, the three vertices of its diagonal triangle, and the double points (if existent) of the involution in which  $l_\infty$  is met by the pencil of conics through  $A, B, C, D$ . From the projective point of view, according to which  $l_\infty$  is any line whatever,  $C^2$  is called the *nine-point* (or the *eleven-point*) conic of the complete quadrangle  $ABCD$  and the line  $l_\infty$ . Derive the analogous theorems for the pencils of conics of Types II-V (cf. § 47, Vol. I).

8. The five diameters† of the complete quadrilaterals formed by leaving out one line at a time from a five-line meet in a point  $A$ , which is the center of the conic tangent to the five lines.

9. The six points  $A$  determined, according to the last exercise, by the six complete five-lines formed by leaving out one line at a time from a six-line are on a conic  $C^2$ .

10. The seven conics  $C^2$  determined, according to the last exercise, by the seven complete six-lines formed by leaving out one line at a time from a seven-line, all pass through three points.

42. **Vectors.** Any ordered pair of points determines a set of pairs all of which are equivalent to it under the group of translations. In order to study the relations between such sets of pairs we introduce the notion of a vector. The term "vector" appears in the literature

\* A conic is said to be inscribed in a given figure if the figure is circumscribed to the conic (cf. § 43, Vol. I).

† Cf. Ex. 4, § 40. This and the following exercises are taken from an article by W. W. Taylor, *Messenger of Mathematics*, Vol. XXXVI (1907), p. 118.

under a multitude of guises, none of which, however, is in serious contradiction with the following abstract definition. In this definition the term "ordered pair of points" is to be understood to include the case of a single point counted twice.

DEFINITION. A *planar field of vectors* (or *vector field*) is any set of objects, the individuals of which are called *vectors*, such that (1) there is one vector for each ordered pair of points in a Euclidean plane, and (2) there is only one vector for any two ordered pairs  $AB$  and  $A'B'$  which are equivalent under the group of translations. A vector corresponding to a coincident pair of points is called a *null vector* or a *zero vector*, and denoted by the symbol  $0$ .

For example, a properly chosen set of matrices would be a vector field according to this definition. So would also the set of all translations including the identity; also a set of classes of ordered point pairs such that two point pairs are in the same class if and only if equivalent under the group of translations. However a vector field be defined, it will be found that, in most applications, only those properties which follow from the definition as stated above are actually used.

A precisely similar state of affairs exists in the definition of a number system. The objects in the particular number system determined for a given space by the methods of Chap. VI, Vol. I, are points, but a number system in general is any set of objects in a proper one-to-one correspondence with this set of points.

In the following discussion we shall suppose that one field of vectors has been selected, and all statements will refer to this one field. Thus, the vector corresponding to the point pair  $AB$  is a definite object, and we shall denote it as "the vector  $AB$ ," or, in symbols,  $\text{Vect}(AB)$ .

Since any point of a Euclidean plane can be carried by a translation to any other point, the set of all vectors is the same as the set of vectors  $OA$ , where

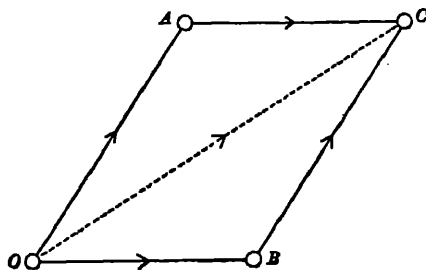


FIG. 28

$O$  is a fixed and  $A$  a variable point. Consequently, the following definition gives a meaning to the operation of "adding" any two vectors.

DEFINITION. If  $O, A, C$  are points of a Euclidean plane, the vector  $OC$  is called the *sum* of the vectors  $OA$  and  $AC$ . In symbols this is

indicated by  $\text{Vect}(OC) = \text{Vect}(OA) + \text{Vect}(AC)$ . The operation of obtaining the sum of two vectors is called *addition* of vectors.

An obvious corollary of this definition is that

$$\text{Vect}(AB) + \text{Vect}(BA) = 0.$$

Hence we define:

DEFINITION. The vector  $\text{Vect}(BA)$  is called the *negative* of the vector  $\text{Vect}(AB)$ , and denoted by  $-\text{Vect}(AB)$ .

THEOREM 14. The operation of addition of vectors is associative; that is, if  $a, b, c$  are vectors,  $(a + b) + c = a + (b + c)$ .

*Proof.* Let the three vectors be  $OA, AB, BC$  respectively; then, by definition, both  $(\text{Vect}(OA) + \text{Vect}(AB)) + \text{Vect}(BC)$  and  $\text{Vect}(OA) + (\text{Vect}(AB) + \text{Vect}(BC))$  are the same as  $\text{Vect}(OC)$ .

DEFINITION. Two vectors are said to be *collinear* if and only if they can be expressed as  $\text{Vect}(OA)$  and  $\text{Vect}(OB)$  respectively, where  $O, A, B$  are collinear points.

THEOREM 15. The sum of two noncollinear vectors  $OA$  and  $OB$  is the vector  $OC$ , where  $C$  is such that  $OACB$  is a parallelogram.

*Proof.* By Theorem 4, the vector  $OB$  is the same as the vector  $AC$ . Hence, by definition, the sum of  $OA$  and  $OB$  is  $OC$ .

THEOREM 16. The sum of two collinear vectors  $OA$  and  $OB$  is a vector  $OC$  such that  $Q(P_{\infty}AO, P_{\infty}BC)$ , where  $P_{\infty}$  is the point at infinity of the line  $AB$ .

*Proof.* Let  $L$  and  $M$  be two points such that  $OBML$  is a parallelogram. Hence  $\text{Vect}(OB) = \text{Vect}(LM)$ . Then, by definition,  $C$  must be such that  $\text{Vect}(LM) = \text{Vect}(AC)$ , that is, such that  $ACML$  is a parallelogram. Let  $L_{\infty}$  be the ideal

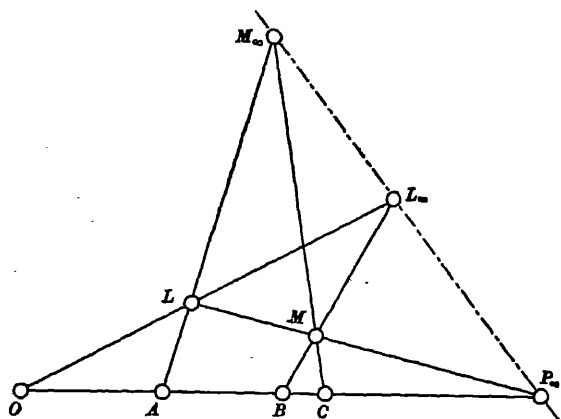


FIG. 29

point of intersection of the lines  $OL$  and  $BM$ , and let  $M_{\infty}$  be the ideal point of intersection of the lines  $AL$  and  $MC$ . The complete quadrangle  $LML_{\infty}M_{\infty}$  determines  $Q(P_{\infty}AO, P_{\infty}BC)$ .

COROLLARY. If  $O, A, B$  are three collinear points, and  $C$  a point such that  $\text{Vect}(OA) + \text{Vect}(OB) = \text{Vect}(OC)$ , then, with respect to any scale (cf. § 48, Vol. I) in which  $P_0$  is  $O$  and  $P_\infty$  the point at infinity of the line  $OA$ ,

$$A + B = C.$$

*Proof.* Cf. Cor. 1, Theorem 1, Chap. VI, Vol. I.

THEOREM 17. The operation of adding vectors is commutative; that is, if  $a$  and  $b$  are vectors,  $a + b = b + a$ .

*Proof.* Let the vectors  $a$  and  $b$  be  $\text{Vect}(OA)$  and  $\text{Vect}(OB)$  respectively. If  $O, A, B$  are noncollinear, the result follows from Theorem 15, and if they are collinear, from Theorem 16.

43. Ratios of collinear vectors. By analogy with the case of addition we should be led to base a definition of multiplication of collinear vectors upon the multiplication of points in § 49, Vol. I. There are, however, a great many ways of defining the product of two vectors, which would not reduce to this sort of multiplication in the case of collinear vectors. Hence, in order to avoid possible confusion we shall not introduce a definition of the multiplication of vectors at present, but only of what we shall call the ratio of two collinear vectors.

DEFINITION. The ratio of two collinear vectors  $OA$  and  $OB$  is the number which corresponds to  $A$  in the scale in which  $P_0$  is  $O$ ,  $P_1$  is  $B$ , and  $P_\infty$  is the point at infinity of the line  $OA$ . It is denoted by

$$\frac{\text{Vect}(OA)}{\text{Vect}(OB)} \text{ or by } \frac{OA}{OB}.$$

It is to be emphasized that the ratio of two collinear vectors as here defined is a number. By comparison with the definition in § 56, Vol. I, we have at once

THEOREM 18. If  $A, B, C, D_\infty$  are collinear points,  $D_\infty$  being ideal,

$$\mathbf{R}(D_\infty A, BC) = \frac{AC}{AB}.$$

Theorem 13, Chap. VI, Vol. I now gives

THEOREM 19. If  $A_1, A_2, A_3, A_4$  are any four collinear ordinary points,

$$\mathbf{R}(A_1 A_2, A_3 A_4) = \frac{A_1 A_2}{A_1 A_4} : \frac{A_2 A_3}{A_3 A_4}.$$

THEOREM 20. If two triangles  $ABC$  and  $A'B'C'$  are such that the sides  $AB, BC, CA$  are parallel to  $A'B', B'C', C'A'$  respectively,

$$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CA}{C'A'}.$$

*Proof.* Suppose that the translation which carries  $A'$  to  $A$  carries  $B'$  to  $B_1$  and  $C'$  to  $C_1$ . Then  $B_1$  is on the line  $AB$  and  $C_1$  on the line  $AC$ , and the line  $B_1C_1$  is parallel to  $BC$ . Thus, if  $B_\infty$  be the point at infinity of the line  $AB$ , and  $C_\infty$  the point at infinity of the line  $AC$ ,

$$B_\infty ABB_1 \overline{\wedge} C_\infty ACC_1.$$

Hence, by Theorem 18, 
$$\frac{AB}{AB_1} = \frac{AC}{AC_1} = \frac{CA}{C_1A},$$

which is, by definition, the same as

$$\frac{AB}{A'B'} = \frac{CA}{C'A'}.$$

In like manner, it follows that

$$\frac{AB}{A'B'} = \frac{BC}{B'C'}.$$

Since we have not defined the product of two vectors, it is necessary to resort to a device in order to compute conveniently with them. This we do as follows:

DEFINITION. With respect to an arbitrary vector  $OA$ , which is called a unit vector, the ratio

$$\frac{OB}{OA},$$

where  $OB$  is any vector collinear with  $OA$ , is called the *magnitude* of  $OB$ .

Observe that the magnitude of  $OB$  is the negative of the magnitude of  $BO$ . Since the magnitude of a vector is a number, there is no difficulty about algebraic computations with magnitudes. In the rest of this section we shall use the symbol  $AB$  to denote the magnitude of the vector  $AB$ . No confusion is introduced by this double use of the symbol, because the ratio of two vectors is precisely the same as the quotient of their magnitudes.

DEFINITION. If  $\Gamma$  is any collineation not leaving  $l_\infty$  invariant, the lines  $\Gamma(l_\infty)$  and  $\Gamma^{-1}(l_\infty)$  are called the *vanishing lines* of  $\Gamma$ . If  $\Pi$  is any projectivity transforming a line  $l$  to a line  $l'$  (which may coincide with  $l$ ), the ordinary points of  $l$  and  $l'$  which are homologous with points at infinity are (if existent) called the *vanishing points* of  $\Pi$ . If  $\Pi$  is an involution transforming  $l$  into itself but not leaving the point at infinity invariant, the vanishing point is called the *center* of the involution.

THEOREM 21. DEFINITION. If  $O$  and  $O'$  are the vanishing points, on  $l$  and  $l'$  respectively, of a projectivity transforming a line  $l$  to a

parallel\* line  $l'$ , and  $X$  is a variable point of  $l$ , and  $X'$  the point of  $l'$  to which  $X$  is transformed, the product  $OX \cdot O'X'$  is a constant, called the power of the transformation.

*Proof.* Let  $P_\infty$  be the point at infinity of  $l$  and  $l'$ ; and let  $X_1$  and  $X_2$  be two values of  $X$ , and  $X'_1$  and  $X'_2$  the points to which they are transformed by the given projectivity. Then, by the fundamental property of a cross ratio,

$$\mathbf{R}(P_\infty O, X_1 X_2) = \mathbf{R}(O' P_\infty, X'_1 X'_2) = \mathbf{R}(P_\infty O', X'_2 X'_1),$$

and hence, by Theorem 18,  $\frac{OX_2}{OX_1} = \frac{O'X'_1}{O'X'_2}$ .

Hence, by the definition of magnitude of vectors,

$$OX_2 \cdot O'X'_1 = OX_1 \cdot O'X'_2.$$

COROLLARY 1. The power of an involution having a center  $O$  and a conjugate pair  $AA_1$  is  $OA \cdot OA_1$ .

COROLLARY 2. Let  $\Pi$  be a homology whose center is an ordinary point  $F$  and whose axis is an ordinary line, and let  $D$  be any point of the vanishing line  $\Pi^{-1}(l_\infty)$ . If  $P$  is a variable point,  $P' = \Pi(P)$ , and  $D'$  is the point in which the line through  $P'$  parallel to  $FD$  meets the vanishing line  $\Pi(l_\infty)$ , then

$$\frac{FP}{FP'} = \frac{DF}{P'D'}.$$

*Proof.* Let  $Q$  and  $Q'$  be the points in which the line  $FP$  meets the vanishing lines  $\Pi^{-1}(l_\infty)$  and  $\Pi(l_\infty)$  respectively. By the theorem,

$$PQ \cdot P'Q' = FQ \cdot FQ';$$

from which we derive successively

$$\frac{PF + FQ}{FQ} = \frac{FP' + P'Q'}{P'Q'},$$

$$\frac{PF}{FQ} = \frac{FP'}{P'Q'},$$

$$\frac{FP}{FP'} = \frac{QF}{P'Q'}.$$

Since  $\Pi$  is a homology, the two vanishing lines are parallel. Hence

$$\frac{QF}{P'Q'} = \frac{DF}{P'D'}.$$

Hence

$$\frac{FP}{FP'} = \frac{DF}{P'D'}.$$

\* With the extension of the definition of congruence in the next chapter the restriction to parallel lines may be removed.

## EXERCISES

1. If a projectivity  $ABCD \overline{\wedge} A'B'C'D'$  is such that the point at infinity of the line  $AB$  corresponds to the point at infinity of the line  $A'B'$ ,

$$\frac{AB}{CD} = \frac{A'B'}{C'D'}.$$

2. If three parallel lines  $a, b, c$  are met by one line in the points  $A', B', C'$  respectively and by another line in  $A''B''C''$  respectively, then

$$\frac{A'B'}{A'C'} = \frac{A''B''}{A''C''}.$$

3. If  $ABCD$  are any four collinear points,

$$AB \cdot CD + AC \cdot DB + AD \cdot BC = 0.$$

4. Six points form a quadrangular set  $Q (A_2B_2C_2, A_1B_1C_1)$  if and only if

$$R(A_1A_2, B_1C_1) \cdot R(B_1B_2, C_1A_1) \cdot R(C_1C_2, A_1B_1) = -1.$$

5. The condition for a quadrangular set may also be written

$$\frac{A_1B_2}{A_2B_1} \cdot \frac{B_1C_2}{B_2C_1} \cdot \frac{C_1A_2}{C_2A_1} = -1.$$

6. If three tangents to a parabola meet two other tangents in  $P_1, P_2, P_3$  and  $Q_1, Q_2, Q_3$  respectively, then

$$\frac{P_1P_2}{P_1P_3} = \frac{Q_1Q_2}{Q_1Q_3}.$$

Conversely, if five lines are such that the points in which two of them meet the other three satisfy this condition, the conic to which the five lines are tangent is a parabola.

7. Let  $O$  be the center of a hyperbola, and  $A_1$  and  $A_2$  the points in which the asymptotes are met by an arbitrary tangent; if another tangent meets the asymptotes  $OA_1, OA_2$  in  $B_1$  and  $B_2$  respectively,

$$\frac{OA_1}{OB_1} = \frac{OA_2}{OB_2}.$$

8. If a fixed tangent  $p$  to a conic at a point  $P$  meets two variable conjugate diameters in  $Q$  and  $Q'$ , then  $PQ \cdot PQ'$  is a constant. Let  $O$  be the center of the conic. If the diameter parallel to  $p$  meets the conic in  $S$ , then

$$PQ \cdot PQ' = -(OS)^2.$$

9. Let  $O_1$  and  $O_2$  be the points of contact of two fixed parallel tangents to a conic. If a variable tangent meets the two fixed tangents in  $X_1$  and  $X_2$  respectively,  $O_1X_1 \cdot O_2X_2$  is constant. If  $O$  is the center of the conic and  $B$  is a point of intersection of the diameter through  $O$  parallel to the fixed tangents,

$$O_1X_1 \cdot O_2X_2 = (OB)^2.$$

## 44. Theorems of Menelaus, Ceva, and Carnot.

THEOREM 22 (MENELAUS). *Three points  $A'$ ,  $B'$ ,  $C'$  of the sides  $BC$ ,  $CA$ ,  $AB$ , respectively, of a triangle are collinear if and only if*

$$\frac{A'B}{A'C} \cdot \frac{B'C}{B'A} \cdot \frac{C'A}{C'B} = 1.$$

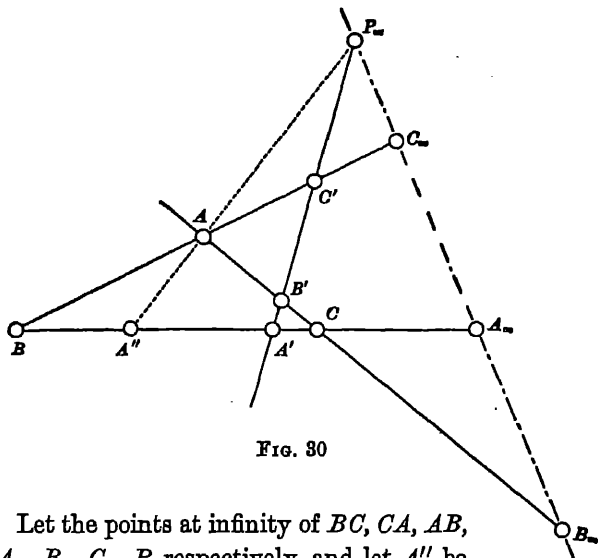


FIG. 80

*Proof.* Let the points at infinity of  $BC$ ,  $CA$ ,  $AB$ ,  $A'B'$  be  $A_\infty$ ,  $B_\infty$ ,  $C_\infty$ ,  $P_\infty$  respectively, and let  $A''$  be the intersection of  $A_\infty B_\infty$  with  $BC$ . Then, supposing  $A'$ ,  $B'$ ,  $C'$  collinear,

$$(B_\infty B' A' C) \frac{P_\infty}{\Lambda} (A_\infty A' A'' C) \text{ and } (C_\infty C' B A) \frac{P_\infty}{\Lambda} (A_\infty A' B A'').$$

Hence  $\frac{B'C}{B'A} = R(B_\infty B', AC) = R(A_\infty A', A'' C) = \frac{A'C}{A'A''},$

and  $\frac{C'A}{C'B} = R(C_\infty C', BA) = R(A_\infty A', B A'') = \frac{A'A''}{A'B}.$

Hence  $\frac{A'B}{A'C} \cdot \frac{B'C}{B'A} \cdot \frac{C'A}{C'B} = \frac{A'B}{A'C} \cdot \frac{A'C}{A'A''} \cdot \frac{A'A''}{A'B} = 1.$

The converse argument is now obvious.

THEOREM 23 (CEVA). *The necessary and sufficient condition for the concurrence of the lines joining the vertices  $A$ ,  $B$ ,  $C$  of a triangle to the points  $A'$ ,  $B'$ ,  $C'$  of the opposite sides is*

$$(4) \quad \frac{A'B}{A'C} \cdot \frac{B'C}{B'A} \cdot \frac{C'A}{C'B} = -1.$$

*Proof.* Let  $C''$  be the point of intersection of the lines  $A'B'$  and  $AB$ . Suppose first that  $C''$  is an ordinary point. Then, by the theorem of Menelaus,

$$(5) \quad \frac{A'B}{A'C} \cdot \frac{B'C}{B'A} \cdot \frac{C''A}{C''B} = 1.$$

The point  $C''$  is harmonically conjugate to  $C'$  with respect to  $A$  and  $B$  if and only if the lines  $AA'$ ,  $BB'$ ,  $CC'$  meet in a point. Thus,

$$(6) \quad \frac{C'A}{C'B} \div \frac{C''A}{C''B} = -1$$

is a necessary and sufficient condition that  $AA'$ ,  $BB'$ ,  $CC'$  concur. But on multiplying (5) by (6) we obtain (4).

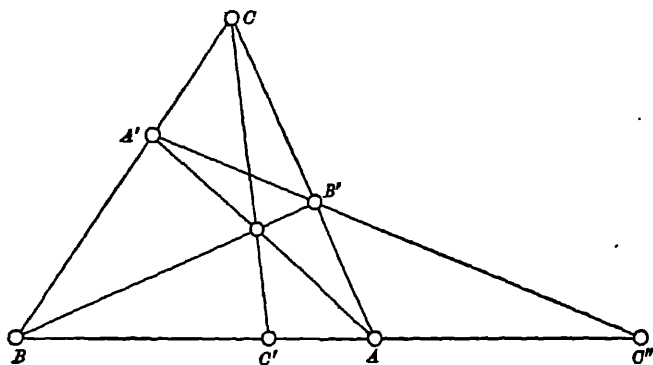


FIG. 31

In case  $C''$  is an ideal point, the line  $A'B'$  is parallel to  $AB$  and, by Theorem 20,

$$(7) \quad \frac{A'B}{A'C} \cdot \frac{B'C}{B'A} = 1.$$

The condition that  $C''$  be harmonically conjugate to  $C'$  with regard to  $A$  and  $B$  now takes the form

$$\frac{C'A}{C'B} = -1.$$

On multiplying this into (7) we again obtain (4).

**THEOREM 24 (CARNOT).** *Three pairs of points,  $A_1A_2$ ,  $B_1B_2$ ,  $C_1C_2$ , respectively, on the sides  $BC$ ,  $CA$ ,  $AB$ , respectively, of a triangle are on the same conic if and only if*

$$(8) \quad \frac{A_1B}{A_1C} \cdot \frac{B_1C}{B_1A} \cdot \frac{C_1A}{C_1B} \cdot \frac{A_2B}{A_2C} \cdot \frac{B_2C}{B_2A} \cdot \frac{C_2A}{C_2B} = 1.$$

*Proof.* Suppose first that the conic reduces to two lines containing  $A_1, B_1, C_1$  and  $A_2, B_2, C_2$  respectively. The formula (8) in this case follows directly from Theorem 22 when we multiply together the conditions that  $A_1, B_1, C_1$  and  $A_2, B_2, C_2$  be respectively collinear.

Now consider any proper conic through  $A_1, A_2, B_1, B_2$  meeting the line  $AB$  in  $C'_1 C'_2$ . By the theorem of Desargues (Theorem 19, Chap. V, Vol. I) the pairs  $AB, C_1 C_2$ , and  $C'_1 C'_2$  are in involution. Hence

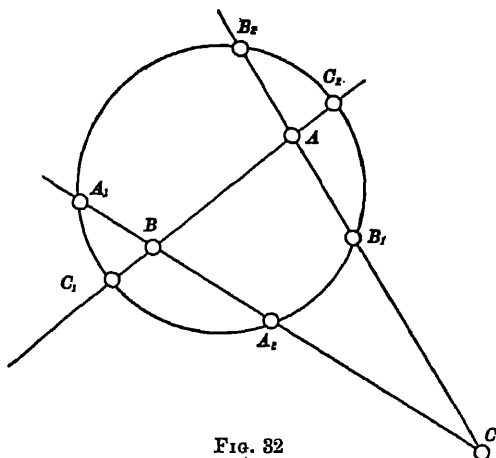


FIG. 32

$$C_1 C'_2 AB \wedge C_2 C'_1 BA \wedge C'_1 C_2 AB,$$

and hence

$$\frac{C_1 A}{C_1 B} : \frac{C'_2 A}{C'_2 B} = \frac{C'_1 A}{C'_1 B} : \frac{C_2 A}{C_2 B},$$

or

$$\frac{C_1 A}{C_1 B} \cdot \frac{C_2 A}{C_2 B} = \frac{C'_1 A}{C'_1 B} \cdot \frac{C'_2 A}{C'_2 B}.$$

Hence the formula (8) is equivalent to the formula obtained from it by substituting  $C'_1, C'_2$  for  $C_1, C_2$  respectively. Hence the formula holds for any conic. The converse argument is now obvious.

The last three theorems are the most important special cases of the "theory of transversals." A few further theorems of this class and some other propositions which can readily be derived from them are stated in the exercises below. Further theorems and references will be found in the *Encyclopädie der Math. Wiss.* III AB 5, § 2, and III C 1, § 23.

### EXERCISES

1. The six lines joining the vertices  $A, B, C$  of a triangle to pairs of points  $A_1, A_2, B_1, B_2, C_1, C_2$  on the respectively opposite sides are tangents to a conic if and only if the relation (8) is satisfied.

2. If the sides  $BC, CA, AB$  of a triangle are tangent to a conic in  $A_1, B_1, C_1$  respectively,

$$\frac{CA_1}{BA_1} \cdot \frac{AB_1}{CB_1} \cdot \frac{BC_1}{AC_1} = -1.$$

3. If a line  $BC$  meets a conic in  $A_1$  and  $A_2$ , and two parallel lines through  $B$  and  $C$ , respectively, meet it in the pairs  $C_1, C_2$  and  $B_1, B_2$  respectively,

$$\frac{A_1B}{A_1C} \cdot \frac{A_2B}{A_2C} \cdot \frac{B_1C}{C_1B} \cdot \frac{B_2C}{C_2B} = 1.$$

4. Let two lines  $a$  and  $b$  through a point  $O$  meet a conic in the pairs  $A_1, A_2$  and  $B_1, B_2$  respectively. If  $O, a, b$  are variable in such a way that  $a$  and  $b$  remain respectively parallel to two fixed lines,

$$\frac{OA_1 \cdot OA_2}{OB_1 \cdot OB_2}$$

is a constant.

5. If the sides of a triangle meet a conic in three pairs of points, the three pairs of lines joining the pairs of points to the opposite vertices of the triangle are tangents to a second conic. State the dual and converse of this theorem.

6. If two points are joined to the vertices of a triangle by six lines, these lines meet the sides in six points (other than the vertices) which are on a conic. Dualize.

7. If a line meets the sides  $A_0A_1, A_1A_2, \dots, A_nA_0$ , respectively, of a simple polygon  $A_0A_1A_2 \dots A_n$  in points  $B_0, B_1, \dots, B_n$  respectively,

$$\frac{A_0B_0}{A_1B_0} \cdot \frac{A_1B_1}{A_2B_1} \dots \frac{A_nB_n}{A_0B_n} = 1.$$

8. If a conic meets the lines  $A_0A_1, A_1A_2, \dots, A_nA_0$ , respectively, in the pairs of points  $B_0C_0, B_1C_1, \dots, B_nC_n$  respectively,

$$\frac{A_0B_0}{A_1B_0} \cdot \frac{A_0C_0}{A_1C_1} \cdot \frac{A_1B_1}{A_2B_1} \cdot \frac{A_1C_1}{A_2C_2} \dots \frac{A_nB_n}{A_0B_n} \cdot \frac{A_nC_n}{A_0C_n} = 1.$$

9. If a conic is tangent to the lines  $A_0A_1, A_1A_2, \dots, A_nA_0$ , respectively, in the points  $B_0, B_1, \dots, B_n$  respectively,

$$\frac{A_0B_0}{A_1B_0} \cdot \frac{A_1B_1}{A_2B_1} \dots \frac{A_nB_n}{A_0B_n} = (-1)^{n-1}.$$

45. **Point reflections.** DEFINITION. A homology of period two whose axis is  $l_\infty$  is called a *point reflection*.

From this definition there follows at once:

**THEOREM 25.** *A point reflection is fully determined by its center. The center is the mid-point of every pair of homologous points. Every two homologous lines are parallel.*

**THEOREM 26.** *The product of two point reflections whose centers are distinct is a translation parallel to the line joining their centers.*

*Proof.* The product obviously leaves fixed all points of  $l_\infty$  and also the line joining the two centers. Let  $C_1$  and  $C_2$  be the two centers,

and let  $P$  be any point not on the line  $C_1C_2$ . Also let  $P'$  be the transform of  $P$  by the point reflection with  $C_1$  as center, and let  $Q$  be the transform of  $P'$  by the point reflection with  $C_2$  as center. Since  $C_1$  is the mid-point of the pair  $PP'$ , and  $C_2$  of the pair  $P'Q$ , the line  $PQ$  is parallel to  $C_1C_2$  (Theorem 12, Cor. 2). Thus the product of the two point reflections leaves invariant all lines parallel to  $C_1C_2$ , and hence is a translation.

**COROLLARY.** *The product of any even number of point reflections is a translation.*

**THEOREM 27.** *Any translation is the product of two point reflections one of which is arbitrary.*

*Proof.* Let  $T$  be any translation,  $C_1$  the center of any point reflection,  $C_3 = T(C_1)$ , and  $C_2$  the mid-point of the pair  $C_1C_3$ . The product of the reflections in  $C_1$  and  $C_2$  is a translation, by Theorem 26, and since it carries  $C_1$  to  $C_3$ , it is the translation  $T$ , by Theorem 3.

**COROLLARY 1.** *The product of any odd number of point reflections is a point reflection.*

*Proof.* Let the given point reflections be  $P_1, P_2, \dots, P_{2n+1}$ . By Theorem 26 the product  $P_1P_2 \cdots P_{2n}$  reduces to a translation, which, by Theorem 27, is the product of two point reflections one of which is  $P_{2n+1}$ . Hence there exists a point reflection  $P$  such that

$$P_1P_2 \cdots P_{2n+1} = PP_{2n+1}P_{2n+1} = P.$$

**COROLLARY 2.** *The product of a translation and a point reflection is a point reflection.*

**COROLLARY 3.** *The set of all point reflections and translations form a group.*

**THEOREM 28.** *The group of point reflections and translations is a self-conjugate subgroup of the affine group.*

*Proof.* It has been proved, in Theorem 11, that if  $T$  is a translation and  $\Sigma$  an affine collineation,  $\Sigma T \Sigma^{-1}$  is a translation. Precisely similar reasoning shows that if  $T$  is a point reflection,  $\Sigma T \Sigma^{-1}$  is a point reflection.

**COROLLARY.** *The group  $G$  of point reflections and translations is self-conjugate under any subgroup of the affine group which contains  $G$ .*

**THEOREM 29.** *With respect to any system of nonhomogeneous coördinates in which  $l_\infty$  is the singular line, the equations of a point reflection have the form*

$$(9) \quad \begin{aligned} x' &= -x + a, \\ y' &= -y + b. \end{aligned}$$

*Proof.* The point reflection whose center is the origin is of the form

$$\begin{aligned} x' &= -x, \\ y' &= -y, \end{aligned}$$

because this transformation evidently leaves  $(0, 0)$  and  $l_\infty$  pointwise invariant and is of period two. Since any other point reflection is the resultant of this one and a translation, it must be of the form (9).

### EXERCISES

1. An ellipse or a hyperbola is transformed into itself by a point reflection whose center is the center of the conic.

2. Let  $[C^2]$  be a system of conics conjugate under the group of translations to a single conic. Under what circumstances is  $[C^2]$  invariant under the group of translations and point reflections?

3. Investigate the subgroups of the group of translations and point reflections.

4. Any odd number of point reflections  $P_1, P_2, \dots, P_n$  satisfy the condition,

$$(P_1 P_2 \cdots P_n)^2 = 1.$$

5. Let  $T$  be the point reflection whose center is the pole of  $l_\infty$  with respect to the  $n$ -point whose vertices are the centers of  $n$  point reflections  $P_1, P_2, \dots, P_n$ . Then\*

$$P_1 T P_2 T P_3 T \cdots P_n T = 1.$$

**46. Extension of the definition of congruence.** **DEFINITION.** Two figures are said to be *congruent* if they are conjugate under the group of translations and point reflections.

This definition is obviously in agreement with that given in § 39. It will be completed in § 57, Chap. IV. The main significance of the present extension of the definition is that it removes any necessity of distinguishing between ordered and nonordered point pairs in statements about congruence.

\* Cf. pp. 46, 84, Vol. I. The center of  $T$  is the "center of gravity" of the centers of  $P_1, \dots, P_n$ . Cf. H. Wiener, *Berichte der Gesellschaft der Wissenschaften zu Leipzig*, Vol. XLV (1893), p. 568.

**THEOREM 30.** *Any ordered point pair  $AB$  is congruent to the ordered point pair  $BA$ .*

*Proof.* Let  $O$  be the mid-point of the ordered point pair  $AB$ . The point reflection with  $O$  as center interchanges  $A$  and  $B$ .

**COROLLARY.** *If a point reflection transforms an ordered point pair  $AB$  to  $A'B'$ ,*

$$\text{Vect}(AB) = -\text{Vect}(A'B').$$

*Proof.* By Theorem 26 the given point reflection is the product of the point reflection in the mid-point of  $AB$  and a translation. The point reflection in the mid-point of  $AB$  interchanges  $A$  and  $B$ , and the translation leaves all vectors unchanged.

**47. The homothetic group.** **DEFINITION.** A homology whose axis is  $l_\infty$  is called a *dilation*. Dilations and translations are both called *homothetic transformations*. Two figures conjugate under a homothetic transformation are said to be *homothetic*.

Homothetic figures are also called, in conformity with definitions introduced later, "similar and similarly placed."

The point reflections are evidently special cases of dilations. Since the product of two perspective collineations (§ 28, Vol. I) having a common axis is a perspective collineation, the set of all homothetic transformations form a group; and by an argument like that used for Theorem 11 this group is self-conjugate under the affine group. Hence we have

**THEOREM 31.** *The set of all homothetic transformations form a group which is a self-conjugate subgroup of the affine group.*

Further theorems on the homothetic group are stated in the exercises below.

### EXERCISES

1. The ratios of parallel vectors are left invariant by the homothetic group.
2. If two point pairs  $AB$  and  $CD$  are transformed by a dilation into  $A'B'$  and  $C'D'$  respectively,

$$\frac{AB}{A'B'} = \frac{CD}{C'D'}.$$

3. If two triangles are homothetic, the lines joining corresponding vertices meet in a point or are parallel.

4. The equations of the homothetic group with respect to any nonhomogeneous coordinate system of which  $l_\infty$  is the singular line are

$$\begin{aligned} x' &= ax + b, \\ y' &= ay + d. \end{aligned} \qquad (a \neq 0)$$

**48. Equivalence of ordered point triads.** Although the theory of congruence as based on the group of translations and point reflections does not yield metric relations between pairs of points unless they are on parallel lines, yet when applied to point triads it yields a complete theory of the equivalence (in area) of triangles.\*

In this section we shall give the definitions and the more important sufficient conditions for equivalence, using methods somewhat analogous to those in the first book of Euclid's Elements. Instead of triangles, however, we shall work with ordered triads of points. This permits the introduction of algebraic signs of areas, though, as we do not need to refer to the interior and exterior of a triangle, we shall not actually employ the word "area." The triads of points which are referred to are all triads of *noncollinear points*.

Our definitions have their origin in the intuitional notions: that any triangle  $ABC$  is equivalent in area to the triangle  $BCA$ , that two triangles are equivalent in area if one can be transformed into the other by a translation or point reflection, and that two triangles which can be obtained by adding equivalent triangles are equivalent.

**DEFINITION.** If  $ABC$  and  $ACD$  are two ordered point triads, and  $B, C$ , and  $D$  are collinear, and  $B \neq D$  (fig. 33), the point triad  $ABD$  is called the *sum* of  $ABC$  and  $ACD$  and is denoted by  $ABC + ACD$  or by  $ACD + ABC$ .

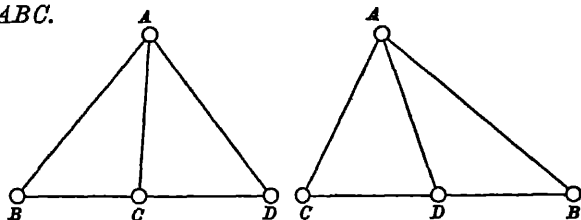


FIG. 33

**DEFINITION.** An ordered point triad  $t$  is said to be *equivalent* to an ordered point triad  $t'$  (in symbols,  $t \simeq t'$ ) (1) if  $t$  can be carried to  $t'$  by a point reflection, or (2) if  $t$  and  $t'$  can be denoted by  $ABC$  and

\* The idea of building up the theory of areas without the aid of a full theory of congruence is due to E. B. Wilson, *Annals of Mathematics*, Vol. V (2d series) (1903), p. 29. His method is quite different from ours, being based on the observation (cf. § 52, below) that an equiaffine collineation is expressible as a product of simple shears. Still another treatment of areas based on the group of translations and employing continuity considerations is outlined by Wilson and Lewis, "The Space-time Manifold of Relativity," *Proceedings of the American Academy of Arts and Sciences*, Vol. XLVIII (1912). We shall return to the subject in later sections.

$BCA$  respectively, or (3) if there exists an ordered point triad  $\bar{t}$  such that  $t \preceq \bar{t}$  and  $\bar{t} \preceq t'$ , or (4) if there exist ordered point triads  $t_1, t_2, t'_1, t'_2$  such that  $t_1 \preceq t'_1, t_2 \preceq t'_2$  and  $t = t_1 + t_2$  and  $t' = t'_1 + t'_2$ . An ordered point triad  $t$  is not said to be equivalent to an ordered point triad  $t'$  unless it follows, by a finite number of applications of the criteria (1), (2), (3), (4), that  $t \preceq t'$ .

Since any translation is a product of two point reflections, Criteria (1) and (3) give

**THEOREM 32.** *Two ordered point triads are equivalent if they are conjugate under the group of translations and point reflections.*

**THEOREM 33.** *If  $A, B$ , and  $C$  are noncollinear points,  $ABC \preceq ABC, ABC \preceq BCA, ABC \preceq CAB$ .*

*Proof.* From (2) of the definition it follows that  $ABC \preceq BCA$  and  $BCA \preceq CAB$ . Hence, by (3),  $ABC \preceq CAB$ . But, by (2),  $CAB \preceq ABC$ . Hence, by (3),  $ABC \preceq ABC$ .

From the last two theorems and from the form of the definition we now have at once

**THEOREM 34.** *If  $t_1 \preceq t_2$ , then  $t_2 \preceq t_1$ .*

**THEOREM 35.** *If  $A, B, C$  are any three noncollinear points and  $O$  the mid-point of the pair  $AB$ , then  $AOC \preceq OBC$ .*

*Proof.* Let  $C'$  be the point to which  $C$  is changed by the translation shifting  $A$  to  $O$ , and let  $M$  be the point of intersection of the non-parallel lines  $BC$  and  $OC'$ . Since  $COBC'$  is a parallelogram,  $M$  is the mid-point of the pairs  $CB$  and  $C'O$ . Thus we have

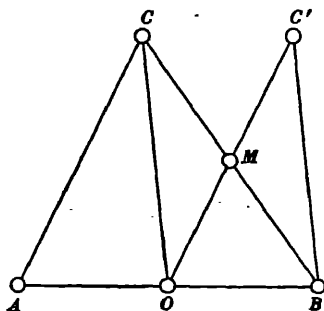


FIG. 34

$$AOC \preceq OBC' \preceq BC'O = BC'M + BMO$$

and

$$OBC = OBM + OMC.$$

But the point reflection with  $M$  as center carries  $OMC$  into  $C'MB$ . Thus

$$OMC \preceq C'MB \preceq BC'M,$$

and

$$OBM \preceq BMO,$$

and hence, by comparison with the equivalences and equations above,

$$AOC \preceq OBC.$$

**THEOREM 36.** *Two ordered point triads  $ABC_1$  and  $ABC_2$ , where  $C_1 \neq C_2$ , are equivalent if the line  $C_1C_2$  is parallel to the line  $AB$ .*

*Proof.* Let  $C_3$  be such that  $B$  is the mid-point of  $C_1C_3$ , and let the line  $C_2C_3$  meet the line  $AB$  in  $O$ , which is an ordinary point because  $C_3$  is not on the line  $C_1C_2$ . It follows (§ 40) that  $O$  is the mid-point of the pair  $C_2C_3$ .

By Theorems 34 and 35,  $ABC_1 \simeq BAC_3 \simeq C_3BA$ . By definition,  $C_3BA = C_3BO + C_3OA$ . By Theorem 35,  $C_3BO \simeq C_2OB$  and  $C_3OA \simeq C_2AO$ . Hence  $C_3BA \simeq C_2AO + C_2OB = C_2AB \simeq ABC_2$ . Hence  $ABC_1 \simeq ABC_2$ .

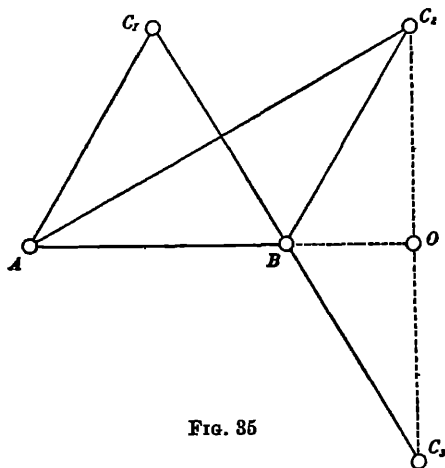


FIG. 35

**COROLLARY.** *If a point  $B'$  is on a line  $OB$  and a point  $C'$  on a different line  $OC$ , and the lines  $BC'$  and  $B'C$  are parallel,  $BOC \simeq B'OC'$ .*

*Proof.* By hypothesis,

$$BOC = BOC' + BC'C$$

$$\text{and } C'B'O = C'B'C + C'BO.$$

But  $C'B'B \simeq C'CB \simeq BC'C$ , by Theorems 36 and 34, and  $C'BO \simeq BOC'$ , by Theorem 34. Hence  $BOC \simeq C'B'O \simeq B'OC'$ .

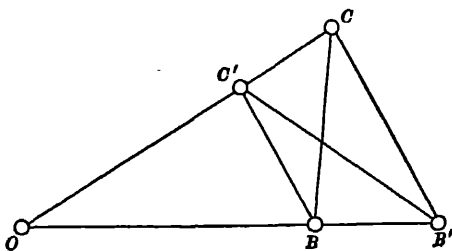


FIG. 36

**THEOREM 37.** *If  $A$ ,  $B$ , and  $C$  are any three noncollinear points, and  $P$  and  $Q$  are any two distinct points, there exists a line  $r$  parallel to  $PQ$  such that if  $R$  is any point of  $r$ ,  $ABC \simeq PQR$ .*

*Proof.* Let  $T$  be the translation such that  $T(A) = P$ , and let  $T(B) = B'$  and  $T(C) = C'$ . If  $B'$  is not on the line  $PQ$ , let  $R'$  be the intersection (fig. 37) of the line through  $C'$  parallel to  $PB'$  with the line through  $P$  parallel to  $QB'$ . If  $B'$  is on the line  $PQ$ , let  $R'$  be the point of intersection with  $PC'$  of the line through  $B'$  parallel to  $QC'$ .



to  $OP$ , meet the line  $OQ$ , and let  $A_1$  be the point in which the line through  $A$ , parallel to  $OP$ , meets the line  $BC$ . Let  $AA_1$  denote the magnitude of the vector  $AA_1$  relative to the unit  $OP$  (§ 43), and  $B_1C_1$  the magnitude of the vector  $B_1C_1$  relative to the unit  $OQ$ . The measure of the ordered triad  $ABC$  is\*

$$AA_1 \cdot B_1C_1$$

and is denoted by  $m(ABC)$ . If the line  $BC$  is parallel to  $OP$ ,  $CA$  is not parallel to  $OP$ , and the measure of  $ABC$  is defined to be  $m(BCA)$ .

If this definition be allowed to apply to any ordered point triad whatever (instead of only to noncollinear triads, cf. § 48), we have  $m(ABC) = 0$  whenever the points  $A, B, C$  are collinear.

**THEOREM 38.** *If  $ABC \simeq A'B'C'$ , then  $m(ABC) = m(A'B'C')$ .*

*Proof.* Let us examine the four criteria in the definition of equivalence in § 48.

(1) In case  $ABC$  is carried to  $A'B'C'$  by a point reflection, each of the vectors  $AA_1$  and  $B_1C_1$  is transformed into its negative (Theorem 30, corollary), and hence the product of their magnitudes is unchanged.

(2) According to the second criterion,  $ABC \simeq BCA$ . Suppose, first, that neither  $BC$  nor  $CA$  is parallel to  $OP$ , and let  $A_1, B_1, C_1$  have the meaning given them in the definition above. Then

$$m(ABC) = AA_1 \cdot B_1C_1.$$

Let  $B_2$  (fig. 38) be the point in which the line through  $B$ , parallel to  $OP$ , meets the line  $CA$ , and let  $A_2$  be the point in which  $OQ$  is met by the parallel to  $OP$  through  $A$ . Then if  $BB_2$  and  $C_1A_2$  represent the magnitudes of the corresponding vectors relative to  $OP$  and  $OQ$  as units,

$$m(BCA) = BB_2 \cdot C_1A_2.$$

By Theorem 20,

$$\frac{AA_1}{B_2B} = \frac{A_1C}{BC}.$$

But since the lines  $CC_1, A_1A_2, BB_1$  are parallel, it follows from § 43 that

$$\frac{A_1C}{BC} = \frac{A_2C_1}{B_1C_1}.$$

Hence

$$\frac{AA_1}{B_2B} = \frac{A_2C_1}{B_1C_1},$$

or  $m(ABC) = AA_1 \cdot B_1C_1 = BB_2 \cdot C_1A_2 = m(BCA)$ .

\* The factor  $\frac{1}{2}$  is lacking in this expression, because we are taking a triangle rather than a parallelogram as the unit.

In case  $BC$  is parallel to  $OP$ , the last clause of the definition states that

$$m(ABC) = m(BCA).$$

In case  $CA$  is parallel to  $OP$ ,  $AB$  and  $BC$  are not parallel to  $OP$ , and hence the argument above shows that

$$m(CAB) = m(ABC).$$

But, by definition,  $m(BCA) = m(CAB).$

Hence  $m(ABC) = m(BCA).$

(3) Corresponding to the fact that if  $t_1 \preceq t_2$  and  $t_2 \preceq t_3$ , then  $t_1 \preceq t_3$ , we have that, since  $m(t)$  is a uniquely defined number, if  $m(t_1) = m(t_2)$ , and  $m(t_2) = m(t_3)$ , then  $m(t_1) = m(t_3).$

(4) Let  $B, C, D$  be three collinear points and  $A$  any point not on the line  $BC$  (fig. 39). In case the line  $BC$  is not parallel to  $OP$ , let  $A_1$  be the point in which the line through  $A$ , parallel to  $OP$ , meets  $BC$ , and let  $B_1, C_1, D_1$  be the points in which the lines through  $B, C, D$  respectively, parallel to  $OP$ , meet  $OQ$ . Then

$$\begin{aligned} m(ABD) &= AA_1 \cdot B_1D_1 \\ &= AA_1 \cdot B_1C_1 + AA_1 \cdot C_1D_1 \\ &= m(ABC) + m(ACD). \end{aligned}$$

In case the line  $BC$  is parallel to  $OP$ , let  $S$  be the point in which  $BC$  meets  $OQ$ , and  $A_2$  be the point in which the line through  $A$ , parallel to  $OP$ , meets  $OQ$ . Then

$$\begin{aligned} m(ABD) &= m(BDA) = BD \cdot SA_2 = BC \cdot SA_2 + CD \cdot SA_2 \\ &= m(BCA) + m(CDA) = m(ABC) + m(ACD). \end{aligned}$$

Thus, in every case, if  $t_1 + t_2 = t_3$ ,  $m(t_1) + m(t_2) = m(t_3).$

Comparing the results proved in these four cases with the definition of equivalence, we have at once that whenever  $t_1 \preceq t_2$ ,  $m(t_1) = m(t_2).$

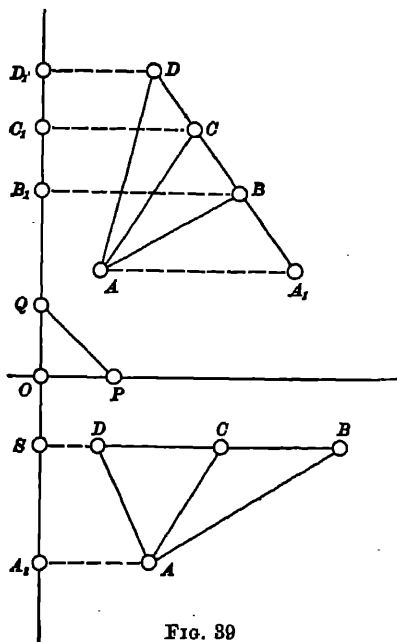


FIG. 39

THEOREM 39. *If  $B, C$ , and  $D$  are collinear points, and the point  $A$  is not on the line  $BC$ ,*

$$\frac{m(\angle ABC)}{m(\angle ABD)} = \frac{BC}{BD}.$$

*Proof.* In case the line  $BC$  is not parallel to  $OP$ , let  $A_1, B_1, C_1$  have the meaning given them in the definition of measure, and let  $D_1$  be the point in which the line through  $D$ , parallel to  $OP$ , meets  $OQ$  (fig. 39). Then

$$\frac{m(\angle ABC)}{m(\angle ABD)} = \frac{AA_1 \cdot B_1C_1}{AA_1 \cdot B_1D_1} = \frac{B_1C_1}{B_1D_1}.$$

But, by § 43,

$$\frac{B_1C_1}{B_1D_1} = \frac{BC}{BD}.$$

In case  $BC$  is parallel to  $OP$ , let  $A_2$  be the point in which the line through  $A$ , parallel to  $OP$ , meets  $OQ$ , and  $S$  the point in which  $BC$  meets  $OQ$ . Then

$$\frac{m(\angle ABC)}{m(\angle ABD)} = \frac{m(\angle BCA)}{m(\angle BDA)} = \frac{BC \cdot SA_2}{BD \cdot SA_2} = \frac{BC}{BD}.$$

COROLLARY 1. *If  $B, C, D, E$  are points no two of which are collinear with a point  $A$ ,*

$$\Re(\angle B, AC, AD, AE) = \frac{m(\angle ABD)}{m(\angle ABE)} \div \frac{m(\angle ACD)}{m(\angle ACE)}.$$

COROLLARY 2. *If  $B, C, D$  are points no two of which are collinear with a point  $A$ , and if  $P_\infty$  is the point at infinity of the line  $CD$  (the latter not being parallel to  $AB$ ),*

$$\Re(\angle AP_\infty, AB, AC, AD) = \frac{m(\angle ABD)}{m(\angle ABC)}.$$

THEOREM 40. *If  $m(\angle ABC) = m(\angle A'B'C') \neq 0$ , then  $ABC \simeq A'B'C'$ .*

*Proof.* By Theorem 37 there exists a point  $C''$  on the line  $A'C'$  such that  $ABC \simeq A'B'C''$ . Hence  $A'B'C' \simeq A'B'C''$ , and by the last theorem,  $C' = C''$ .

In consequence of the last two theorems the unit point triad may be replaced by any equivalent triad without changing the measure of any triad.

THEOREM 41. *If  $ABC \simeq ABC'$ , and  $C \neq C'$ , the line  $CC'$  is parallel to the line  $AB$ .*

*Proof.* The unit triad  $OPQ$  may be chosen so that  $OP$  is parallel to  $AB$ . Then if  $C_1$  is the point in which the line through  $C$ , parallel to  $OP$ , meets  $OQ$ , and  $B_1$  the point in which  $AB$  meets  $OQ$ ,

$$m(ABC) = AB \cdot B_1 C_1.$$

If  $C'_1$  is the point in which the line through  $C'$ , parallel to  $OP$ , meets  $OQ$ ,

$$m(ABC') = AB \cdot B_1 C'_1.$$

By Theorem 38,  $m(ABC) = m(ABC')$ , and hence  $C_1 = C'_1$ . Hence the line  $CC'$  is parallel to  $AB$ .

**THEOREM 42.** *If  $ABC \simeq AB'C'$ , and  $B'$  is on the line  $AB$ , and  $C'$  on the line  $AC$ , then the line  $BC'$  is parallel to the line  $B'C$ .*

*Proof.* By the corollary of Theorem 36, if  $C''$  is a point of  $AC'$  such that  $BC''$  is parallel to  $B'C$ , then

$$ABC \simeq AB'C''.$$

By Theorem 41 the only points  $\bar{C}$  such that  $ABC \simeq AB'\bar{C}$  are on the line through  $C''$ , parallel to  $AB'$ . Hence  $C' = C''$ .

It is notable that although the sufficient conditions for equivalence given in § 48 are all proved on the basis of Assumptions A, E,  $H_0$ , the discussion of the ratios of vectors, and hence all the necessary conditions for equivalence, involve Assumption P in their proofs. This is essential,\* as we can show by proving that Assumption P is a logical consequence of these theorems, together with the previous theorems on equivalence. As was pointed out in § 3, Assumption P is a logical consequence of the theorem of Pappus, Theorem 21, § 36, Vol. I. When one of the lines of the configuration is taken as  $l_\infty$ , this theorem assumes the form:

*If a simple hexagon  $AB'CA'BC'$  is such that  $A, B, C$  are on one line and  $A', B', C'$  on another line, and if  $AB'$  is parallel to  $A'B$  and  $BC'$  parallel to  $B'C$ , then  $CA'$  is parallel to  $C'A$ .*

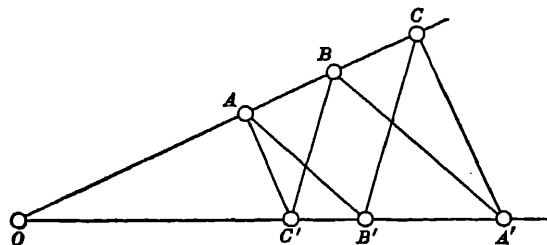


FIG. 40

In case the lines containing  $ABC$  and  $A'B'C'$ , respectively, are parallel, this can be proved from the Desargues theorem on perspective triangles; so that we are interested only in the

\* The rôle of Assumption P (or rather of the equivalent theorem of Pappus) in the theory of areas was first determined in a definite way by D. Hilbert, *Grundlagen der Geometrie*, Chap. IV.

case when  $AB$  and  $A'B'$  intersect in a point  $O$ . By Theorem 36, since  $AB$  is parallel to  $A'B$ ,  $OAA' \simeq OBB'$ ; and since  $BC'$  is parallel to  $B'C$ ,  $OBB' \simeq OCC'$ . By the definition (3) of equivalence it follows that  $OAA' \simeq OCC'$ . But by Theorem 42 this implies that  $AC'$  is parallel to  $A'C$ .

This is perhaps the simplest way of proving the fundamental theorem of projective geometry if it be desired to base projective geometry upon elementary Euclidean geometry (cf. Ex. 3, § 54).

The notion of measure can be extended to any ordered set of  $n$  points, i.e. (cf. § 14, Vol. I) to any *simple  $n$ -point*. The details of this discussion are left to the reader. An outline is furnished by the problems below. The principal references are to A. F. Möbius, *Der barycentrische Calcul*, §§ 1, 17, 18, 165; *Werke*, Vol. I, pp. 23, 39, 200; Vol. II, p. 485. See also the *Encyclopädie der Math. Wiss.*, III AB 9, § 12. It is to be borne in mind in using these references that our hypotheses are narrower than those used by the previous writers.

### EXERCISES

1. For any three points  $A, B, C$ ,

$$m(ABC) + m(ACB) = 0.$$

2. For any four points  $O, A, B, C$ ,

$$m(ABC) = m(OAB) + m(OBC) + m(OCA).$$

3. For any  $n$  points  $A_1, A_2, \dots, A_n$  the number

$$m(OA_1A_2) + m(OA_2A_3) + \dots + m(OA_{n-1}A_n) + m(OA_nA_1)$$

is the same for all choices of the point  $O$ . We define it to be the *measure* of the simple  $n$ -point  $A_1A_2 \dots A_n$  and denote it by  $m(A_1A_2 \dots A_n)$ .

4.  $m(A_1A_2 \dots A_{n-1}A_n) = m(A_2A_3 \dots A_nA_1)$ .

5.  $m(A_1A_2 \dots A_n) + m(A_1A_nA_{n+1} \dots A_{n+k}) = m(A_1A_2 \dots A_{n+k})$ .

6. Derive a formula for  $m(A_1A_2 \dots A_n)$  analogous to the definition of  $m(ABC)$  in terms of vectors collinear with two arbitrary vectors  $OP$  and  $OQ$ .

7. Prove the converse propositions to those stated in the exercises in § 48.

8. If  $ABCD$  and  $A'B'C'D'$  are two parallelograms whose sides are respectively parallel,

$$\frac{m(ABCD)}{m(A'B'C'D')} = \frac{AB}{A'B} \cdot \frac{BC}{B'C}.$$

9. The variable parallelogram two of whose sides are the asymptotes of a hyperbola and one vertex of which is on the hyperbola has a constant measure.

10. If a variable pair of conjugate diameters meets a conic in point pairs  $AA', BB'$ , the parallelogram whose sides are the tangents at  $A, A', B, B'$  has a constant measure. The parallelogram  $ABA'B'$  also has a constant measure.

**50. The equiaffine group.** THEOREM 43. *If two equivalent ordered point triads  $t_1$  and  $t_2$  are transformed by an affine collineation into  $t'_1$  and  $t'_2$ , then  $t'_1 \preceq t'_2$ .*

*Proof.* It is necessary merely to verify that the relation used in each of the criteria (1),  $\dots$ , (4) in the definition of equivalence (§ 48) is unaffected by an affine collineation. For Criterion (1) this reduces to Theorem 28. For Criteria (2), (3), (4) it is a consequence of the fact that an affine collineation transforms ordered triads into ordered triads and collinear points into collinear points.

THEOREM 44. *If an affine collineation transforms one ordered point triad into an equivalent point triad, it transforms every ordered point triad into an equivalent point triad.*

*Proof.* It follows from Theorem 43 that if  $ABC$  is transformed by a given collineation into an equivalent ordered point triad  $A'B'C'$ , then every point triad equivalent to  $ABC$  is transformed into a point triad equivalent to  $A'B'C'$  and thus into one equivalent to  $ABC$ . By Theorem 37 any ordered point triad whatever is equivalent to some point triad  $ADC$ , where  $D$  is on the line  $AB$ . Hence the present theorem will be proved if we can show that  $ADC$  is transformed into an equivalent point triad.

Denote the point to which  $D$  is transformed by the given collineation by  $D'$ . By Theorem 39,

$$\frac{m(ADC)}{m(ABC)} = \frac{AD}{AB} \text{ and } \frac{m(A'D'C')}{m(A'B'C')} = \frac{A'D'}{A'B'}.$$

By § 43, 
$$\frac{AD}{AB} = R(P_\infty A, BD),$$

where  $P_\infty$  is the point at infinity of the line  $AB$ . But since the given collineation is affine,  $P_\infty$  is transformed to the point at infinity  $P'_\infty$  of the line  $A'B'$ , and

$$R(P_\infty A, BD) = R(P'_\infty A', B'D') = \frac{A'D'}{A'B'} = \frac{m(A'D'C')}{m(A'B'C')}.$$

Since  $m(ABC) = m(A'B'C')$ , it follows that  $m(ADC) = m(A'D'C')$ . Hence

$$ADC \preceq A'D'C'.$$

DEFINITION. Any affine collineation which transforms an ordered point triad into an equivalent point triad is said to be *equiaffine*.

**THEOREM 45.** *The equiaffine collineations form a self-conjugate subgroup of the affine group.*

*Proof.* By the last theorem an equiaffine collineation transforms every ordered point triad into an equivalent point triad. Hence, by Condition (3) in the definition of equivalence, the product of two equiaffine collineations is equiaffine. By Theorem 43,  $\Sigma T \Sigma^{-1}$  is equiaffine whenever  $T$  is equiaffine and  $\Sigma$  affine.

**THEOREM 46.** *Let  $A, B, A', B'$  be points such that  $A \neq B$  and  $A' \neq B'$ ; let  $a$  be a line on  $A$  but not on  $B$ , and let  $a'$  be a line on  $A'$  but not on  $B'$ . There is one and only one equiaffine collineation transforming  $A$  to  $A'$ ,  $B$  to  $B'$ , and  $a$  to  $a'$ .*

*Proof.* Let  $C$  be any point distinct from  $A$  on  $a$ . By Theorem 37, there is a point  $C'$  on the line  $a'$  such that

$$ABC \simeq A'B'C'.$$

By Theorem 1 there is one and only one affine transformation carrying  $A, B, C$  to  $A', B', C'$  respectively, and by definition this transformation is equiaffine. By Theorem 41,  $C'$  is the only point on  $a'$  such that  $ABC \simeq A'B'C'$ . Hence (Theorem 44) there is only one equiaffine transformation carrying  $A, B, a$  into  $A', B', a'$  respectively.

### EXERCISE

Any affine collineation leaves invariant the ratio of the measures of any two point triads.

**\*51. Algebraic formula for measure. Barycentric coördinates.** Consider a nonhomogeneous coördinate system in which  $l_\infty$  is the singular line. Let the unit of measure for ordered triads be  $OPQ$ , where  $O = (0, 0)$ ,  $P = (1, 0)$ ,  $Q = (0, 1)$ . Let  $A = (a_1, a_2)$ ,  $B = (b_1, b_2)$ ,  $C = (c_1, c_2)$ ; the line through  $A$ , parallel to  $OP$ , consists of the points  $(a_1 + \lambda, a_2)$ , where  $\lambda$  is arbitrary, and the line  $BC$  has the equation (§ 64, Vol. I),

$$\begin{vmatrix} x & y & 1 \\ b_1 & b_2 & 1 \\ c_1 & c_2 & 1 \end{vmatrix} = 0.$$

In case the line  $BC$  is not parallel to  $OP$ , and therefore  $b_2 \neq c_2$ , the point  $A_1$  which appears in the definition of measure (§ 49) is  $(a_1 + \lambda, a_2)$ , where  $\lambda$  satisfies

$$\begin{vmatrix} \lambda & 0 & 0 \\ b_1 & b_2 & 1 \\ c_1 & c_2 & 1 \end{vmatrix} + \begin{vmatrix} a_1 & a_2 & 1 \\ b_1 & b_2 & 1 \\ c_1 & c_2 & 1 \end{vmatrix} = 0.$$

Hence 
$$AA_1 = \frac{-1}{(b_2 - c_2)} \cdot \begin{vmatrix} a_1 & a_2 & 1 \\ b_1 & b_2 & 1 \\ c_1 & c_2 & 1 \end{vmatrix}.$$

The points  $B_1$  and  $C_1$  of the definition of measure are  $(0, b_2)$  and  $(0, c_2)$ , respectively, so that

$$B_1C_1 = c_2 - b_2.$$

Hence

$$(10) \quad m(ABC) = \begin{vmatrix} a_1 & a_2 & 1 \\ b_1 & b_2 & 1 \\ c_1 & c_2 & 1 \end{vmatrix}.$$

That the same result holds good in case  $BC$  is parallel to  $OP$  is readily verified.

Now if  $A, B, C$  are transformed to  $A', B', C'$  respectively by a transformation

$$(11) \quad \begin{aligned} x' &= \alpha_1 x + \beta_1 y + \gamma_1, \\ y' &= \alpha_2 x + \beta_2 y + \gamma_2, \end{aligned} \quad \Delta = \begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix} \neq 0$$

of the affine group,

$$(12) \quad m(A'B'C') = \begin{vmatrix} \alpha_1 a_1 + \beta_1 a_2 + \gamma_1 & \alpha_2 a_1 + \beta_2 a_2 + \gamma_2 & 1 \\ \alpha_1 b_1 + \beta_1 b_2 + \gamma_1 & \alpha_2 b_1 + \beta_2 b_2 + \gamma_2 & 1 \\ \alpha_1 c_1 + \beta_1 c_2 + \gamma_1 & \alpha_2 c_1 + \beta_2 c_2 + \gamma_2 & 1 \end{vmatrix} \\ = \begin{vmatrix} a_1 & a_2 & 1 \\ b_1 & b_2 & 1 \\ c_1 & c_2 & 1 \end{vmatrix} \cdot \begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix}.$$

Hence we have

**THEOREM 47.** *A transformation (11) of the affine group is equiaffine if and only if\**

$$\begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix} = 1.$$

Let  $A = (a_1, a_2)$ ,  $B = (b_1, b_2)$ ,  $C = (c_1, c_2)$  be the vertices of any triangle, and  $P = (x, y)$  any point. In the homogeneous coördinates for which  $x_1/x_0 = x$ ,  $x_2/x_0 = y$ , these points may be written  $A = (1, a_1, a_2)$ , etc. Hence by the result established in § 27 for the three-dimensional case, the numbers proportional to

$$\xi_0 = \begin{vmatrix} 1 & x & y \\ 1 & b_1 & b_2 \\ 1 & c_1 & c_2 \end{vmatrix}, \quad \xi_1 = \begin{vmatrix} 1 & a_1 & a_2 \\ 1 & x & y \\ 1 & c_1 & c_2 \end{vmatrix}, \quad \xi_2 = \begin{vmatrix} 1 & a_1 & a_2 \\ 1 & b_1 & b_2 \\ 1 & x & y \end{vmatrix}$$

may be regarded as homogeneous coördinates of  $P$  in a system for which  $ABC$  is the triangle of reference.

\* By comparison with § 30 this condition yields the result that, in an ordered space, the equiaffine collineations are all direct.

This is a particular one of the homogeneous coördinate systems for which  $ABC$  is the triangle of reference, and of course corresponds to a particular choice of the point  $(1, 1, 1)$ . Other particular systems may be obtained by replacing  $(1, a_1, a_2)$  by  $(k, ka_1, ka_2)$  and like changes. The coördinates written down, however, have (in view of (10)) the remarkable property that

$$\xi_0 = m(PBC), \quad \xi_1 = m(APC), \quad \xi_2 = m(ABP).$$

Also, in view of Ex. 2, § 49, they satisfy the condition

$$\xi_0 + \xi_1 + \xi_2 = m(ABC)$$

for all ordinary points  $P$ . If  $ABC$  be taken as the unit of measure, this condition assumes the form

$$\xi_0 + \xi_1 + \xi_2 = 1.$$

Since all ordinary points satisfy this condition, the equation

$$\xi_0 + \xi_1 + \xi_2 = 0,$$

which can always be satisfied by properly chosen homogeneous coördinates, must represent  $l_\infty$ . Therefore the point  $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ , which is polar to  $l_\infty$  relatively to the triangle  $ABC$ , must be the point of intersection of the medians of this triangle.

DEFINITION. Given a homogeneous coördinate system with respect to which the line at infinity has the equation

$$x_0 + x_1 + x_2 = 0,$$

the three numbers  $x_0, x_1, x_2$ , which are homogeneous coördinates of an ordinary point  $P$  and satisfy the condition

$$x_0 + x_1 + x_2 = 1,$$

are called the *barycentric coördinates* of  $P$ , relative to the triangle  $x_0 = 0, x_1 = 0, x_2 = 0$ .

### EXERCISES

1. Defining the barycentric coördinates of a point  $P$ , relative to a triangle  $ABC$ , as

$$\xi_0 = \frac{m(ABP)}{m(ABC)}, \quad \xi_1 = \frac{m(BCP)}{m(ABC)}, \quad \xi_2 = \frac{m(CAP)}{m(ABC)},$$

prove that a line is represented by a linear equation.

2. If  $A, B, C, D$  are four fixed points of a conic, and  $P$  a variable point, the ratio

$$\frac{m(ABP) \cdot m(CDP)}{m(ADP) \cdot m(CBP)}$$

is constant (cf. Cor. 1, Theorem 89).

3. Show that the equation of a conic through five points  $A, B, C, D, E$  may be written in the form

$$(ADE)(BCE)(ABX)(CDX) - (ABE)(CDE)(ADX)(BCX) = 0,$$

where  $(ADE)$  stands for

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ d_1 & d_2 & d_3 \\ e_1 & e_2 & e_3 \end{vmatrix},$$

and the other parenthetical triads have analogous meanings.

**\*52. Line reflections.** DEFINITION. A homology of period two whose center is on  $l_2$  is called a *line reflection*; if its center is  $L$  and its axis  $l$ , we shall denote the line reflection by  $\{Ll\}$ .

This definition could also be expressed by saying that a line reflection is a transformation having an axis such that (1) if  $P'$  be the transform of a point  $P$  and  $P \neq P'$ , the mid-point of the pair  $PP'$  is on the axis of the reflection; and (2) if  $P_1$  and  $P'_1$  are any other pair of homologous points, the line  $P_1P'_1$  is parallel to  $PP'$ .

**THEOREM 48.** *A product of two line reflections is an equiaffine collineation.*

*Proof.* Let the given line reflections be  $\{L_1l_1\}$  and  $\{L_2l_2\}$ . Let  $l$  be any line meeting both  $l_1$  and  $l_2$ , and let  $L$  be any point at infinity not on  $l$ . Then

$$\{L_2l_2\} \cdot \{L_1l_1\} = \{L_2l_2\} \cdot \{Ll\} \cdot \{Ll\} \cdot \{L_1l_1\}.$$

Let  $A$  be the point of intersection of  $l$  and  $l_1$ ,  $B$  any other point of  $l_1$ ,  $C$  any other point of  $l$ ,  $C_1$  the point to which  $C$  is transformed by  $\{L_1l_1\}$ , and  $O$  the point in which the line  $CC_1$  meets  $l_1$ . Since  $O$  is the mid-point of  $CC_1$ , Theorem 35

gives in case  $A \neq O \neq B$ ,

$$COB \simeq C_1BO,$$

$$CAO \simeq C_1OA.$$

Since  $CAO + COB = CAB$ ,

and  $C_1BO + C_1OA = C_1BA$ ,

it follows that

$$CAB \simeq C_1BA.$$

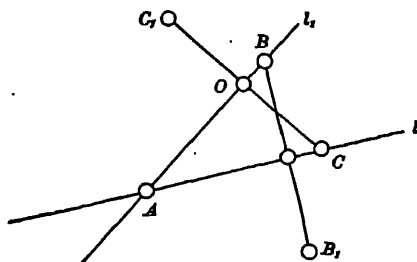


FIG. 41

In case  $A=O$  or  $O=B$  the same result follows directly from Theorem 35.

In like manner, if  $B_1$  be the point to which  $B$  is transformed by  $\{Ll\}$ ,

$$CAB \simeq CB_1A.$$

Hence

$$C_1BA \simeq CB_1A.$$

The product  $\{Ll\} \cdot \{L_1l_1\}$  transforms  $C_1BA$  to  $CB_1A$  and is therefore an equiaffine collineation. In like manner,  $\{L_2l_2\} \cdot \{Ll\}$  is also equiaffine. Hence the product  $\{L_2l_2\} \cdot \{L_1l_1\}$  is equiaffine.

**THEOREM 49.** *An equiaffine collineation is a product of two line reflections.*

*Proof.* Let  $\Gamma$  be any equiaffine collineation. If there be any point which is not on an invariant line of  $\Gamma$ , let  $A_1$  be such a point. Let  $A_0, A_2, A_3$  be defined by the conditions

$$\Gamma(A_0) = A_1, \quad \Gamma(A_1) = A_2, \quad \Gamma(A_2) = A_3.$$

By the hypothesis on  $A_1$  the points  $A_0, A_1, A_3$  are noncollinear, and by the hypothesis that  $\Gamma$  is equiaffine

$$A_0A_1A_2 \simeq A_1A_2A_3 \simeq A_3A_1A_2.$$

Hence, by Theorem 41, the line  $A_0A_3$  is parallel to  $A_1A_2$ , or else  $A_0 = A_3$ .

Let  $M_1$  be the mid-point of the pair  $A_0A_3$ , and  $M_2$  of the pair  $A_1A_2$ . Let  $L_1$  be the point at infinity of the line  $A_0A_3$ ,  $L_2$  of the line  $A_1A_2$ , and  $K$  of the line  $A_1A_3$ . Since  $A_0A_3$  is parallel to  $A_1A_2$ , it follows that

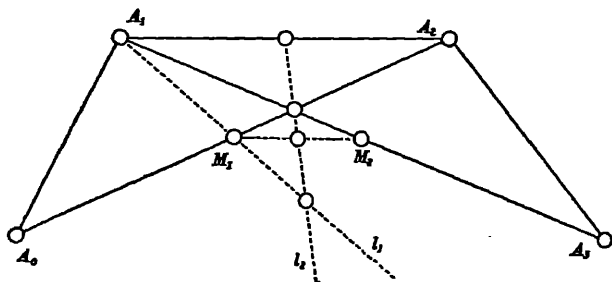


FIG. 42

$A_0A_3L_1 \stackrel{L_2}{\wedge} A_3A_1K$ , and hence, by the definition of mid-point, that  $M_1, M_2$ , and  $L_2$  are collinear. Since  $A_0, A_2$ , and the point at infinity of the line  $A_0A_2$  are transformed by  $\Gamma$  to  $A_1, A_3$ , and the point at infinity of the line  $A_1A_3$ ,  $\Gamma(M_1) = M_2$ .

Let  $l_1$  be the line  $A_1M_1$ , and  $l_2$  the line joining the mid-point of  $A_1A_2$  to the mid-point of  $A_1M_2$ . By the above,

$$\{L_1l_1\}(A_0A_1M_1) = A_2A_1M_1,$$

and

$$\{L_2l_2\}(A_3A_1M_1) = A_1A_2M_2.$$

Hence

$$\{L_2l_2\} \cdot \{L_1l_1\}(A_0A_1M_1) = A_1A_2M_2.$$

But since  $\Gamma(A_0A_1M_1) = A_1A_2M_2$ , it follows, by Theorem 1, that

$$\Gamma = \{L_2l_2\} \cdot \{L_1l_1\}.$$

In case there is no point not on an invariant line of  $\Gamma$ , the invariant lines all meet in a point  $O$ . For the point of intersection of any two of them is invariant, and any three nonconcurrent ordinary lines have at least two ordinary points in common. Thus we should be led to a contradiction with Theorem 46 if the invariant lines were not concurrent.

Let  $A_1$  be a point which is not invariant, and let  $A_2 = \Gamma(A_1)$ . Also let  $B_1$  be another point which is not invariant and not on the line  $A_1A_2$ , and let  $\Gamma(B_1) = B_2$ . The lines  $A_1A_2$  and  $B_1B_2$  necessarily meet in  $O$ .

If  $O$  is ordinary, then since any line through it is invariant, all points of  $l_\infty$  are invariant, and hence  $A_1B_1$  is parallel to  $A_2B_2$ . Since  $\Gamma$  is equiaffine,

$$A_1B_1O \simeq A_2B_2O.$$

Hence, by Theorem 42,  $A_1B_2$  and  $A_2B_1$  are parallel, and  $A_1B_1A_2B_2$  is a parallelogram. Hence  $O$  is the mid-point of  $A_1A_2$  and  $B_1B_2$ , and  $\Gamma$  is a point reflection.

Let  $a$  be the line  $A_1A_2$  and  $A$  the point at infinity of  $a$ , and let  $b$  be the line  $B_1B_2$  and  $B$  the point at infinity of  $b$ . The product  $\{Ab\} \cdot \{Ba\}$  transforms  $A_1, B_1, O$  into  $A_2, B_2, O$  respectively, and hence is  $\Gamma$ .

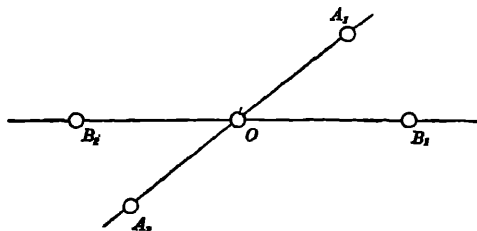


FIG. 48

If  $O$  is an ideal point, let  $l$  be the line  $A_1B_1$ , and let  $m$  be the line joining the mid-points of  $A_1A_2$  and  $B_1B_2$ . Then  $\{Om\} \cdot \{Ol\}$  transforms  $O, A_1, B_1$  into  $O, A_2, B_2$  respectively, and hence, by Theorem 46, is  $\Gamma$ .

**COROLLARY 1.** *An equiaffine collineation  $\Gamma$  such that  $A, \Gamma(A)$  and  $\Gamma^2(A)$  are collinear for all choices of  $A$  is either a point reflection or a translation or an elation whose center is at infinity and whose axis is an ordinary line.*

*Proof.* In the argument above it was proved that if the point  $O$  is ordinary,  $\Gamma$  is a point reflection; and that if  $O$  is ideal,  $\Gamma = \{Om\} \cdot \{Ol\}$ . If  $m$  and  $l$  are parallel,  $\Gamma$  is evidently a translation; and if  $m$  and  $l$  are not parallel, it is an elation with  $O$  as center and the line joining  $O$  to the point  $lm$  as axis.

DEFINITION. An elation whose center is at infinity and whose axis is an ordinary line is called a *simple shear*.

COROLLARY 2. If  $\Gamma = \{L_2 l_2\} \cdot \{L_1 l_1\}$ , then for every line  $l$  concurrent with  $l_1$  and  $l_2$  which is not a double line of  $\Gamma$  there exist points  $L$  and  $M$  and a line  $m$  such that

$$\Gamma = \{Mm\} \cdot \{Ll\}.$$

There also exist a point  $M'$  and a line  $m'$  such that

$$\Gamma = \{Ll\} \cdot \{M'm'\}.$$

If  $l$  be taken as variable,

$$[l] \wedge [L] \wedge [m] \wedge [M] \wedge [M'] \wedge [m'].$$

*Proof.* The first conclusion follows from the arbitrariness in the choice of  $A_1$  in the proof of the theorem above. The second conclusion follows from the first, combined with the fact that

$$\Gamma^{-1} = \{L_1 l_1\} \cdot \{L_2 l_2\}.$$

The projectivities follow from the constructions given in the proof of theorem for  $A_0, A_2, M$ , etc.

COROLLARY 3. If  $\Gamma = \{L_2 l_2\} \cdot \{L_1 l_1\}$ , then for every point  $L$  of  $l_\infty$  which is not a double point of  $\Gamma$ , there exists a point  $M$  of  $l_\infty$  and two lines  $l$  and  $m$  concurrent with  $l_1$  and  $l_2$  such that

$$\Gamma = \{Mm\} \cdot \{Ll\}.$$

There also exist a point  $M'$  and a line  $m'$  such that

$$\Gamma = \{Ll\} \cdot \{M'm'\}.$$

THEOREM 50. The set of all affine collineations which are products of line reflections form a group. Every transformation of this group is either an equiaffine transformation or the product of an equiaffine transformation by a line reflection.

*Proof.* By Theorems 48 and 49 the product of an even number of line reflections is equiaffine and reduces to a product of two line reflections. Hence the product of an odd number of line reflections reduces to a product of three line reflections. The statements above follow in an obvious way from this.

## EXERCISES

1. Let the points at infinity of  $l_1, l_2, l$  respectively in Theorem 49, Cor. 2, be denoted by  $L'_1, L'_2, L'$ . If the points  $L_1, L'_1, L_2, L'_2$  are distinct, the pairs  $L_1L'_1, L_2L'_2, LL'$  are in involution.

2. In case  $L_1$  is on  $l_2$  and  $L_2$  is not on  $l_1$ ,  $\{L_2l_2\} \cdot \{L_1l_1\} = T$  is a collineation of Type II (cf. § 40, Vol. I), parabolic on  $l_\infty$  and of period two on the line joining  $L_1$  to the point of intersection of  $l_1$  and  $l_2$ . If  $l$  be any line, except  $l_2$ , through the point  $l_1l_2$ ,  $P$  the point in which  $l$  meets  $l_\infty$ , and  $L$  the harmonic conjugate of  $L_1$  with respect to  $P$  and  $T(P)$ ,

$$T = \{Ll_2\} \cdot \{L_1l_1\}.$$

If  $M$  be the harmonic conjugate of  $L_1$  with respect to  $P$  and  $T^{-1}(P)$ ,  $T = \{L_1l_1\} \cdot \{Ml_2\}$ .

3. The product  $\{L_2l_2\} \cdot \{L_1l_1\}$  is a point reflection if and only if  $L_1$  is on  $l_2$  and  $L_2$  on  $l_1$ . A point reflection with  $O$  as center is the product of any two line reflections  $\{L_1l_1\}$  and  $\{L_2l_2\}$  for which  $l_1$  is on  $O$ ,  $l_2$  on  $O$ ,  $L_1$  on  $l_2$ , and  $L_2$  on  $l_1$ .

4. The product  $\{L_2l_2\} \cdot \{L_1l_1\}$  is a translation if and only if  $L_1 = L_2$  and  $l_1$  is parallel to  $l_2$ . The ideal point  $L_1$  is the center of the translation. If  $T$  is any translation,  $T_\infty$  its center,  $P_1$  any ordinary point,  $P = T(P_1)$ ,  $P_2$  the mid-point of the pair  $PP_1$ , and  $p_1$  and  $p_2$  two parallel lines through  $P_1$  and  $P_2$  respectively,  $T = \{T_\infty, p_2\} \cdot \{T_\infty, p_1\}$ .

5. The product  $\{L_2l_2\} \cdot \{L_1l_1\}$  is a simple shear if  $L_1 \neq L_2$  and  $l_1 = l_2$ , or if  $L_1 = L_2$  and  $l_1$  intersects  $l_2$  in an ordinary point, but not in any other case.

6. Let  $\Sigma$  be a simple shear whose axis is  $l$  and whose center is  $L$ . Let  $P_1$  be any point of  $l_\infty$ ,  $P = \Sigma(P_1)$ , and  $P_2$  the harmonic conjugate of  $L$  with respect to  $P$  and  $P_1$ . Then  $\Sigma = \{P_2l\} \cdot \{P_1l\}$ . If  $p_1$  be any line meeting  $l$  in an ordinary point,  $p = \Sigma(p_1)$ , and  $p_2$  the harmonic conjugate of  $l$  with respect to  $p$  and  $p_1$ ,

$$\Sigma = \{Lp_2\} \cdot \{Lp_1\}.$$

7. Let  $PP_1P_2P_3P_4$  be a simple pentagon. Let  $C_1, C_2, C_3, C_4, C_5$  be the mid-points of the pairs  $PP_1, P_1P_2, P_2P_3, P_3P_4, P_4P$  respectively. If the line  $PP_1$  is parallel to  $P_3P_4$ , and  $PP_4$  is parallel to  $P_1P_2$ , the three lines  $C_1C_4, C_2C_5, PC_3$  are concurrent or parallel. Discuss the degenerate cases.

8. Every equiaffine transformation is either the identity or a point reflection or an elation whose center is at infinity (i.e. a translation or a simple shear) or expressible as a product of two elations whose centers are at infinity.

9. Prove Cors. 2 and 3 of Theorem 49 directly, without using the theory of equivalence.

10. A necessary and sufficient condition that a planar collineation be the product of two harmonic homologies is that it transform ordered point triads into equivalent point triads relative to a fixed line of the collineation regarded as  $l_\infty$  (E. B. Wilson, *Annals of Mathematics*, Vol. V, 2d series (1903), p. 45).

11. Let us denote an involution whose double points are  $L$  and  $M$  by  $\{LM\}$ . If  $I_1 = \{L_1M_1\}$  and  $I_2 = \{L_2M_2\}$  are two distinct involutions on the same line, then for every point  $L_3$  of this line,  $L_3$  not being a double point of  $I_1 \cdot I_2$ , there exists a unique point  $M_3$  and involution  $\{L_3M_3\}$  such that if we denote  $\{L_3M_3\}$  by  $I_3$  and  $\{L_4M_4\}$  by  $I_4$ ,

$$I_3I_2I_1 = I_4, \text{ and } I_2I_1 = I_3I_4.$$

The pairs  $L_1M_1$ ,  $L_2M_2$ ,  $L_3M_3$ ,  $L_4M_4$  are all pairs of the same involution, unless the pairs  $L_1M_1$  and  $L_2M_2$  have a point in common, in which case all four pairs have this point in common.

12. The projectivities on a line which are expressible in the form  $\{L_1M_1\} \cdot \{L_2M_2\}$  form a group.

The last two exercises connect with the following algebraic considerations. An involution in a net of rationality is always of the form (§ 54, Vol. I)

$$x' = \frac{ax + b}{cx - a},$$

where  $a, b, c, d$  are rational. The double points are the roots of

$$cx^2 - 2ax - b = 0,$$

and both will be rational if  $k$  is rational in

$$a^2 + bc = k^2.$$

Now any projectivity is the product of two involutions, a double point of one of which may be chosen arbitrarily. The projectivity may therefore be written

$$x'' = \frac{a' \frac{ax + b}{cx - a} + b'}{c' \frac{ax + b}{cx - a} - a'} = \frac{(aa' + b'c)x + (a'b - ab')}{(ac' - a'c)x + (bc' + aa')},$$

and so has the determinant

$$\begin{aligned} aa'bc' + a^2a'^2 + bb'cc' + b'caa' - (aa'bc' - a^2b'c' - a'^2bc + aa'b'c) \\ = a^2(a'^2 + b'c') + bc(b'c' + a'^2) = k^2k'^2, \end{aligned}$$

where  $k'^2 = a'^2 + b'c'$ . Hence (1) the product of two involutions whose double points have rational coordinates is a projectivity whose determinant is a perfect square; and (2) if the determinant of a projectivity is a perfect square, and one of two involutions of which it is a product has rational double points, then the other has rational double points. Hence there is a subgroup of the group of collineations of a linear net of rationality generated by the involutions with rational double points. This is the group of transformations whose determinants are perfect squares.

\*53. Algebraic formulas for line reflections. Let us employ the nonhomogeneous coördinates for which  $l_\infty$  is the singular line and the corresponding homogeneous coördinates for which

$$\frac{x_1}{x_0} = x, \quad \frac{x_2}{x_0} = y.$$

The line  $l_\infty$  now has the equation  $x_0 = 0$ , and the equations (1) of the affine group become

$$(13) \quad \begin{aligned} x'_0 &= x_0, \\ x'_1 &= c_1 x_0 + a_1 x_1 + b_1 x_2, \\ x'_2 &= c_2 x_0 + a_2 x_1 + b_2 x_2, \end{aligned} \quad \Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0.$$

On the line  $l_\infty$  this effects the transformation

$$\begin{aligned} x'_1 &= a_1 x_1 + b_1 x_2, \\ x'_2 &= a_2 x_1 + b_2 x_2. \end{aligned}$$

According to § 54, Vol I, this is an involution if and only if  $a_1 = -b_2$ . Thus  $a_1 = -b_2$  is a necessary condition that (13) represent a line reflection.

The ordinary double points of (13) are given by the following equations, in which we have put  $a = a_1 = -b_2$ .

$$(14) \quad \begin{aligned} (a-1)x + b_1 y + c_1 &= 0, \\ a_2 x - (a+1)y + c_2 &= 0. \end{aligned}$$

If (13) is to be a line reflection, it must have a line of fixed points. Hence the two equations (14) must represent a single ordinary line, which requires

$$(15) \quad 0 = \begin{vmatrix} a-1 & b_1 \\ a_2 & -(a+1) \end{vmatrix} = \begin{vmatrix} a-1 & c_1 \\ a_2 & c_2 \end{vmatrix} = \begin{vmatrix} b_1 & c_1 \\ -(a+1) & c_2 \end{vmatrix}.$$

The first of these conditions is equivalent to  $\Delta = -1$ .

Since the coefficients of  $x$  and  $y$  in (14) cannot all vanish, the conditions (15) are also sufficient that (14) represent a single ordinary line. Hence

**THEOREM 51.** *A transformation of the form*

$$(16) \quad \begin{aligned} x' &= ax + b_1 y + c_1, \\ y' &= a_2 x - ay + c_2, \end{aligned}$$

*is a line reflection if and only if*

$$\Delta = \begin{vmatrix} a & b_1 \\ a_2 & -a \end{vmatrix} = -1, \quad \begin{vmatrix} a-1 & c_1 \\ a_2 & c_2 \end{vmatrix} = \begin{vmatrix} b_1 & c_1 \\ -(a+1) & c_2 \end{vmatrix} = 0.$$

From this it follows that a product of two line reflections is such that  $\Delta = 1$ , and a product of three line reflections is such that  $\Delta = -1$ . By Theorems 47 and 49 any transformation for which  $\Delta = 1$  is a product of two line reflections. Any transformation  $T$  for which  $\Delta = -1$ , when multiplied by a line reflection  $\Lambda$  yields a transformation  $\Sigma$  for which  $\Delta = 1$ , i.e. an equiaffine transformation. From  $T\Lambda = \Sigma$  follows  $T = \Sigma\Lambda$ . Hence  $T$  is a product of three line reflections. Thus we have (cf. Theorem 47)

**THEOREM 52.** *The group of affine transformations which are products of line reflections has the equations*

$$\begin{aligned} x' &= a_1x + b_1y + c_1, \\ y' &= a_2x + b_2y + c_2, \end{aligned} \quad \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 = 1.$$

### EXERCISES

1. The set of all affine transformations which are products of equiaffine transformations by dilations form a group which is a self-conjugate subgroup of the affine group. Its equations are

$$\begin{aligned} x' &= a_1x + b_1y + c_1, \\ y' &= a_2x + b_2y + c_2, \end{aligned} \quad \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = k^2,$$

where  $k$  is any number in the geometric number system.

2. The set of all affine transformations which are products of line reflections and dilations form a group which is self-conjugate under the affine group. Its equations are

$$\begin{aligned} x' &= a_1x + b_1y + c_1, \\ y' &= a_2x + b_2y + c_2, \end{aligned} \quad \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \pm k^2,$$

where  $k$  is any number in the geometric number system.

**54. Subgroups of the affine group.** We give below a list of the principal subgroups of the affine group which we have considered in this chapter and in § 30 of Chap. II. These are all self-conjugate subgroups. We also include the groups which will be considered in the next chapter in connection with the Euclidean geometry.

The groups are all described by means of the conditions which must be imposed on the coefficients of the equations of the affine group to reduce it to each of the other groups. In some spaces, i.e. when the variables and coefficients are in certain number systems, these groups are not all distinct. However, they are all distinct in case the variables and coefficients are ordinary rational numbers.

With respect to a system of nonhomogeneous coördinates of which  $l_{\infty}$  is the singular line, the equations of the affine group are

$$(1) \quad \begin{aligned} x' &= a_1 x + b_1 y + c_1 \\ y' &= a_2 x + b_2 y + c_2, \end{aligned}$$

where

$$\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0.$$

The principal subgroups connected with the affine geometry are:

$$(2) \quad \Delta > 0;$$

the transformations satisfying this condition are direct (§ 30).

$$(3) \quad \Delta = k^2,$$

where  $k$  is in the geometric number system (§ 53, Ex. 1).

$$(4) \quad \Delta = \pm k^2,$$

where  $k$  is in the geometric number system (§ 53, Ex. 2).

$$(5) \quad \Delta^2 = 1;$$

these are products of two or of three line reflections (Theorem 52).

$$(6) \quad \Delta = 1,$$

the equiaffine group (§ 51).

$$(7) \quad a_2 = b_1 = 0, \quad a_1 = b_2,$$

the homothetic group (§ 47).

$$(8) \quad a_2 = b_1 = 0, \quad a_1 = b_2, \quad a_1^2 = 1,$$

the group of translations and point reflections (§ 45).

$$(9) \quad a_2 = b_1 = 0, \quad a_1 = b_2 = 1,$$

the group of translations (§ 38).

The principal groups connected with the Euclidean geometry are:

$$(10) \quad a_1^2 + a_2^2 = b_1^2 + b_2^2 \neq 0, \quad a_1 b_1 + a_2 b_2 = 0,$$

the Euclidean group (§§ 55 and 62). Its transformations are called similarity transformations.

$$(11) \quad a_1^2 + a_2^2 = b_1^2 + b_2^2 \neq 0, \quad a_1 b_1 + a_2 b_2 = 0, \quad \Delta > 0,$$

the direct similarity transformations.

$$(12) \quad a_1^2 + a_2^2 = b_1^2 + b_2^2 \neq 0, \quad a_1 b_1 + a_2 b_2 = 0, \quad \Delta = k^2,$$

where  $k$  is in the geometric number system.

$$(13) \quad a_1^2 + a_2^2 = b_1^2 + b_2^2 \neq 0, \quad a_1 b_1 + a_2 b_2 = 0, \quad \Delta = \pm k^2,$$

where  $k$  is in the geometric number system.

$$(14) \quad a_1^2 + a_2^2 = 1, \quad a_1 = \pm b_2, \quad a_2 = \mp b_1,$$

the group of displacements and symmetries (§ 62).

$$(15) \quad a_1^2 + a_2^2 = 1, \quad a_1 = b_3, \quad a_2 = -b_1,$$

the group of displacements.

The relations among these groups may be indicated by the following diagram, in which we have included only those groups which are distinct in case of the real geometry. A dotted line indicates that the lower of the two groups joined is a subgroup of the upper, and a solid line that it is a self-conjugate subgroup.

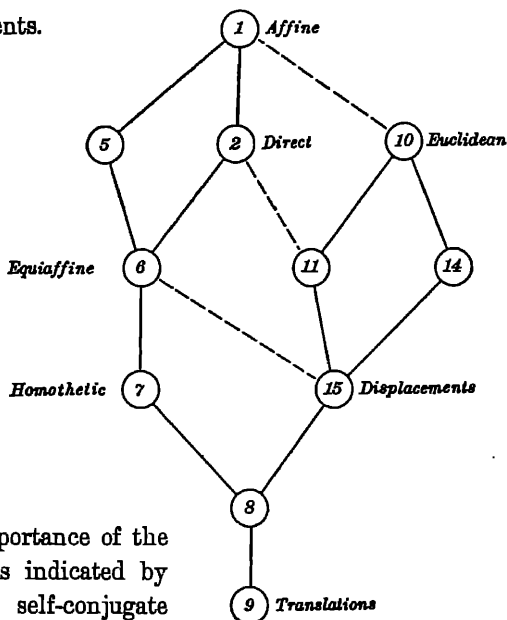


FIG. 44

The fundamental importance of the group of translations is indicated by the fact that it is a self-conjugate subgroup of each of the other groups.

### EXERCISES

1. Supposing the number of points on a line to be  $p + 1$ , what is the number of transformations in each of the groups listed above?

2. Supposing the geometric number system to be (a) the ordinary real, or (b) the ordinary complex number system, how many parameters are there in the equations for each of the groups listed above?

3. Prove that the plane affine geometry as a separate science could be based on the following assumptions with regard to undefined elements, called points, and undefined classes of points, called lines:

I. Two points are contained in one and only one line.

II. For any line  $l$  and any point  $P$ , not on  $l$ , there is one and only one line containing  $P$  and not containing any point of  $l$ .

III. Every line contains at least two points.

IV. There exist at least three noncollinear points.

V. The special case of the Pappus theorem given in the fine print in § 49; or Theorem 41.

## CHAPTER IV

### EUCLIDEAN PLANE GEOMETRY

**55. Geometries of the Euclidean type.** We come now to the extension of the definition of congruence which was promised in §§ 39 and 46. This requires the consideration of groups which are not self-conjugate under the affine group. Not being self-conjugate, these groups are not determined uniquely by the affine group, and hence our definitions will contain a further arbitrary element.

DEFINITION. Let  $I$  be an arbitrary but fixed involution on  $l_\infty$ . This involution shall be called the *absolute* or *orthogonal involution*. The group of all projective collineations leaving  $I$  invariant shall be called a *parabolic\* metric group*. The transformations of the group shall be called *similarity transformations*. Two figures conjugate under the group shall be said to be *similar*. The geometry corresponding to the group shall be called the *parabolic metric geometry*.

The absolute involution is supposed to be fixed throughout the rest of the discussion, but of course there are as many parabolic metric groups as there are choices of  $I$ . We nevertheless speak of *the* parabolic metric group in order to emphasize the fact that we are fixing attention on one group.

In case the plane in which we are working is a real plane and the absolute involution is without double points, the parabolic metric geometry is the Euclidean geometry. It is for this reason that we refer to the parabolic metric geometries as geometries of the Euclidean type.

The investigations in the following sections are arranged in order of increasing specialization. First we consider a perfectly general involution,  $I$ , in a projective plane satisfying  $A, E, P, H_0$ . Then we consider a particular type of involution in an ordered plane, and finally limit the plane to be the real plane.

\* The reason for the term "parabolic" in this connection is explained in a later chapter, where the elliptic and hyperbolic metric groups are defined.

When the plane and the involution are fully specialized, it is a theorem (§ 70) that the real plane is contained in a complex plane in which the absolute involution has double points. Thus the theorems on the general type of involution (where the possible existence of double points is taken into account) come to have a new application.

**56. Orthogonal lines.** DEFINITION. Two lines are said to be *orthogonal* or *perpendicular* to each other if and only if they meet  $l_\infty$  in conjugate points of the absolute involution.

The following consequences of this definition are obvious:

**THEOREM 1.** *The pairs of perpendicular lines through any point,  $O$ , are the pairs of an involution. Through any point there is one and but one line perpendicular to a given line. A line perpendicular to one of two parallel lines is perpendicular to the other. Two lines perpendicular to the same line are parallel.*

DEFINITION. In case the absolute involution  $I$  has two double points,  $I_1$  and  $I_2$ , they are called the *circular points*. Any line through  $I_1$  or  $I_2$  is called an *isotropic line* or a *minimal line*.

Any isotropic line has the property of being perpendicular to itself. The circular points are so called because all ordinary points of any circle (cf. § 60) are on a conic through  $I_1$  and  $I_2$ . The ordinary points of the conic section referred to in the following lemma will later be proved to be on a circle.

DEFINITION. A homology of period two whose center  $L$  is on  $l_\infty$ , and whose axis  $l$  meets  $l_\infty$  in the point conjugate to the center with regard to the absolute involution, is called an *orthogonal line reflection*, and is denoted by  $\{LL\}$ .

Since the center of a homology is not a point of the axis, the center cannot be a double point of the orthogonal involution, nor can the axis pass through such a point. An orthogonal line reflection is of course a special case of a line reflection as defined in § 52.

**LEMMA.** *Let  $O$  and  $P_1$  be two points not collinear with either double point of the absolute involution. There is one and only one conic,  $C^2$ , having  $O$  as center, passing through  $P_1$ , and having the pairs of the absolute involution as pairs of conjugate points.*

*Proof.* Let  $P_2$  be the harmonic conjugate of  $P_1$  with respect to  $O$  and the point at infinity,  $P_\infty$ , of the line  $OP_1$ . Any conic containing  $P_1$

and having  $O$  as center must contain  $P_2$ , by the definition of center. Let  $X$  be a variable point of  $l_\infty$ , and  $Y$  the conjugate of  $X$  in the absolute involution. Any of the triangles  $OX Y$  must be self-polar to any conic satisfying the required conditions. But if  $P$  is the point of intersection of the lines  $P_1 X$  and  $P_2 Y$ , and  $Q$  the point of intersection of  $P_1 X$  and  $O Y$ ,

$$P_2 O P_1 P_x \frac{Y}{\Lambda} P Q P_1 X,$$

and hence the points  $P_1$  and  $P$  are harmonically conjugate with respect to  $X$  and  $Q$ . Hence  $P$  must be on any conic through  $P_1$  with regard to which  $X$  is the pole of  $Q Y$ . Hence  $P$  must be on any conic satisfying the hypotheses of the lemma.

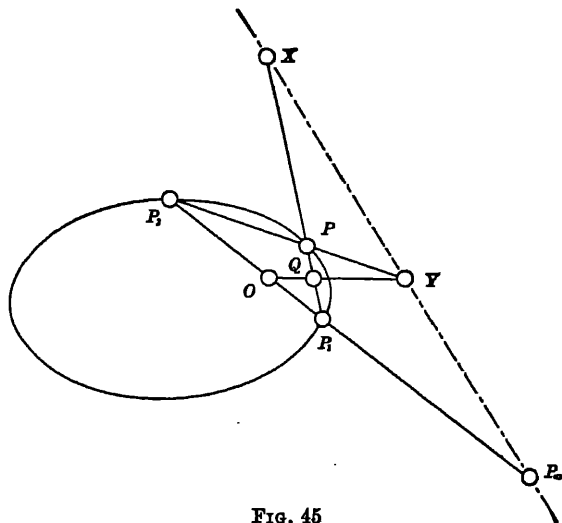


FIG. 45

Since  $P_1[X] \bar{\Lambda} P_2[Y]$ , the points  $P$ , together with  $P_1$  and  $P_2$ , constitute a unique conic (§ 41, Vol. I); and this conic, by its construction, satisfies the condition required by the lemma.

**COROLLARY.** *In case the absolute involution has double points the conic  $C^2$  passes through them.*

**THEOREM 2.** *An orthogonal line reflection leaves the absolute involution invariant.*

*Proof.* If  $l$  is the axis of an orthogonal line reflection and  $L$  its center, let  $O$  be any point on  $l$  and  $P_1$  any point not on  $l$ . The conic  $C^2$  (cf. Lemma), which contains  $P_1$ , has  $O$  as center, and has the absolute involution as an involution of conjugate points, must have  $L$  and  $l$  as pole and polar. Hence, by the definition of pole and polar (§ 44, Vol. I)  $C^2$  is transformed into itself by the harmonic homology having  $L$  and  $l$  as center and axis. Hence the absolute involution is transformed into itself by the orthogonal line reflection  $\{Ll\}$ .

**THEOREM 3.** *The product of two orthogonal line reflections whose axes are parallel is a translation parallel to any line perpendicular to the axes.*

*Proof.* Let the given line reflections be  $\{L_1 l_1\}$  and  $\{L_2 l_2\}$ . Their axes meet in a point  $L'$  of  $l_\infty$ , and  $L_1$  and  $L_2$  must be conjugate to  $L'$  with respect to the absolute involution. Hence  $L_1 = L_2$ . The product therefore leaves all points on  $l_\infty$  invariant and also all lines through  $L_1$ . Hence it is a translation parallel to any line through  $L_1$ .

**THEOREM 4.** *A translation,  $T$ , whose center is not a double point of the absolute involution, is a product of two orthogonal line reflections,  $\{Ll_1\}$ ,  $\{Ll_2\}$ , where  $L$  is the center of the translation. If  $O$  is an arbitrary ordinary point and  $P$  the mid-point of the pair  $O$  and  $T(O)$ ,  $l_1$  may be chosen as  $OL'$  and  $l_2$  as  $PL'$ , where  $L'$  is the conjugate of  $L$  with respect to the absolute involution. Or  $l_1$  may be chosen as  $PL'$  and  $l_2$  as the line joining  $T(O)$  to  $L'$ . A translation whose center is a double point of the absolute involution is a product of four orthogonal line reflections.*

*Proof.* If  $l_1 = OL'$  and  $l_2 = PL'$ , the reflection  $\{Ll_1\}$  leaves  $O$  invariant and  $\{Ll_2\}$  carries  $O$  to  $T(O)$ . Hence the translation  $\{Ll_2\} \cdot \{Ll_1\}$  carries  $O$  to  $T(O)$ , and, by Theorem 3, Chap. III, is identical with  $T$ .

If  $l_1 = PL'$  and  $l_2 = QL'$ , where  $Q = T(O)$ , the reflection  $\{Ll_1\}$  carries  $O$  to  $Q$  and  $\{Ll_2\}$  leaves  $Q$  invariant. Hence, as before,  $\{Ll_2\} \cdot \{Ll_1\} = T$ .

A translation whose center is a double point of the absolute involution can be expressed as a product of two translations with arbitrary points of  $l_\infty$  as centers (Theorem 8, Chap. III), and hence is expressible as a product of four orthogonal line reflections.

**DEFINITION.** If the axes of two orthogonal line reflections intersect in an ordinary point,  $O$ , the product is called a *rotation about  $O$* , and the point  $O$  is called its *center*.

**THEOREM 5.** *A rotation which is the product of two orthogonal line reflections whose axes are orthogonal is a point reflection.*

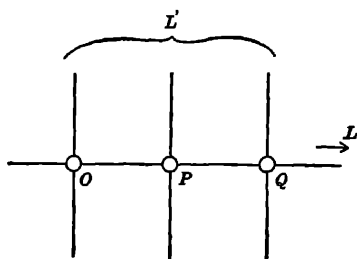


FIG. 46

*Proof.* Let the two line reflections be  $\{L_1l_1\}$  and  $\{L_2l_2\}$  and let  $O$  be the point of intersection of  $l_1$  and  $l_2$ . Since  $l_1$  and  $l_2$  are orthogonal,  $L_1$  is on  $l_2$  and  $L_2$  on  $l_1$ . The product  $\{L_2l_2\} \cdot \{L_1l_1\}$  therefore leaves  $O$  and every point of  $l_\infty$  invariant. Moreover, it is of period two on the axis of either of the line reflections. Hence it is a homology of period two with  $O$  as center and  $l_\infty$  as axis, i.e. a point reflection.

**DEFINITION.** If a line  $l$  is perpendicular to a line  $m$ , the point of intersection of the two lines is called the *foot* of the perpendicular  $l$ . A line  $l$  is said to be the *perpendicular bisector* of a pair of points  $A$  and  $B$  if it is perpendicular to the line  $AB$  and its foot is the mid-point of the pair  $AB$ .

**DEFINITION.** A simple quadrangle  $ABCD$  is said to be a *rectangle* if and only if the lines  $AB$  and  $CD$  are perpendicular to  $AD$  and  $BC$ .

### EXERCISES

1. A parallelogram  $ABCD$  is a rectangle if and only if the lines  $AB$  and  $AD$  are perpendicular.

2. The perpendicular bisectors of the point pairs  $AB$ ,  $BC$ ,  $CA$  of a triangle  $ABC$  meet in a point.

3. The perpendiculars from the vertices of a triangle to the opposite sides meet in a point.

4. The lines through the vertices of a triangle parallel to the transforms of the opposite sides by a fixed orthogonal line reflection are concurrent.

**57. Displacements and symmetries. Congruence.** **DEFINITION.** The product of an even number of orthogonal line reflections is called a *displacement*. The product of an odd number of orthogonal line reflections is called a *symmetry*.

**THEOREM 6.** *The set of all displacements form a self-conjugate subgroup of the parabolic metric group.*

*Proof.* That the displacements form a group is evident because (cf. § 26, Vol. I): (1) the identity is a displacement, being the product of any orthogonal line reflection by itself; (2) the inverse of a product of orthogonal line reflections is the product of the same set of line reflections taken in the reverse order; (3) the product of an even number of orthogonal line reflections by an even number of orthogonal line reflections is, by definition, a displacement.

The group of displacements is contained in the parabolic metric group by Theorem 2.

If  $\{Ll\}$  is an orthogonal line reflection,  $\Sigma$  a similarity transformation, and  $L' = \Sigma(L)$ ,  $l' = \Sigma(l)$ , then  $\Sigma \cdot \{Ll\} \cdot \Sigma^{-1}$  is a harmonic homology with  $L'$  as center and  $l'$  as axis. But since  $L$  and the point at infinity of  $l$  are paired in the absolute involution, so are  $L'$  and the point at infinity of  $l'$ . Hence  $\Sigma \cdot \{Ll\} \cdot \Sigma^{-1} = \{L'l'\}$  is an orthogonal line reflection.

If  $\Lambda_1$  and  $\Lambda_2$  are any two line reflections  $\Sigma\Lambda_1\Lambda_2\Sigma^{-1} = \Sigma\Lambda_1\Sigma^{-1}\Sigma\Lambda_2\Sigma^{-1}$ . A similar argument shows that  $\Sigma\Lambda_1\Lambda_2 \cdots \Lambda_n \cdot \Sigma^{-1}$  is a product of  $n$  orthogonal line reflections whenever  $\Lambda_1, \dots, \Lambda_n$  are orthogonal line reflections and  $\Sigma$  is in the parabolic metric group. Hence the group of displacements is a self-conjugate subgroup of the parabolic metric group.

**COROLLARY 1.** *The set of all displacements and symmetries form a self-conjugate subgroup of the parabolic metric group.*

**DEFINITION.** Two figures such that one can be transformed into the other by a displacement are said to be *congruent*. Two figures such that one can be transformed into the other by a symmetry are said to be *symmetric*.

**COROLLARY 2.** *If a figure  $F_1$  is congruent to a figure  $F_2$ , and  $F_2$  to a figure  $F_3$ , then  $F_1$  is congruent to  $F_3$ .*

**COROLLARY 3.** *If a figure  $F_1$  is symmetric with a figure  $F_2$ , and  $F_2$  is symmetric with a figure  $F_3$ , then  $F_1$  is congruent to  $F_3$ .*

**COROLLARY 4.** *If a figure  $F_1$  is symmetric with a figure  $F_2$ , and  $F_2$  is congruent to a figure  $F_3$ , then  $F_1$  is symmetric with  $F_3$ .*

Since translations and point reflections leave the absolute involution invariant, the definition of congruence given in this section includes the definitions in §§ 39 and 46 as special cases. Theorem 6 shows that the theory of congruence and symmetry in general belongs to the geometry of the parabolic metric group. It must be remembered, however, that the theory of congruence of point pairs on parallel lines belongs to the affine group. In other words, the part of the theory of congruence developed in Chap. III is independent of the choice of the absolute involution.

In case the absolute involution has double points, the theory of congruence of point pairs on the minimal lines (§ 56) is different from that on other lines. As will appear in the following sections the

theory on any line which is not minimal is essentially the same as that developed in Chap. III on the basis afforded by the group of translations and point reflections. On a minimal line, however, the set of points  $[P]$  such that  $OP_0$  is congruent to  $OP$  consists of all points on this line except the point  $O$ . For let  $I_1$  and  $I_2$  denote the double points of the absolute involution,  $I_1$  being the one on the line  $OP_0$ . Let  $Q$  be a point of the line  $OI_2$  distinct from  $O$  and from  $I_2$ , and let  $P$  be any point of  $OI_1$  distinct from  $O$  and from  $I_1$ . If  $\Lambda_1$  be the orthogonal line reflection whose center is the point at infinity of the line  $P_0Q$  and whose axis passes through  $O$ , and  $\Lambda_2$  be the orthogonal line reflection whose center is the point at infinity of the line  $QP$  and whose axis passes through  $O$ , we have  $\Lambda_1(P_0) = Q$  and  $\Lambda_2(Q) = P$ . Hence the rotation  $\Lambda_2\Lambda_1$  transforms  $P_0$  to  $P$ . Combining transformations of the form  $\Lambda_2\Lambda_1$  with translations it is clear that we have

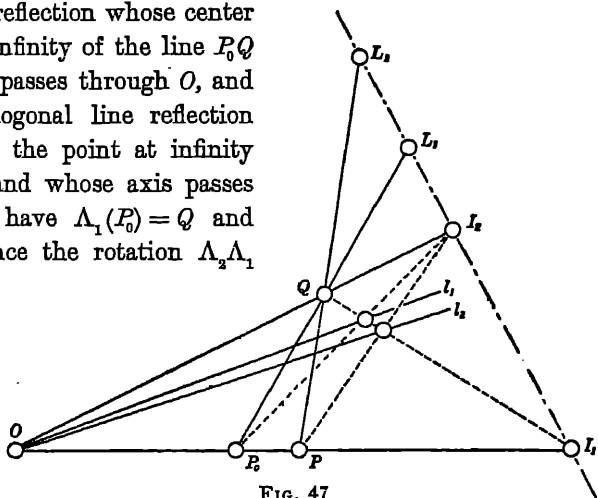


FIG. 47

**THEOREM 7.** *Any pair of points on a minimal line is congruent to any other pair of points on the same line.*

For example, if a mid-point of a pair  $AB$  were defined to be a point  $C$  such that  $AC$  is congruent to  $CB$ , we should have that whenever the line  $AB$  is minimal, the point  $C$  may be any point on this line different from  $A$  and  $B$ . The theorems on mid-points in Chap. III would in general have exceptional cases. It is to avoid this difficulty that we have adopted the definition of mid-point given in § 40, Chap. III. A similar remark applies to the definition of ratio of collinear point pairs in § 43, Chap. III.

**DEFINITION.** A parallelogram  $ABCD$  whose sides do not pass through double points of the absolute involution and in which the point pair  $AB$  is congruent to the point pair  $AD$  is called a *rhombus*. A rhombus which is also a rectangle is called a *square*.

## EXERCISES

1. Prove that the group of displacements and symmetries could be defined as the group of all collineations leaving invariant the set of all conics obtainable by translations from a fixed central conic.

2. The parabolic metric group consists of all projective collineations transforming the group of displacements into itself.

3. Two point pairs on nonminimal lines are symmetric if and only if they are congruent.

4. The perpendicular bisector of a point pair  $AB$  contains all points  $P$  such that  $AP$  is congruent to  $BP$ .

5. The simple quadrangle  $ABCD$  is a rhombus if and only if the lines  $AC$  and  $BD$  are the perpendicular bisectors of the point pairs  $BD$  and  $AC$  respectively.

6. A parallelogram  $ABCD$  is a rectangle if and only if the point pair  $AC$  is congruent to the point pair  $BD$ .

7. Specialize the quadrangle-quadrilateral configuration (§ 18, Vol. I) to the case where the vertices of the quadrangle are the vertices of a square.

**58. Pairs of orthogonal line reflections.** THEOREM 8. *If  $\Lambda_1, \Lambda_2, \Lambda_3$  are three orthogonal line reflections whose axes pass through a point  $O$  (ordinary or ideal), the product  $\Lambda_3\Lambda_2\Lambda_1$  is an orthogonal line reflection whose axis passes through  $O$ .*

*Proof.* In case the three axes are parallel, the product  $\Lambda_3\Lambda_2$  is a translation, and so by Theorem 4 is expressible in the form  $\Lambda_4\Lambda_1$ , where  $\Lambda_4$  is an orthogonal line reflection whose axis is parallel to the other axes. Hence

$$\Lambda_3\Lambda_2\Lambda_1 = \Lambda_4\Lambda_1\Lambda_1 = \Lambda_4.$$

In case two of the axes are not parallel, the third axis must pass

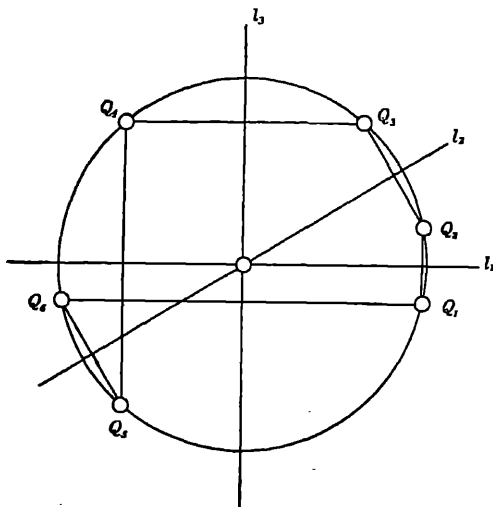


FIG. 48

through their common point  $O$ . Let  $P$  be any point not collinear with  $O$  and a circular point. Let  $C^2$  be the conic, existent and unique according to the lemma of § 56, which passes through  $P$ , has  $O$  as center, and has the absolute involution as an involution of conjugate

points. If  $Q_1$  be any point of  $C^2$ , let  $\Lambda_1(Q_1) = Q_2$ ,  $\Lambda_2(Q_2) = Q_3$ ,  $\Lambda_3(Q_3) = Q_4$ ,  $\Lambda_1(Q_4) = Q_5$ ,  $\Lambda_2(Q_5) = Q_6$ .

According to this construction the line  $Q_1Q_2$  is parallel to  $Q_4Q_5$  and  $Q_2Q_3$  to  $Q_5Q_6$ , where in case  $Q_i = Q_j$ , the line  $Q_iQ_j$  is taken to mean the tangent to  $C^2$  at  $Q_i$ . Hence, by Pascal's theorem (Chap. V, Vol. I) or one of its degenerate cases, it follows that  $Q_3Q_4$  is parallel to  $Q_6Q_1$ . Hence

$$\Lambda_3(Q_6) = Q_1$$

and

$$(\Lambda_3\Lambda_2\Lambda_1)^2(Q_1) = Q_1.$$

Since  $Q_1$  is an arbitrary point of  $C^2$ ,

$$(\Lambda_3\Lambda_2\Lambda_1)^2 = 1.$$

The transformation  $\Lambda_3\Lambda_2\Lambda_1$  is not the identity, because it cannot leave invariant a point, different from  $O$ , of the axis of  $\Lambda_1$  unless  $\Lambda_2 = \Lambda_3$ , and in the latter case the product is equal to  $\Lambda_1$ . Since  $\Lambda_3\Lambda_2\Lambda_1$  leaves invariant the line  $Q_1Q_4$  (or the tangent at  $Q_1$ , if  $Q_1 = Q_4$ ), it leaves invariant the point at infinity of this line and also the line through  $O$  perpendicular to it. As  $\Lambda_3\Lambda_2\Lambda_1$  is of period two, it follows that it is an orthogonal line reflection.

**COROLLARY 1.** *If  $\Lambda_1$ ,  $\Lambda_2$ , and  $\Lambda_3$  are any three orthogonal line reflections whose axes meet in a point or are parallel, there exists an orthogonal line reflection  $\Lambda_4$  such that  $\Lambda_2\Lambda_1 = \Lambda_3\Lambda_4$ , and an orthogonal line reflection  $\Lambda_5$  such that  $\Lambda_2\Lambda_1 = \Lambda_5\Lambda_3$ .*

*Proof.* By the theorem,  $\Lambda_4$  exists such that

$$\Lambda_3\Lambda_2\Lambda_1 = \Lambda_4.$$

Hence

$$\Lambda_2\Lambda_1 = \Lambda_3\Lambda_4.$$

In like manner,  $\Lambda_5$  exists such that

$$\Lambda_2\Lambda_1\Lambda_3 = \Lambda_5.$$

Hence

$$\Lambda_2\Lambda_1 = \Lambda_5\Lambda_3.$$

**COROLLARY 2.** *The product of any odd number of orthogonal line reflections whose axes meet in a point or are parallel is an orthogonal line reflection.*

*Proof.* By the theorem, whenever  $n \geq 3$ , the product of  $n$  orthogonal line reflections whose axes are concurrent reduces to a product of  $n - 2$ . Thus, if  $n$  is odd, the number of line reflections can be reduced by successive steps to one.

If  $n$  is even, this process reduces the number of line reflections in the product to two. Thus we have

**COROLLARY 3.** *The product of any even number of orthogonal line reflections is a rotation in case their axes meet in a point, and is a translation in case the axes are parallel.*

**COROLLARY 4.** *An orthogonal line reflection is not a displacement.*

**COROLLARY 5.** *The set of all rotations having a common center is a commutative group.*

*Proof.* A rotation is defined as a product of two orthogonal line reflections whose axes meet in an ordinary point. So, by definition, the identity is a rotation, and the inverse of a rotation  $\Lambda_2\Lambda_1$  is the rotation  $\Lambda_1\Lambda_2$ . The product of two rotations is a rotation by Cor. 3. Hence the rotations having a given point as center form a group. To show that any two of these rotations are commutative amounts to showing that

$$(1) \quad \Lambda_4\Lambda_3\Lambda_2\Lambda_1 = \Lambda_2\Lambda_1\Lambda_4\Lambda_3$$

whenever the  $\Lambda$ 's are orthogonal line reflections whose axes concur. By the theorem we have

$$\Lambda_4\Lambda_3\Lambda_2 = \Lambda_2\Lambda_3\Lambda_4,$$

and hence

$$(2) \quad \Lambda_4\Lambda_3\Lambda_2\Lambda_1 = \Lambda_2\Lambda_3\Lambda_4\Lambda_1.$$

But since

$$\Lambda_3\Lambda_4\Lambda_1 = \Lambda_1\Lambda_4\Lambda_3,$$

$$\Lambda_2\Lambda_3\Lambda_4\Lambda_1 = \Lambda_2\Lambda_1\Lambda_4\Lambda_3,$$

which combined with (2) gives (1).

**THEOREM 9.** *Any displacement leaving a point  $O$  invariant is a rotation about  $O$ .*

*Proof.* The given displacement is a product of an even number,  $n$ , of orthogonal line reflections,  $\Lambda_n \cdots \Lambda_1$ . Let  $\Lambda'_i$  be the line reflection whose axis is the line through  $O$  parallel to the axis of  $\Lambda_i$ . Then the product  $T_i = \Lambda_i\Lambda'_i$  is a translation (Theorem 3) and

$$\Lambda_i = T_i\Lambda'_i.$$

Thus

$$\Lambda_n \cdots \Lambda_1 = T_n\Lambda'_n \cdots T_1\Lambda'_1,$$

where each  $T_i$  is a translation. But by Cor. 2, Theorem 11, Chap. III, if  $\Sigma$  is any affine collineation,  $T_i\Sigma = \Sigma T'_i$ , where  $T'_i$  is a translation or the identity. Hence

$$\Lambda_n \cdots \Lambda_1 = \Lambda'_n \cdots \Lambda'_1 T'_n \cdots T'_1.$$

But since  $\Lambda_n \cdots \Lambda_1$  and  $\Lambda'_n \cdots \Lambda'_1$  leave  $O$  invariant, the product  $T'_n \cdots T'_1$  leaves  $O$  invariant, and hence, by Theorem 3, Chap. III, is the identity. Hence

$$\Lambda_n \cdots \Lambda_1 = \Lambda'_n \cdots \Lambda'_1,$$

where  $\Lambda'_1, \dots, \Lambda'_n$  are orthogonal line reflections whose axes pass through  $O$ . By Cor. 3, Theorem 8,  $\Lambda'_n \cdots \Lambda'_1$  is a rotation about  $O$ .

**59. The group of displacements.** THEOREM 10. *Let  $O$  be an arbitrary point. Any displacement can be expressed in the form  $PT$ , where  $P$  is a rotation about  $O$  and  $T$  a translation.*

*Proof.* By precisely the argument used in the last theorem the given displacement can be expressed in the form

$$\Lambda'_{2n} \cdots \Lambda'_1 T'_{2n} \cdots T'_1,$$

where  $\Lambda'_i$  ( $i = 1, \dots, 2n$ ) is an orthogonal line reflection whose axis passes through  $O$ , and  $T'_i$  ( $i = 1, \dots, 2n$ ) is a translation or the identity. The product  $T'_{2n} \cdots T'_1$  is, by Theorem 6, Chap. III, a translation. By Cor. 3, Theorem 8,  $\Lambda'_{2n} \cdots \Lambda'_1$  is a rotation or a translation. Since it leaves  $O$  invariant, it is a rotation.

**COROLLARY 1.** *Any displacement can also be expressed in the form  $T'P'$ , where  $T'$  is a translation and  $P'$  a rotation with  $O$  as center.*

**COROLLARY 2.** *Any symmetry is a product of a line reflection whose axis contains an arbitrary point and a translation.*

**THEOREM 11.** *Any displacement, except a translation having a double point of the absolute involution as center, is a product of two orthogonal line reflections.*

*Proof.* Let  $O$  be an arbitrary point. By the last theorem the given displacement reduces to  $PT$ , where  $T$  is a translation and  $P$  a rotation about  $O$ . If the center,  $L$ , of  $T$  is not a double point of the absolute involution, by Theorem 4,

$$T = \{Ll_2\} \cdot \{Ll_1\},$$

where  $l_1$  and  $l_2$  meet  $l_\infty$  in the conjugate of  $L$  relative to the absolute involution and where  $l_2$  passes through  $O$ . By Cor. 1, Theorem 8, there exists an orthogonal line reflection  $\{Mm\}$  such that

$$P = \{Mm\} \cdot \{Ll_2\}.$$

Hence

$$\begin{aligned} PT &= \{Mm\} \cdot \{Ll_2\} \cdot \{Ll_2\} \cdot \{Ll_1\} \\ &= \{Mm\} \cdot \{Ll_1\}. \end{aligned}$$

If  $P$  is not the identity, it is clear that  $m$  and  $l_1$  cannot be parallel, and hence  $PT$  is a rotation.

In case  $T$  is a translation whose center is a double point of the absolute involution, it can be expressed (Theorem 8, Chap. III) as a product of two translations  $T_1, T_2$  whose centers are not double points of the absolute involution. Hence, if  $P$  is not the identity,  $PT_2$  is a rotation, and thus  $PT_2T_1$  is also a rotation. In case  $P$  is the identity, we have the exceptional case noted in the theorem.

**COROLLARY.** *A displacement is either a rotation or a translation.*

The following two theorems have the same relation to the parabolic metric group and the group of displacements, respectively, that the fundamental theorem of projective geometry (Assumption  $P$ ) has to the projective group on a line.

**THEOREM 12.** *A transformation of the parabolic metric group leaving invariant two ordinary points not collinear with a double point of the absolute involution is either an orthogonal line reflection or the identity.*

*Proof.* Denote the given fixed points by  $O$  and  $P$ , and let  $C^2$  be the conic through  $P$  having  $O$  as center and the absolute involution as an involution of conjugate points. Since  $C^2$  is uniquely determined by these conditions (cf. the lemma in § 56), it is left invariant by the given transformation  $\Gamma$ . Now  $\Gamma$  leaves  $O, P$ , and the point at infinity of the line  $OP$  invariant. Hence the line  $OP$  is point-wise invariant, and every line  $l$  perpendicular to it is transformed into itself. Since  $C^2$  is also invariant and each of the lines perpendicular to  $OP$  meets  $C^2$  in at most two points,  $\Gamma$  is either the identity or of period two. If of period two, it is evidently an orthogonal line reflection.

**THEOREM 13.** *A displacement leaving invariant a point  $O$  and a line  $l$  containing  $O$  but not containing a double point of the absolute involution is either the identity or a point reflection with  $O$  as center.*

*Proof.* Let  $P$  be any ordinary point of  $l$  distinct from  $O$ , and let  $C^2$  be the conic through  $P$  having  $O$  as center and the absolute involution as an involution of conjugate points. A displacement leaving  $O$  invariant, being a product of two orthogonal line reflections whose axes meet in  $O$ , must leave  $C^2$  invariant. Hence it either leaves  $P$  invariant or transforms it into the other point in which the line  $OP$  meets  $C^2$ . In the first case the transformation must, by Theorem 12

and Cor. 4, Theorem 8, reduce to the identity. In the second case the given displacement, which we shall denote by  $\Delta$ , multiplied by the orthogonal line reflection  $\Lambda$  whose axis is the line through  $O$  perpendicular to  $OP$ , leaves  $P$  invariant. Hence, by Theorem 12,

$$\Delta\Lambda = \Lambda',$$

where  $\Lambda'$  is a line reflection having  $OP$  as axis or the identity. Hence

$$\Delta = \Lambda'\Lambda.$$

Since  $\Delta$  cannot be a line reflection,  $\Lambda'$  cannot be the identity. Since the axes of  $\Lambda$  and  $\Lambda'$  are perpendicular,  $\Delta$  is a point reflection.

### EXERCISES

1. A displacement which carries a point  $A$  to a point  $B$  and has a point  $O$  (ordinary or not) as center is, if the line  $OA$  is not minimal, the product of an orthogonal line reflection whose axis is  $OA$  followed by one whose axis is the line joining  $O$  to the mid-point of the pair  $AB$ .

2. If three of the perpendicular bisectors of the point pairs  $AB, BC, CD, DA$  of a simple quadrangle meet in a point, the fourth perpendicular bisector passes through this point.

\*3. Any affine transformation which leaves a central conic invariant is a line reflection whose center and axis are pole and polar with regard to the conic or a product of two such line reflections.

\*4. In case the absolute involution is without double points, the group of displacements can be defined as the group of transformations common to the parabolic metric group and the equiaffine group. Thus two ordered point triads are congruent if they are both equivalent and similar. Develop the theory of congruence on this basis, and show what difficulties arise in case the absolute involution has double points.

**60. Circles.** DEFINITION. A *circle* is the set of all points  $[P]$  such that the point pairs  $OP$ , where  $O$  is a fixed point, are all congruent to a fixed point pair  $OP_0$ , provided that the line  $OP_0$  does not contain a double point of the absolute involution. The point  $O$  is called the *center* of the circle.

Since the displacements form a group, it is clear that  $P_0$  may be any one of the points  $P$ . It has already been proved (§ 57) that if the line  $OP_0$  contained an invariant point of the absolute involution, the set  $[P]$  would consist of all ordinary points, except  $O$ , of the line  $OP_0$ .

**THEOREM 14.** *A circle consists of the ordinary points of a conic section having the pairs of the absolute involution as pairs of conjugate points. The center of the circle is the pole of  $l_\infty$  with respect to the circle.*

*Proof.* Let  $O$  be the center of the circle and  $P_0$  any point of the circle. The circle consists of all points obtainable from  $P_0$  by displacements which leave  $O$  invariant. If one of the line reflections of which each of these displacements is a product be taken to have  $OP_0$  as axis (Cor. 1, Theorem 8), it follows that the circle consists of the points obtainable from  $P_0$  by orthogonal line reflections whose axes pass through  $O$ . But the system of points so obtained is identical by construction with the ordinary points of the conic referred to in the lemma of § 56.

**COROLLARY.** *In case the absolute involution has no double points, every circle is a conic section. In case the circular points exist, they and the points of any circle form a conic section.*

**THEOREM 15.** *The ordinary points of any proper conic, with regard to which the pairs of the absolute involution are pairs of conjugate points, form a circle.*

*Proof.* A conic  $C^2$  with regard to which the pairs of the absolute involution are conjugate points cannot be a parabola, since all points of  $l_\infty$  are conjugate to the point of contact of a parabola. Hence  $C^2$  has an ordinary point  $O$  as center. Let  $P$  be any point of  $C^2$ . By definition there is one and only one circle through  $P$  which has  $O$  as a center. By Theorem 14, this circle is a conic through  $P$  having  $O$  as center and the pairs of the absolute involution as pairs of conjugate points. By the lemma of § 56 there is only one such conic. Hence the circle through  $P$  with  $O$  as center contains the ordinary points of  $C^2$ .

**THEOREM 16.** *Three noncollinear points, no two of which are on a minimal line, are contained in one and only one circle.*

*Proof.* Let the three points be  $P_0, P_1$ , and  $P_2$ . Let  $L_\infty$  be the point at infinity of the line  $P_0P_1$  and  $l$  the perpendicular bisector of the point pair  $P_0P_1$ . The polar of  $L_\infty$  with regard to any circle through  $P_0$  and  $P_1$  must, by Theorem 14, pass through the mid-point of  $P_0P_1$  and the conjugate of  $L_\infty$  in the absolute involution. Hence the polar of  $L_\infty$  with regard to any circle through  $P_0$  and  $P_1$  must be  $l$ . In like manner, the polar of the point at infinity  $M_\infty$  of the line  $P_1P_2$  with regard to any circle containing  $P_1$  and  $P_2$  must be the perpendicular bisector  $m$  of  $P_1P_2$ . Since the points  $P_0, P_1, P_2$  are not collinear,  $l$  and  $m$  intersect in an ordinary point  $O$ , which must be the pole of

$L_\infty M_\infty = l_\infty$  with regard to any circle through  $P_0$ ,  $P_1$ , and  $P_2$ . Since, by definition, there is one and only one circle through  $P$  with  $O$  as center, there cannot be more than one circle through  $P_0$ ,  $P_1$ , and  $P_2$ .

Since the product of the orthogonal line reflection with  $OP_0$  as axis by that with  $l$  as axis transforms the point pair  $OP_0$  into the point pair  $OP_1$ , the circle through  $P_0$  with  $O$  as center contains  $P_1$ . A like argument shows that it contains  $P_2$ . Hence there is one circle containing  $P_0$ ,  $P_1$ , and  $P_2$ .

Observe that we do not prove at this stage

that a circle has a point on every line through its center. This could not be done without further hypotheses on the nature of the plane than we are making at present.

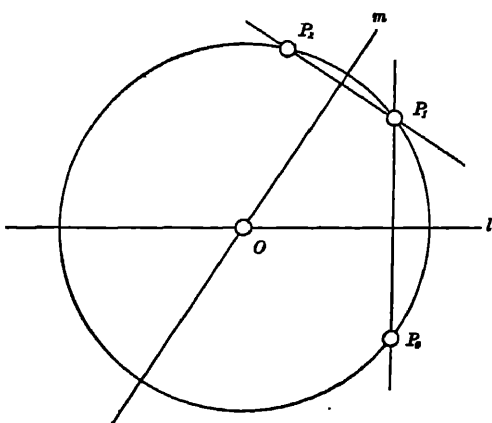


FIG. 49

### EXERCISES

1. The locus of the points of intersection of the lines through a point  $A$  with the perpendicular lines through a point  $B$ , not on a minimal line through  $A$ , is a circle whose center is the mid-point of the pair  $AB$ .

2. A tangent to a circle is perpendicular to the diameter through the point of contact.

3. Any two conjugate diameters of a circle are orthogonal.

4. If the tangents at two points  $A$  and  $B$  of a circle meet in a point  $O$ , the pairs  $OA$  and  $OB$  are congruent.

5. If  $l$  is the perpendicular bisector of a point pair  $AB$ , then the circles through  $A$  and  $B$  meet  $l$  in pairs of an involution whose center (§ 43) is the mid-point of  $AB$ .

6. The system of all circles having a common center meet any line in the pairs of an involution.

7. A parallelogram which circumscribes a circle must be a rhombus.

8. A parallelogram inscribed in a circle is a rectangle.

9. If two circles have two points in common, the pair of tangents at one common point is symmetric to the pair of tangents at the other.

10. The feet of the perpendiculars from any point of a circle to the sides of an inscribed triangle are collinear.

**61. Congruent and similar triangles.** Two of the three fundamental criteria for the congruence of triangles can be derived at the present stage. The third criterion, that in terms of "two sides and the included angle," essentially involves order relations and is given in § 63.

In the following theorems we shall restrict attention to triangles none of whose sides pass through double points of the absolute involution. The sides of a triangle  $ABC$  which are opposite to the vertices  $A, B, C$  are denoted by  $a, b, c$  respectively. It will be observed that instead of angles we refer to ordered line pairs.

**THEOREM 17.** *Two triangles  $ABC$  and  $A'B'C'$  are congruent in such a way that  $A$  corresponds to  $A'$  and  $B$  to  $B'$  if the point pair  $AB$  is congruent to the point pair  $A'B'$  and the ordered line pairs  $ca$  and  $cb$  are congruent to the ordered line pairs  $c'a'$  and  $c'b'$  respectively.*

*Proof.* By hypothesis, there is a displacement  $\Gamma$  carrying  $A$  and  $B$  to  $A'$  and  $B'$  respectively. Let  $\Gamma(a) = a''$ ,  $\Gamma(b) = b''$ , and  $\Gamma(C) = C''$ . If  $a'' \neq a'$ , we should have the ordered line pair  $c'a'$  congruent to  $c'a''$ , and hence there

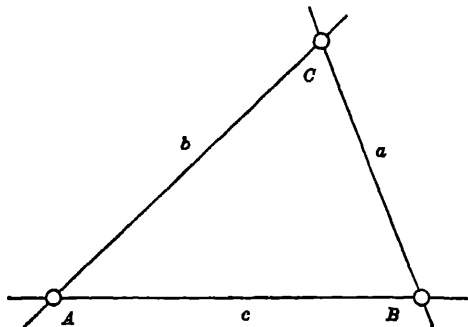


FIG. 50

would be a transformation leaving  $B'$  and  $c'$  invariant and carrying  $a'$  to  $a''$ , but this transformation, by Theorem 13, would be the identity or a point reflection with  $B'$  as center contrary to the assumption that  $a'' \neq a'$ . In like manner it follows that  $b'' = b'$ , and hence that  $C'' = C'$ .

**THEOREM 18.** *If in two triangles  $ABC$  and  $A'B'C'$  the point pairs  $AB, BC, CA$  are congruent, respectively, to  $A'B', B'C', C'A'$ , the pair of lines  $bc$  is congruent to the pair of lines  $b'c'$ . The two triangles are either congruent or symmetric.*

*Proof.* By hypothesis, there is a displacement which carries  $A'B'$  to  $AB$ . Let  $C''$  be the point into which  $C$  is carried by this displacement. Let  $C'''$  be the point to which  $C''$  is carried by the orthogonal line reflection of which  $AB$  is axis. Now if  $C$  were not identical with  $C''$  or  $C'''$ , we should have three congruent point pairs  $AC, AC'', AC'''$  and

three other congruent point pairs  $BC, BC'', BC'''$ . That is, there would be two circles, one with  $A$  as center and one with  $B$  as center, having three points in common. If  $C, C'', C'''$  were collinear, or if two of them were on a minimal line, this would contradict Theorem 14; otherwise it would contradict Theorem 16.

The conclusions of the theorem are now obvious.

The theorems converse to the above are not difficult and are stated in the exercises below. The theorems on similar triangles (Exs. 3, 4, 5) are proved in an analogous way, using Theorem 12 instead of Theorem 13. For these theorems we used the following definition:

**DEFINITION.** Two figures are said to be *directly similar* if and only if one can be transformed into the other by a similarity transformation which effects on  $l_\infty$  the same transformation as some displacement. A transformation of this sort is called a *direct similarity transformation*.

### EXERCISES

1. If two ordered point triads are congruent, the corresponding ordered point pairs and line pairs are congruent.

2. If two ordered point triads are symmetric, the corresponding point pairs are congruent and the corresponding ordered line pairs are symmetric.

3. If the ordered line pairs  $ab, bc, ca$  are congruent, respectively, to the ordered line pairs  $a'b', b'c', c'a'$ , the ordered triad  $abc$  is directly similar to the ordered triad  $a'b'c'$ .

4. If the ordered line pairs  $ab, bc, ca$  are symmetric, respectively, to the ordered line pairs  $a'b', b'c', c'a'$ , the ordered triad  $abc$  is similar to the ordered triad  $a'b'c'$ .

5. If two ordered triads  $abc$  and  $a'b'c'$  are directly similar, the ordered pairs  $ab, bc, ca$  are congruent to  $a'b', b'c', c'a'$  respectively. If the ordered triads are similar but not directly similar, the ordered pairs  $ab, bc, ca$  are symmetric to  $a'b', b'c', c'a'$  respectively.

6. A direct similarity transformation is a product of a displacement and a homology.

**62. Algebraic formulas for certain parabolic metric groups.** Adopting a system of nonhomogeneous coördinates  $(x, y)$  for which  $l_\infty$  is the singular line, and a system of homogeneous coördinates for which

$$\frac{x_1}{x_0} = x, \quad \frac{x_2}{x_0} = y,$$

the line  $l_\infty$  has the equation  $x_0 = 0$ , and any involution on it can be written in the form (§§ 54, 58, Vol. I),

$$x_0 = 0, \quad ax_1\bar{x}_1 + bx_1\bar{x}_2 + bx_2\bar{x}_1 + cx_2\bar{x}_2 = 0.$$

If the coördinate system be chosen so that  $(0, 1, 0)$  and  $(0, 0, 1)$  are conjugate points in this involution, the bilinear equation reduces to

$$(3) \quad ax_1\bar{x}_1 + cx_2\bar{x}_2 = 0.$$

Here the point  $(0, 1, 1)$  is paired with the point  $(0, c, -a)$ . In case the involution contains two pairs of points which are harmonically conjugate, one pair may be chosen as  $(0, 1, 0)$  and  $(0, 0, 1)$  and the other pair as  $(0, 1, 1)$  and  $(0, 1, -1)$ . In that case (3) reduces to

$$(4) \quad x_1\bar{x}_1 + x_2\bar{x}_2 = 0.$$

For the rest of this section we assume that the absolute involution contains two pairs of points which are harmonically conjugate with respect to each other. Such involutions exist in every plane satisfying Assumption  $H_0$ , since any two distinct collinear pairs of points determine an involution. Hence this assumption is no restriction on the nature of the plane in which we are working. It is, moreover, easy to replace the formulas which we shall obtain from (4) by the more general but more cumbersome formulas based on (3).

The equations of the transformation required to change (3) into (4) are

$$x'_0 = x_0, \quad x'_1 = \sqrt{c}x_1, \quad x'_2 = \sqrt{a}x_2.$$

Hence it is clear that in the complex geometry (§ 5) every involution may be reduced to the form (4), and in the real geometry only those involutions can be reduced to this form which are such that  $a/c > 0$ . The involutions of the latter type are direct (§ 18).

The equations of the affine group are

$$(5) \quad \begin{aligned} x'_0 &= x_0, \\ x'_1 &= c_1x_0 + a_1x_1 + b_1x_2, \\ x'_2 &= c_2x_0 + a_2x_1 + b_2x_2, \end{aligned}$$

and if the involution (4) is to be transformed into itself, all pairs  $x_1, x_2$  and  $\bar{x}_1, \bar{x}_2$  which satisfy

$$x_1\bar{x}_1 + x_2\bar{x}_2 = 0$$

must also satisfy

$$(a_1x_1 + b_1x_2)(a_1\bar{x}_1 + b_1\bar{x}_2) + (a_2x_1 + b_2x_2)(a_2\bar{x}_1 + b_2\bar{x}_2) = 0,$$

which is the same as

$$(a_1^2 + a_2^2)x_1\bar{x}_1 + (a_1b_1 + a_2b_2)(x_1\bar{x}_2 + x_2\bar{x}_1) + (b_1^2 + b_2^2)x_2\bar{x}_2 = 0.$$

Hence

$$\begin{cases} a_1^2 + a_2^2 = b_1^2 + b_2^2 \neq 0, \\ a_1b_1 + a_2b_2 = 0, \end{cases}$$

are the necessary and sufficient conditions that (5) leave (4) invariant. Combining these two equations, we obtain

$$a_1^2 a_2^2 + a_2^4 - b_1^2 a_2^2 - a_1^2 b_1^2 = 0$$

or 
$$(a_1^2 + a_2^2)(a_2^2 - b_1^2) = 0.$$

Thus we infer  $a_2 = \pm b_1$  and  $a_1 = \mp b_2$ . Hence

**THEOREM 19.** *The equations of the parabolic metric group are*

$$(6) \quad \begin{aligned} x' &= \alpha x + \beta y + \gamma, \\ y' &= \epsilon(-\beta x + \alpha y) + \gamma, \end{aligned}$$

where  $\epsilon^2 = 1$ .

Any conic section has an equation of the form (§ 66, Vol. I)

$$(7) \quad a_{00}x_0^2 + a_{11}x_1^2 + a_{22}x_2^2 + 2a_{01}x_0x_1 + 2a_{02}x_0x_2 + 2a_{12}x_1x_2 = 0,$$

which determines on the line  $x_0 = 0$  an involution whose double elements satisfy

$$a_{11}x_1^2 + a_{22}x_2^2 + 2a_{12}x_1x_2 = 0.$$

Comparing with (4), we have that a circle must satisfy the condition

$$a_{11} = a_{22} \neq 0, \quad a_{12} = 0.$$

If this circle is to have (1, 0, 0) as center, i.e. as pole of  $x_0 = 0$ , the equation (7) must also satisfy the condition

$$a_{01} = 0 = a_{02}.$$

Thus, returning to nonhomogeneous coördinates, the equation of a circle with the origin as center must be of the form\*

$$(8) \quad x^2 + y^2 = k.$$

According to § 59, the transformations of the parabolic metric group leaving such a circle invariant are all displacements or symmetries, and, moreover, all displacements and symmetries leaving the origin invariant leave this circle invariant. Substituting (6) in (8), we see that a displacement or symmetry leaving the origin invariant is of the form

$$\begin{aligned} x' &= \alpha x + \beta y, & \alpha^2 + \beta^2 &= 1. \\ y' &= \epsilon(-\beta x + \alpha y), \end{aligned}$$

\*This argument does not prove that every equation of this form represents a circle. The answer to this question depends on the value of  $k$ .

Since any displacement or symmetry is expressible as the resultant of one leaving the origin invariant and a translation (Theorem 10, Cor. 1), we have

THEOREM 20. *The equations of the group of displacements and symmetries are*

$$(9) \quad \begin{aligned} x' &= \alpha x + \beta y + \gamma_1, \\ y' &= \epsilon(-\beta x + \alpha y) + \gamma_2, \end{aligned}$$

where  $\alpha^2 + \beta^2 = 1$  and  $\epsilon^2 = 1$ .

By § 54, Vol. I, a transformation of the form (9) effects an involution on  $l_\infty$  if and only if  $\epsilon = -1$ . By Theorem 10, Cor. 2, any symmetry leaving the origin invariant is a line reflection. Hence

THEOREM 21. *The displacements are the transformations of the type (9) for which  $\epsilon = 1$  and the symmetries those for which  $\epsilon = -1$ .*

### EXERCISES

1. The equation of a circle containing the point  $(a_2, b_2)$  and having the point  $(a_1, b_1)$  as center is

$$(x - a_1)^2 + (y - b_1)^2 = (a_2 - a_1)^2 + (b_2 - b_1)^2.$$

2. Two lines  $ax + by + c = 0$  and  $a'x + b'y + c' = 0$  are orthogonal if and only if  $aa' + bb' = 0$ .

3. In case the absolute involution has double points, the equiaffine transformations of the parabolic metric group are of the form (9), where  $\alpha^2 + \beta^2 = \epsilon$  and  $\epsilon = \pm 1$ .

63. **Introduction of order relations.** Let us now assume that the plane which we are considering is an ordered plane in the sense of § 15. We may therefore apply the results of Chap. II, particularly of §§ 28–30. Let us also assume that the absolute involution satisfies the condition referred to in § 62, that there exist two pairs of points conjugate with regard to the absolute involution which separate each other harmonically. By Theorem 9, Chap. II, and its corollaries, it follows that any two pairs of the absolute involution separate each other, and that the absolute involution has no double points.\* This result may conveniently be put in the following form:

THEOREM 22. *Two pairs of perpendicular lines intersecting in the same point separate each other. No line is perpendicular to itself.*

\* The geometry arising from the hyperbolic case has been studied by Wilson and Lewis in the article referred to in § 48.

The restrictions which we have just introduced enable us to state the fundamental theorem (Theorem 13) about the group of displacements in the following more precise form:

**THEOREM 23.** *The only displacement leaving a ray invariant is the identity.*

*Proof.* Let  $A$  be the origin and  $B$  any point of the ray. Since any collineation preserves order relations,  $A$  is transformed into itself. Since the line  $AB$  is invariant, the displacement is a point reflection or the identity (Theorem 13). But a point reflection would change  $B$  into a point of the ray opposite to the ray  $AB$ , and thus not leave the ray  $AB$  invariant.

With the aid of this theorem we can complete the set of fundamental theorems on congruent triangles, the first two of which were given in § 61.

**THEOREM 24.** *Two triangles  $ABC$  and  $A'B'C'$  are congruent if the point pairs  $AB$ ,  $AC$  and the angle  $\angle CAB$  are congruent respectively to the point pairs  $A'B'$ ,  $A'C'$  and the angle  $\angle C'A'B'$ .*

*Proof.* Since the angle  $\angle CAB$  is congruent to the angle  $\angle C'A'B'$ , there exists a displacement  $\Delta_1$  carrying  $A$  to  $A'$  and the rays  $AC$  and  $AB$  to  $A'C'$  and  $A'B'$  respectively. Since the point pair  $AB$  is congruent to  $A'B'$ , there is also a displacement  $\Delta_2$  carrying  $A$  to  $A'$  and  $B$  to  $B'$ , and since  $AC$  is congruent to  $A'C'$ , there is a displacement  $\Delta_3$  carrying  $A$  to  $A'$  and  $C$  to  $C'$ . By Theorem 23,  $\Delta_1 = \Delta_2$  and  $\Delta_1 = \Delta_3$ . Hence the displacement  $\Delta_1$  carries the triangle  $ABC$  to  $A'B'C'$ .

### EXERCISES

1. Two triangles  $ABC$  and  $A'B'C'$  are congruent if the point pair  $AB$  is congruent to the point pair  $A'B'$  and the angles  $\angle CAB$  and  $\angle CBA$  are congruent respectively to the angles  $\angle C'A'B'$  and  $\angle C'B'A'$ .

2. If two triangles  $ABC$  and  $A'B'C'$  are congruent in such a way that  $A$  corresponds to  $A'$  and  $B$  to  $B'$ , the angles  $\angle ABC$ ,  $\angle BCA$ ,  $\angle CAB$  are congruent to the angles  $\angle A'B'C'$ ,  $\angle B'C'A'$ ,  $\angle C'A'B'$  respectively.

3. If two triangles  $ABC$  and  $A'B'C'$  are symmetric in such a way that  $A$  corresponds to  $A'$  and  $B$  to  $B'$ , the angles  $\angle ABC$ ,  $\angle BCA$ ,  $\angle CAB$  are congruent to the angles  $\angle C'B'A'$ ,  $\angle A'C'B'$ ,  $\angle B'A'C'$  respectively.

4. Let  $A$ ,  $B$ ,  $C$  be three collinear points and  $P_\infty$  the point at infinity of the line joining them;  $B$  is between  $A$  and  $C$  if and only if

$$0 < R(P_\infty A, CB) < 1.$$

5. An orthogonal line reflection interchanges the two sides of its axis.

\* Note that an angle is an *ordered* pair of rays (§ 28).

64. **The real plane.** Let us finally assume that we are dealing with the geometry of reals. In consequence, we have the theorem (§ 4) that any one-dimensional projectivity which alters sense (i.e. for which  $\Delta < 0$ ) has two double elements. This may be put into the following form as a theorem of the affine geometry.

**THEOREM 25.** *If  $A_1$  and  $A_2$  are any two points of an ellipse, any line  $l$ , meeting the line  $A_1A_2$  in a point between  $A_1$  and  $A_2$ , meets the ellipse in two points.*

*Proof.\** Let us denote the given ellipse by  $E^2$ , and let  $A$  be a variable point on it. Let  $L_1$  and  $L_2$  be the points in which  $l$  is met by  $A_1A$  and  $A_2A$  respectively, and let  $Q_1$  and  $Q_2$  be the points in which  $l_\infty$  is met by  $A_1A$  and  $A_2A$  respectively. Also let  $Q_s$  be the point in which  $A_1L_2$  meets  $l_\infty$ . By construction, and by the definition of a conic,

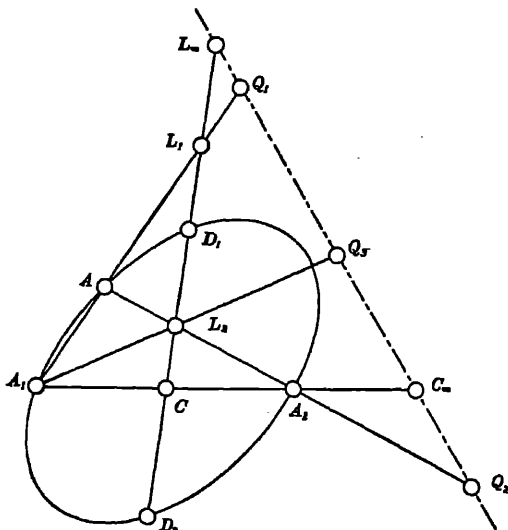


FIG. 51

$$(10) \quad [L_1] \stackrel{A}{\underset{\wedge}{=}} [Q_1] \stackrel{A}{\underset{\wedge}{=}} A_1[A] \stackrel{A}{\underset{\wedge}{=}} A_2[A] \stackrel{A}{\underset{\wedge}{=}} [Q_2] \stackrel{A_2}{\underset{\wedge}{=}} [L_2] \stackrel{A_1}{\underset{\wedge}{=}} [Q_s].$$

The projectivity  $[Q_1] \stackrel{A}{\underset{\wedge}{=}} [Q_2]$  is direct, because, by the remark at the beginning of this section, if the projectivity altered sense it would have two double points, and these, by the definition of the projectivity, would be points of intersection of  $l_\infty$  with  $E^2$ , contrary to the hypothesis that  $E^2$  is an ellipse.

Let  $C$  and  $C_\infty$  be the points of intersection of  $A_1A_2$  with  $l$  and  $l_\infty$  respectively. Also let  $L_\infty$  be the point at infinity of  $l$ . Then, by the hypothesis that  $C$  is between  $A_1$  and  $A_2$ ,

$$S(C_\infty CA_2) \neq S(C_\infty CA_1).$$

\*A simpler proof of this theorem, which, however, involves more preliminary theorems, is given in the next chapter (§ 76).

But, by construction,  $A_1 C A_2 C_\infty \frac{L_2}{\wedge} Q_3 L_\infty Q_2 C_\infty$ .

Hence, by Theorem 6, Chap. II,

$$S(C_\infty L_\infty Q_2) \neq S(C_\infty L_\infty Q_3).$$

But the points  $C_\infty$ ,  $L_\infty$ ,  $Q_2$  are carried to  $C_\infty$ ,  $L_\infty$ ,  $Q_3$ , respectively, by the projectivity  $[Q_2] \overline{\wedge} [Q_3]$ , indicated in (10). Hence the projectivity  $[Q_2] \overline{\wedge} [Q_3]$  is opposite. Since  $[Q_1] \overline{\wedge} [Q_2]$  is direct,  $[Q_1] \overline{\wedge} [Q_3]$  is opposite. From this, since  $Q_1$  and  $Q_3$  are carried by a perspectivity with  $A_1$  as center to  $L_1$  and  $L_2$  respectively, it follows (Theorem 6, Chap. II) that the projectivity

$$[L_1] \overline{\wedge} [L_2]$$

is opposite. By the remark at the beginning of the section this projectivity must therefore have two double points, and by the definition of the projectivity these double points must be points of intersection of  $l$  with  $E^2$ .

COROLLARY 1. *The points in which  $l$  meets the ellipse are separated by  $A_1$  and  $A_2$  relative to the order relations on the ellipse.*

*Proof.* Let  $D_1$  and  $D_2$  (fig. 51) be the two points in which  $l$  meets the ellipse, and let  $A$ ,  $A_1$ ,  $A_2$ , etc. have the meanings given them in the proof of the theorem. Then since the projectivity  $[L_1] \overline{\wedge} [L_2]$  is opposite,

$$S(D_1 D_2 L_1) \neq S(D_1 D_2 L_2).$$

Hence the lines  $AD_1$  and  $AD_2$  separate the lines  $AA_1$  and  $AA_2$ , which, according to the definition in § 20, implies that the pair of points  $D_1 D_2$  separates the pair  $A_1 A_2$  on the ellipse.

COROLLARY 2. *The points in which  $l$  meets the ellipse are on opposite sides of the line  $A_1 A_2$ .*

*Proof.* Let  $a$  be the tangent at  $A_1$ . By the first corollary the lines  $a$  and  $A_1 A_2$  separate the lines  $A_1 D_1$  and  $A_1 D_2$ . Hence, if  $A'$  denote the point in which  $a$  meets  $D_1 D_2$ ,  $D_1$  and  $D_2$  separate  $A'$  and  $C$ . Now  $A'$  is not between  $D_1$  and  $D_2$ , because if it were, the line  $a$  would meet the ellipse in two points instead of only in one. Hence  $C$  is between  $D_1$  and  $D_2$ , and hence  $D_1$  and  $D_2$  are on opposite sides of  $l$ .

THEOREM 26. *A rotation which transforms a given circle into itself transforms any triad of points on the circle into a triad of points in the same sense relatively to the order relations on the circle.*

*Proof.* Let the given triad of points be  $A, B, C$ , let  $O$  be any other point of the circle, and let  $A_\infty, B_\infty, C_\infty$  be the points at infinity of the lines  $OA, OB, OC$  respectively; let  $O', A', B', C', A'_\infty, B'_\infty, C'_\infty$  be the points to which  $O, A, B, C, A_\infty, B_\infty, C_\infty$ , respectively, are carried by the given rotation; let  $A''_\infty, B''_\infty, C''_\infty$  be the points at infinity of the lines  $OA', OB', OC'$  respectively.

The given rotation effects on  $l_\infty$  a transformation which is the product of two hyperbolic involutions. Hence  $S(A_\infty B_\infty C_\infty) = S(A'_\infty B'_\infty C'_\infty)$ . As in the proof of Theorem 25, the projectivity  $A'_\infty B'_\infty C'_\infty \bar{\wedge} A''_\infty B''_\infty C''_\infty$  is direct because otherwise it would have double points and these would be common to the circle and  $l_\infty$ . Hence  $S(A'_\infty B'_\infty C'_\infty) = S(A''_\infty B''_\infty C''_\infty)$  and, therefore,  $S(A_\infty B_\infty C_\infty) = S(A''_\infty B''_\infty C''_\infty)$ . Projecting from  $O$ , we have, by the definition of sense on a conic (§ 20), that

$$S(ABC) = S(A'B'C').$$

Theorem 26, which is here proved only for a real space, can be proved for any ordered space by the methods of the next chapter. This theorem states one of the most intuitively immediate properties of a rotation. In fact, most of the older discussions of the notions of sense describe sense, without further explanation, as "sense of rotation."

### EXERCISES

1. If  $\angle AOB$  is any angle, and  $PQ$  any ray, there is one and only one ray  $PR$  on a given side of the line  $PQ$  such that  $\angle AOB$  is congruent or symmetric to  $\angle QPR$ .

\*2. Prove that Theorem 25 is not true in a space satisfying Assumptions A, E, H, Q.

65. **Intersectional properties of circles.** THEOREM 27. *If  $A$  and  $B$  are any two distinct points, then on any ray having a point  $O$  as origin there is one and only one point  $P$  such that the pair  $AB$  is congruent to the pair  $OP$ .*

*Proof.* Let  $B_1$  be the point to which  $B$  is carried by the translation which carries  $A$  to  $O$ . The circle through  $B_1$  with  $O$  as center contains all points  $Q$  such that  $OQ$  is congruent to  $AB$ . Let  $B_2$  be the point to which  $B_1$  is transformed by a point reflection with  $O$  as center. Then since  $O$  is between  $B_1$  and  $B_2$ , any line  $l$  through  $O$  (and distinct from  $OB_1$ ) must meet the circle in two points, according to Theorem 25. But by Theorem 23 neither of the rays on  $l$  which have  $O$  as origin can contain more than one point of the circle. Hence each of these

rays contains just one point of the circle. Hence each ray with  $O$  as origin contains a single point  $P$  such that  $AB$  is congruent to  $OP$ .

Combining this theorem with Theorem 23, we have

**THEOREM 28.** *There is one and only one displacement carrying a given ray to a given ray.*

This result characterizes the group of displacements in the same way that the proposition that there is a unique projectivity of a one-dimensional form carrying any ordered triad of elements to any ordered triad characterizes the one-dimensional projective group.

**THEOREM 29.** *If two circles are such that the line joining their centers meets them in two point pairs which separate each other, the circles have two points in common, one on each side of the line joining the centers.*

*Proof.* Let the two circles be  $C_1^2$  and  $C_2^2$ , and let them meet the line joining the centers in the pairs  $P_1Q_1$  and  $P_2Q_2$  respectively. Let  $A$  be the center (§ 43) of the involution  $\Gamma$  in which  $P_1Q_1$  and  $P_2Q_2$  are pairs, and let  $a$  be the perpendicular to the line  $P_1P_2$  at  $A$ .

Since  $P_1$  and  $Q_1$  separate  $P_2$  and  $Q_2$ , the ordered triads  $P_1Q_1P_2$  and  $Q_1P_1Q_2$  are in the same sense. The involution  $\Gamma$  interchanges these two triads and hence transforms any triad into a triad in the

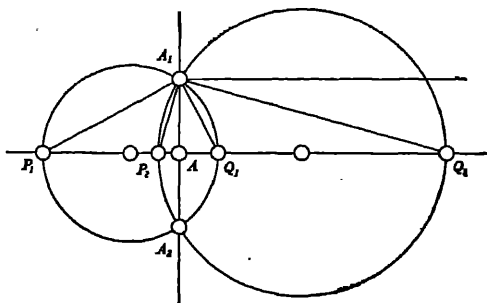


FIG. 52

same sense. Hence  $A$  is between  $P_1$  and  $Q_1$ . Hence, by Theorem 25, the line meets the circle  $C_1^2$  in two points  $A_1$  and  $A_2$ ; and by the second corollary of this theorem,  $A_1$  and  $A_2$  are on opposite sides of the line  $P_1Q_1$ .

The lines  $A_1P_1$  and  $A_1Q_1$  are orthogonal since  $P_1$  and  $Q_1$  are the ends of the diameter of a circle through  $A_1$ . The line  $A_1A$  is orthogonal to the line through  $A_1$  parallel to  $P_1Q_1$ . Hence the involution  $\Gamma$  is perspective with the involution of pairs of orthogonal lines through  $A_1$ . Hence  $A_1$ ,  $P_2$ , and  $Q_2$  are on a circle whose center is on the line  $P_1Q_1$ . By Theorem 16 this circle must be  $C_2^2$ . Hence  $C_1^2$  and  $C_2^2$  have  $A_1$  in common. A similar argument shows that  $A_2$  is on  $C_1^2$  and  $C_2^2$ .

66. **The Euclidean geometry. A set of assumptions.** In the geometry of reals the coefficients of the formulas derived in § 62 are real numbers. The formulas given for displacements in that section are the well-known equations for the "rigid motions" of elementary Euclidean geometry. Hence *the geometry of the parabolic metric group in a real plane is the Euclidean geometry.*

This result can also be established by considering a set of postulates from which the theorems of Euclidean geometry are deducible and proving that these postulates are theorems of the parabolic metric geometry. It then follows that all the theorems of Euclidean geometry are true in the parabolic metric geometry.

As a set of assumptions for Euclidean geometry of three dimensions we may choose the ordinal assumptions I-IX which are stated in § 29, together with the assumptions of congruence (X-XVI) stated below. For our immediate purpose, however, a set of assumptions for Euclidean plane geometry is needed. To obtain such a set we merely replace VII and VIII by the following:

VII. *All points are in the same plane.*

Thus our set of postulates for Euclidean plane geometry is I-VI, VII, IX-XVI.

Assumptions X-XVI make use of a new undefined relation between ordered point pairs which is indicated by saying " $AB$  is congruent to  $CD$ ." It must be verified that the new assumptions are valid when this relation is identified with the relation of congruence defined above.

X. *If  $A \neq B$ , then on any ray whose origin is a point  $C$  there is one and only one point  $D$  such that  $AB$  is congruent to  $CD$ .*

*Proof.* This is the same as Theorem 27.

XI. *If  $AB$  is congruent to  $CD$  and  $CD$  is congruent to  $EF$ , then  $AB$  is congruent to  $EF$ .*

*Proof.* This is a consequence of the fact that the displacements form a group.

XII. *If  $AB$  is congruent to  $A'B'$ , and  $BC$  is congruent to  $B'C'$  and  $\{ABC\}$  and  $\{A'B'C'\}$ , then  $AC$  is congruent to  $A'C'$ .*

*Proof.* By Theorem 28, there is a unique displacement which carries  $A$  and  $B$  to  $A'$  and  $B'$  respectively. This displacement carries

$C$  to a point  $C'$  such that  $\{A'B'C'\}$ , because a collineation preserves order relations. Moreover, the point  $C'$  so obtained is such that  $BC$  is congruent to  $B'C'$  and  $AC$  to  $A'C'$ ; and, by Theorem 27, there is only one point  $C'$  in the order  $\{A'B'C'\}$  such that  $BC$  is congruent to  $B'C'$ .

XIII.  $AB$  is congruent to  $BA$ .

*Proof.*  $AB$  is transformed into  $BA$  by the point reflection whose center is the mid-point of  $AB$ .

XIV. If  $A, B, C$  are three noncollinear points and  $D$  is a point in the order  $\{BCD\}$ , and if  $A'B'C'$  are three noncollinear points and  $D'$  is a point in the order  $\{B'C'D'\}$  such that the point pairs  $AB, BC, CA, BD$  are respectively congruent to  $A'B', B'C', C'A', B'D'$ , then  $AD$  is congruent to  $A'D'$ .

*Proof.* Since  $AB$  is congruent to  $A'B'$ , there exists a displacement  $\Delta$  which carries  $AB$  to  $A'B'$ . Let  $\Delta(C) = C_1$ ,  $\Delta(D) = D_1$ . Also let  $C_2$  and  $D_2$  be the points to which  $C_1$  and  $D_1$  are transformed by the orthogonal line reflection having  $A'B'$  as axis.

According to § 57, the pair  $BC$  is congruent to  $B'C_1$  and to  $B'C_2$ ;  $CA$  to  $C_1A'$  and  $C_2A'$ ;  $BD$  to  $B'D_1$  and  $B'D_2$ ; and  $AD$  to  $A'D_1$  and  $A'D_2$ . It follows that  $C'$  must coincide with  $C_1$  or  $C_2$ , for

otherwise there would be two circles, one with  $A'$  as center and the other with  $B'$  as center, containing the three points  $C_1, C_2, C'$ .

If  $C' = C_1$ , it follows, by Theorem 23, that  $D' = D_1$ , and hence that  $AD$  is congruent to  $A'D'$ . If  $C' = C_2$ , it follows, similarly, that  $D' = D_2$ , and hence that  $AD$  is congruent to  $A'D'$ .

DEFINITION. If  $O$  and  $X_0$  are two points of a plane  $\alpha$ , then the set of points  $[X]$  of  $\alpha$  such that  $OX$  is congruent to  $OX_0$  is called a circle.

XV. If the line joining the centers of two coplanar circles meets them in pairs of points,  $P_1Q_1$  and  $P_2Q_2$  respectively, such that  $\{P_1P_2Q_1\}$  and  $\{P_1Q_1Q_2\}$ , the circles have two points in common, one on each side of the line joining the centers.

*Proof.* This is the same as Theorem 29.

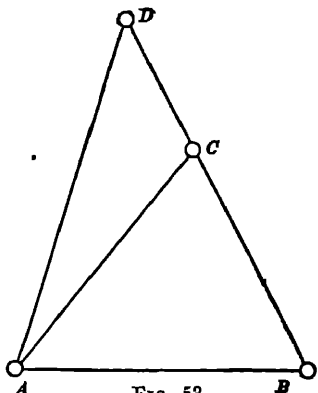


FIG. 53

XVI. If  $A, B, C$  are three points in the order  $\{ABC\}$  and  $B_1, B_2, B_3, \dots$  are points in the order  $\{ABB_1\}, \{AB_1B_2\}, \dots$  such that  $AB$  is congruent to each of the point pairs  $BB_1, B_1B_2, \dots$ , then there are not more than a finite number of the points  $B_1, B_2, \dots$  between  $A$  and  $C$ .

*Proof.* Let  $B_\infty$  be the point at infinity of the line  $AB$ . Then  $B_1$  is the harmonic conjugate of  $A$  with respect to  $B$  and  $B_\infty$ ,  $B_2$  is the harmonic conjugate of  $B$  with respect to  $B_1$  and  $B_\infty$ ; and so on. Thus  $A, B, B_1, B_2, \dots$  form a harmonic sequence of which  $B_\infty$  is the limit-point. Since  $C$  has a finite coördinate, the result follows from § 8, Chap. I.

The set of assumptions I-XVI is not categorical. It provides merely for the existence of such irrational points as are needed in constructions involving circles and lines (see § 77, below). It can be made categorical by adding Assumption XVII, § 29. It must be noted, however, that when XVII is added, X-XVI become redundant in the sense that it is possible to introduce ideal elements and then bring in the congruence relations by means of the definitions in this and the preceding chapters.

In order to convince himself that the assumptions given above are a sufficient basis for the theorems of Euclid, the reader should carry out the deduction from these assumptions of some of the fundamental theorems in Euclid's Elements. An outline of this process will be found in the monograph on the subject from which the assumptions have been quoted.\*

In making a rigorous deduction of the theorems of elementary geometry, either from the assumptions above or from the general projective basis, it is necessary to derive a number of theorems which are not mentioned in Euclid or in most elementary texts. These are mainly theorems on order and continuity. They involve such matters as the subdivision of the plane into regions by means of curves, the areas of curvilinear figures, etc., all of which are fundamental in the applications of geometry to analysis, and vice versa. In so far as these theorems relate to circles, they have been partially treated in §§ 64-65 and will be further discussed in the next chapter. The methods used for the more general theorems on order and continuity, however, are less closely related to the elementary part of projective geometry and will therefore be postponed to a later chapter.

\* Foundations of Geometry, by Oswald Veblen, in Monographs on Modern Mathematics, edited by J. W. A. Young, New York, 1911.

67. *Distance.* In § 43 we have defined the magnitude of a vector  $OB$  as its ratio to a unit vector  $OA$  collinear with it; but in the affine geometry the magnitudes of noncollinear vectors are absolutely unrelated. In the parabolic metric geometry we introduce the additional requirement that any two unit vectors  $OA$  and  $O'A'$  shall be such that the point pair  $OA$  is congruent to the point pair  $O'A'$ .

Thus, if a given unit vector  $OA$  is fixed and  $C^a$  is the circle through  $A$  with  $O$  as center, any other unit vector must be expressible in the form  $\text{Vect}(OP)$ , where  $P$  is a point of the circle. This gives two choices for the unit vector of any system of collinear vectors, and each of the two possible unit vectors is the negative of the other. Therefore, while it is possible under our convention to compare the absolute values of the magnitudes of noncollinear vectors, there is no relation at all between their algebraic signs. This corresponds to the fact that there is no unique relation between particular sense classes on two nonparallel lines.

Formulas in which the magnitudes of noncollinear vectors appear must, if they state theorems of the Euclidean geometry, be such that their meaning is unchanged when the unit vector on any line is replaced by its negative. This condition is satisfied, for example, in Exs. 2 and 4, § 71.

The ratio of two collinear vectors is invariant under the affine group; the magnitude of a vector is invariant under the group of translations; but the absolute value of the magnitude of a vector, according to our last convention, is invariant under the group of displacements. The last invariant may be defined directly in terms of point pairs as follows:

DEFINITION. Let  $AB$  be an arbitrary pair of distinct points which shall be referred to as the *unit of distance*. If  $P$  and  $Q$  are any two points, let  $C$  be a point of the ray  $AB$  such that the pair  $AC$  is congruent to the pair  $PQ$ . The ratio

$$\frac{AC}{AB}$$

is called the *distance from  $P$  to  $Q$* , and denoted by  $\text{Dist}(PQ)$ . If  $L$  is any point and  $l$  any line, the distance from  $L$  to the foot of the perpendicular to  $l$  through  $L$  is called the *distance from  $L$  to  $l$* .

It follows directly from the theorem above that  $\text{Dist}(PQ)$  is uniquely defined and positive whenever  $P \neq Q$ , and zero whenever  $P = Q$ . From the corresponding theorems on the magnitudes of vectors there follows the theorem that if  $\{ABC\}$ , then

$$\text{Dist}(AB) + \text{Dist}(BC) = \text{Dist}(AC).$$

Other properties of the distance-function are stated in the exercises.

The notion of the length (or circumference) of a circle may be defined as follows: Let  $P_1, P_2, \dots, P_n$  be  $n$  points in the order  $\{P_1 P_2 \dots P_n\}$  on a circle, and let

$$p = \text{Dist}(P_1 P_2) + \text{Dist}(P_2 P_3) + \dots + \text{Dist}(P_n P_1).$$

It can easily be proved that for a given circle  $C^2$ , the numbers  $p$  obtained from all possible ordered sets of points  $P_1, P_2, \dots, P_n$ , for all values of  $n$ , do not exceed a certain number.

DEFINITION. The number  $c$ , which is the smallest number larger than all values of  $p$ , is called the *length* or *circumference* of the circle  $C^2$ .

The proof of the existence of the number  $c$  will be omitted for the reasons explained below. The existence of  $c$  having been established, it follows without difficulty that if  $c$  and  $c'$  are the lengths of two circles with centers  $O$  and  $O'$ , respectively, and passing through points  $P$  and  $P'$ , respectively,

$$\frac{c}{c'} = \frac{\text{Dist}(OP)}{\text{Dist}(O'P')}.$$

Choosing the point pair  $O'P'$  as the unit of distance and denoting the constant  $c'$  by  $2\pi$ , this gives the formula

$$(11) \quad c = 2\pi \cdot \text{Dist}(OP).$$

The theory of the lengths of curves in general could be developed at the present stage without any essential difficulty. This subject, however, is very different (in respect to method, at least) from the other matters which we are considering, and therefore will be passed over with the remark that, starting with the theory of distance here developed, all the results of this branch of geometry may be obtained as applications of the integral calculus. Even the theory of the length of circles which we have summarized in the paragraphs above involves the ideas, if not the methods, of the calculus.

### EXERCISES

1. Two point pairs  $AB$  and  $CD$  are congruent if and only if  $\text{Dist}(AB) = \text{Dist}(CD)$ .
2. If  $A, B, C$  are noncollinear points,  $\text{Dist}(AB) + \text{Dist}(BC) > \text{Dist}(AC)$ .
3. Two triangles  $ABC$  and  $A'B'C'$  are similar in such a way that  $A$  corresponds to  $A'$ ,  $B$  to  $B'$ , and  $C$  to  $C'$  if and only if

$$\frac{\text{Dist}(AB)}{\text{Dist}(A'B')} = \frac{\text{Dist}(AC)}{\text{Dist}(A'C')} = \frac{\text{Dist}(BC)}{\text{Dist}(B'C')}.$$

4. Relative to a coordinate system in which the axes are at right angles, the distance between two points  $(x_1, y_1), (x_2, y_2)$  is

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2},$$

the positive determination of the radical being taken. The distance from a point  $(x_1, y_1)$  to a line  $ax + by + c = 0$  is the numerical value of

$$\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}.$$

**68. Area.** The area of a triangle, as distinguished from the measure of an ordered point triad, may be defined as follows:

DEFINITION. Relative to a unit triad  $OPQ$  (§ 49) such that the lines  $OP$  and  $OQ$  are orthogonal and the point pairs  $OP$  and  $OQ$  are congruent to the unit of distance, the positive number

$$\frac{1}{2} |m(ABC)|$$

is called the *area of the triangle*  $ABC$ , and denoted by  $a(ABC)$ .

As was brought out in Chap. III, the theory of measure of polygons belongs properly to the affine geometry. But the standard formula for the area of a triangle in terms of base and altitude (Ex. 1, below) involves the ideas of distance and perpendicularity and hence belongs to the parabolic metric geometry. It should be noticed that this formula assumes that the side of the triangle which is regarded as the base does not pass through a double point of the absolute involution. This condition is satisfied under the hypotheses of §§ 63, 64, but is not always satisfied in a complex plane; whereas the definitions of equivalence and measure as given in Chap. III are entirely free of such restrictions.

The theory of areas in general depends on considerations of order and continuity which we have not yet developed, and which, like the theory of lengths of curves, belongs essentially to another branch of geometry than that with which we are concerned in this chapter. We shall, however, outline the definition of the area of an ellipse from the point of view of elementary geometry, because the derivation of the area of an ellipse from that of the circle affords rather an interesting application of one of the theorems about the affine group.

Let  $P_1, P_2, \dots, P_n$  be any finite number of points in the order  $\{P_1 P_2 \dots P_n\}$  on an ellipse  $E^2$  with a point  $O$  as center, and let

$$A = a(OP_1 P_2) + a(OP_2 P_3) + \dots + a(OP_n P_1).$$

It can easily be proved that there exists a finite number,  $a(E^2)$ , which is the smallest number which is greater than all values of  $A$  formed according to the rule above.

DEFINITION. The number  $a(E^2)$  is called the *area* of the ellipse.

In case  $E^2$  is a circle,  $C^2$ , it is easy to prove that

$$a(C^2) = \pi r^2,$$

where  $\pi$  is the constant defined above and  $r = \text{Dist}(OP_1)$ .

Now suppose  $E^2$  is an ellipse with two perpendicular conjugate diameters  $OA$  and  $OB$  which meet  $E^2$  in  $A$  and  $B$  respectively, and let  $C^2$  be the circle through  $A$  with  $O$  as center, and let  $C$  be the point in which the ray  $OB$  meets  $C^2$ . The homology  $T$  with  $O.A$  as axis and the point at infinity of  $OB$  as center, which transforms  $B$  to  $C$ , is an affine transformation carrying the ellipse  $E^2$  to the circle  $C^2$ . This homology transforms the triangle  $OAB$  to the triangle  $OAC$ ; and the areas of these triangles satisfy the relation

$$\frac{a(OAC)}{a(OAB)} = \frac{\text{Dist}(OC)}{\text{Dist}(OB)} = k.$$

It follows, by § 50, that the homology transforms any triangle into one whose area is  $k$  times as large. By the definition of the area of an ellipse, therefore,

$$\frac{a(C^2)}{a(E^2)} = \frac{\text{Dist}(OC)}{\text{Dist}(OB)}.$$

Denoting  $\text{Dist}(OA)$  by  $a$  and  $\text{Dist}(OB)$  by  $b$ , this gives

$$a(E^2) = \frac{\pi a^2 b}{a} = \pi ab.$$

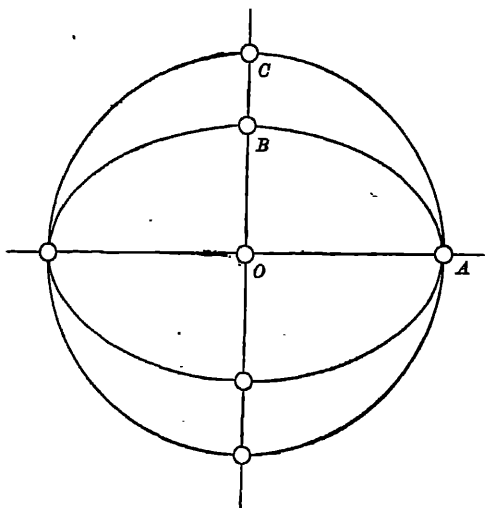


FIG. 54

### EXERCISES

1. The numerical value of the measure of a point triad  $ABC$  is equal to  $\text{Dist}(AB) \cdot \text{Dist}(CC')$ , where  $C'$  is the foot of the perpendicular from  $C$  to the line  $AB$ .

2. If  $abcd$  is a simple quadrilateral whose vertices are on a conic and  $P$  is a variable point of the conic,

$$\frac{\text{Dist}(Pa) \cdot \text{Dist}(Pc)}{\text{Dist}(Pb) \cdot \text{Dist}(Pd)}$$

is a constant (cf. Ex. 2, § 51).

3. If a projective collineation carries a variable point  $M$  and two fixed lines  $a, b$  to  $M', a', b'$  respectively, the number

$$\frac{\text{Dist}(Ma)}{\text{Dist}(Mb)} + \frac{\text{Dist}(M'a')}{\text{Dist}(M'b')}$$

is a constant.

4. Let  $F$  be the center of a homology  $\Gamma$  and  $l$  the vanishing line,  $\Gamma^{-1}(l_\infty)$ . If  $P$  is a variable point and  $Q = \Gamma(P)$ ,

$$\frac{\text{Dist}(FP)}{\text{Dist}(Pl)} = k \cdot \text{Dist}(FQ),$$

where  $k$  is a constant.

5. The area of an ellipse is  $\pi a/2$ , where  $a$  is the area of any inscribed parallelogram whose diagonals are conjugate diameters.

6. Among all simple quadrilaterals circumscribed to an ellipse, the ones whose sides are tangent at the ends\* of conjugate diameters have the least area.

7. Among all simple quadrilaterals inscribed in an ellipse, the ones whose vertices are the ends of conjugate diameters have the greatest area.

8. Of all ellipses inscribed in a parallelogram, the one which has the lines joining the mid-points of opposite sides as a pair of conjugate diameters has the greatest area.

9. Of all ellipses circumscribed to a parallelogram, the smallest is the one having the diagonals as conjugate diameters.

**69. The measure of angles.** The unit of distance may be chosen arbitrarily, because any point pair can be transformed under the parabolic metric group into any other point pair. It is otherwise with angles or line pairs, because, for example, an orthogonal line pair cannot be transformed into a nonorthogonal pair. Therefore the systems of measurement for angles obtained by choosing different units are, in general, essentially different. We shall give an outline of the generally adopted system of measurement, basing it upon properties of the group of rotations leaving a point  $O$  invariant.

Let  $P_0$  be an arbitrary point different from  $O$ , and  $C^2$  the circle through  $P_0$  with  $O$  as center. Let  $P_1$  (fig. 55) be the point different from  $P_0$  in which the line  $P_0O$  meets  $C^2$ , and let  $P_{\frac{1}{2}}$  and  $P_{\frac{3}{2}}$  be the points in which the perpendicular to  $P_0O$  at  $O$  meets  $C^2$ . By Cor. 1, Theorem 25, these points are in the order  $\{P_0P_{\frac{1}{2}}P_1P_{\frac{3}{2}}\}$  on the circle. Let  $\sigma$  denote the segment  $\overline{P_0P_{\frac{1}{2}}}$ . Any line through  $O$  meets  $C^2$  in two points which are separated by  $P_0$  and  $P_1$ , and hence meets  $\sigma$  in a unique point. Let  $P_{\frac{1}{2}}$  be the point in which the line through  $O$  perpendicular to  $P_0P_{\frac{1}{2}}$  meets  $\sigma$ . And, in general, let  $[P_{\frac{1}{2^n}}]$ ,  $n = 1, 2, \dots$  be the set such that  $P_{\frac{1}{2^n}}$  is the point in which the line through  $O$  perpendicular to  $P_0P_{\frac{1}{2^{n-1}}}$  meets  $\sigma$ .

\*The ends of a diameter are the points in which it meets the conic.

The line  $OP_{\frac{1}{2}}$  obviously meets the line  $P_0P_{\frac{1}{2}}$  in the mid-point of the pair  $P_0P_{\frac{1}{2}}$ , and the mid-point is between  $P_0$  and  $P_{\frac{1}{2}}$ . Hence, by Cor. 1, Theorem 25, we have the order relation  $\{P'P_0P_{\frac{1}{2}}P_{\frac{1}{2}}\}$ , where  $P'$  denotes, for the moment, the point not on  $\sigma$  in which the line  $OP_{\frac{1}{2}}$  meets the circle. Since  $O$  is between  $P_{\frac{1}{2}}$  and  $P'$ , the same corollary gives  $\{P_0P_{\frac{1}{2}}P'\}$ .

Since  $P_{\frac{1}{2}}$  is on the segment  $\sigma$ , we have either  $\{P_0P_{\frac{1}{2}}P_{\frac{1}{2}}P_1\}$  or  $\{P_0P_{\frac{1}{2}}P_1P_1\}$ . The second of these alternatives, however, when combined with  $\{P'P_0P_{\frac{1}{2}}P_{\frac{1}{2}}\}$ , would imply  $\{P'P_0P_{\frac{1}{2}}P_{\frac{1}{2}}P_1\}$ , contrary to  $\{P_0P_{\frac{1}{2}}P_1P_1\}$ . Hence  $\{P_0P_{\frac{1}{2}}P_{\frac{1}{2}}P_1\}$  is impossible, and we must have  $\{P_0P_{\frac{1}{2}}P_1P_1\}$ . In like manner it is proved that  $\{P_0P_{\frac{1}{2}}P_{\frac{1}{2}}P_1\}$  and, in general, that

$$\{P_0 \cdots P_{\frac{1}{2^n-1}} P_{\frac{1}{2^n-1}} \cdots P_{\frac{1}{2^n-1}} P_1\}.$$

Let  $\Pi$  denote the rotation (a point reflection in this case) which leaves  $O$  fixed and transforms  $P_0$  to  $P_1$ , and let  $\Pi^{\frac{1}{2^n}}$  denote the rotation transforming  $P_0$  to  $P_{\frac{1}{2^n}}$ . The rotation  $\Pi^{\frac{1}{2}}$ , being the product of the orthogonal line reflection with  $OP_{\frac{1}{2}}$  as axis followed by that with  $OP_{\frac{1}{2}}$  as axis, carries the point pair  $OP_{\frac{1}{2}}$  to the point pair  $OP_1$ . Hence\*

$$(\Pi^{\frac{1}{2}})^2 = \Pi.$$

In like manner it follows that

$$(\Pi^{\frac{1}{2^n}})^2 = \Pi^{\frac{1}{2^{n-1}}}.$$

\* The symbol  $A^n$ , where  $A$  is any transformation and  $n$  a positive integer, has been defined in § 24, Vol. I.

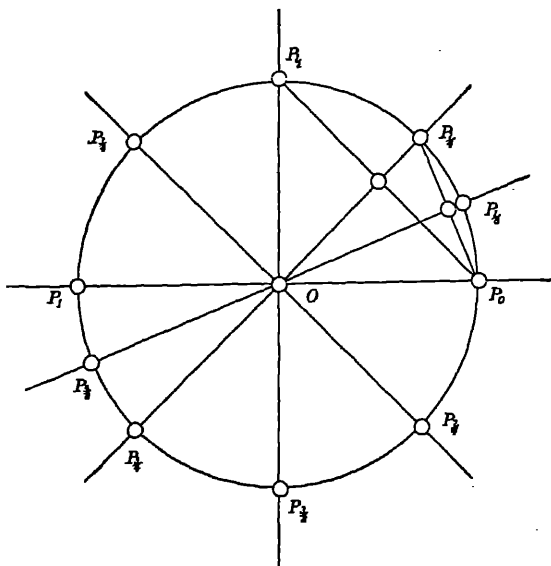


FIG. 55

Let us denote  $(\Pi^{\frac{1}{2^n}})^m$  by  $\Pi^{\frac{m}{2^n}}$ , where  $m$  is any positive or negative integer, and  $\Pi^{\frac{m}{2^n}}(P_0)$  by  $P_{\frac{m}{2^n}}$ .

Now all rotations are direct (Theorem 26). Hence  $S(P_0 P_{\frac{1}{2}} P_{\frac{1}{4}}) = S(P_{\frac{1}{2}} P_{\frac{1}{4}} P_0) = S(P_{\frac{1}{4}} P_0 P_{\frac{1}{2}})$ . Combining these relations with  $\{P_0 P_{\frac{1}{2}} P_{\frac{1}{4}} P_0\}$ , we have the order relation  $\{P_0 P_{\frac{1}{2}} P_{\frac{1}{4}} P_0 P_{\frac{1}{2}} P_0\}$ , and in general, by a like argument,

$$\{P_0 P_{\frac{1}{2^n}} P_{\frac{2}{2^n}} P_{\frac{3}{2^n}} \cdots P_1\}.$$

Hence we have  $\{P_0 P_{\frac{m}{2^n}} P_{\frac{m'}{2^n}} P_0\}$ , whenever  $0 < \frac{m}{2^n} < \frac{m'}{2^n} < 1$ , as can easily be seen on reducing the two fractions to a common denominator.

Since  $\Pi^2 = 1$ , it follows that whenever  $m/2^n$  is expressible in the form  $2k + \alpha$ ,  $k$  being an integer,

$$(12) \quad \Pi^{2k+\alpha} = \Pi^\alpha \quad \text{and} \quad P_{2k+\alpha} = P_\alpha.$$

DEFINITION. Let  $\pi$  be the constant defined in § 67, (11). The number  $\alpha \cdot \pi$ , where  $\alpha = m/2^n$ , is called the *measure* of any angle congruent to  $\angle P_0 P_\alpha P_0$ . An angle whose measure is  $\alpha\pi$  is also said to be equal to  $2\alpha$  right angles.

The measure of an angle is indeterminate according to this definition. In fact, according to (12), whenever the measure of an angle is  $\beta$ , it is also  $2k\pi + \beta$ , where  $k$  is any positive or negative integer. This indetermination can be removed by requiring that the measure  $\beta$  chosen for any angle shall always satisfy a condition of the form  $0 \equiv \beta < 2\pi$ , or  $-\pi < \beta \leq \pi$ .

Since the rays  $OP_{\frac{m}{2^n}}$  do not include all rays with  $O$  as center, the definition just given does not determine the measures of all angles. The required extension may be made by means of elementary continuity considerations, the details of which we shall omit. The essential steps required are: (1) to prove that if  $\bar{P}$  be any point in the order  $\{P_0 \bar{P} P_{\frac{1}{2}} P_0\}$ , there exists a positive integral value of  $n$  such that  $\{P_0 P_{\frac{1}{2^n}} \bar{P} P_{\frac{1}{2^n}} P_0\}$ ; (2) hence to prove that if  $P$  be any point on the circle not of the form  $P_{\frac{m}{2^n}}$ , the points of the form  $P_{\frac{m}{2^n}}$  fall into two classes,  $[P_\alpha]$  and  $[P_\beta]$ , such that  $\{P_0 P_\alpha P P_\beta\}$ , and there is no point, except  $P$ , on every segment  $\overline{P_\alpha P_\beta}$  of the circle; (3) having required that  $0 < \alpha < \beta < 2$ , to define  $\Pi^{2k+\alpha}$  (where  $k$  is an integer, positive, negative, or zero, and  $\alpha$  is the number

such that  $\alpha < x < \beta$  for all  $\alpha$ 's and  $\beta$ 's) as the rotation about  $O$  carrying  $P_0$  to  $P$ ; (4) to show that if  $x$  is a rational number  $m/n$ ,  $(\Pi^n)^m = \Pi^m$ ; (5) to define measure of angle as above, but with the restriction that  $\alpha = m/2^n$  removed; (6) to prove that the measure of the sum of two angles differs from the sum of the measures by  $2k\pi$ , the sum being defined as below.

DEFINITION. If  $a, b, c$  are any three rays having a common origin, but not necessarily distinct, any angle  $\angle a_1c_1$  congruent to  $\angle ac$  is said to be the *sum* of any two angles  $\angle a_2b_2$  and  $\angle b_2c_2$  such that  $\angle a_2b_2$  is congruent to  $\angle ab$  and  $\angle b_2c_2$  is congruent to  $\angle bc$ . The sum  $\angle a_1c_1$  is denoted by  $\angle a_2b_2 + \angle b_2c_2$ .

For some purposes it is desirable to have a conception of angle according to which any two numbers are the measures of distinct angles. This may be obtained as follows:

DEFINITION. A ray associated with an integer, positive, negative, or zero, is called a *numbered ray*. An ordered pair of numbered rays having the same origin is called a *numbered angle*. If the measure of an angle  $\angle hk$  in the earlier sense is  $\alpha$ , where  $0 \leq \alpha < 2\pi$ , the measure of a numbered angle in which  $h$  is associated with  $m$ , and  $k$  with  $n$ , is

$$2(n - m)\pi + \alpha.$$

Defining the sum of two numbered angles in an obvious way, it is clear that the sum of two numbered angles has a measure which is the sum of their measures.

The trigonometric functions can now be defined, following the elementary textbooks, as the ratios of certain distances multiplied by  $\pm 1$  according to appropriate conventions. This we shall take for granted in the future as having been carried out.

**70. The complex plane.** Instead of the assumption in § 64, we could assume that the Euclidean plane is obtained by leaving out one line from the complex projective plane ( $A, E, J$ , or  $A, E, H, C, \bar{R}, I$ ). All the results of Chap. III and of the present chapter up to § 63 are applicable to this case. The rest of the theory, however, is essentially different from that of the real plane, because the absolute involution necessarily has two double points and because a line does not satisfy the one-dimensional order relations. Thus the minimal lines play a principal rôle and must be regarded as exceptional in the statement of a large class of theorems; and another large class of theorems of elementary geometry (those involving order relations) disappears entirely.

For the present, therefore, we shall confine attention to the geometry of reals, but shall make use, whenever we find it convenient to do so, of the fact (§ 6) that a real space  $S$  may be regarded as immersed in a complex space,  $S'$ , in such a way that every line  $l$  of  $S$  is contained in a unique line  $l'$  of  $S'$ . As a direct consequence it follows that any conic  $C^2$  of  $S$  is a subset of the points of a unique conic of  $S'$ . For any five points of  $C^2$ , regarded as points of  $S'$ , determine a unique conic of  $S'$  which, by construction (§ 41, Vol. I), contains all points of  $C^2$  and is uniquely determined by any five of its points. Similar reasoning will show that any plane  $\pi$  of  $S$  is contained in a unique plane  $\pi'$  of  $S'$ ; and like remarks may be made with regard to any one-, two-, or three-dimensional form.

A like situation arises with respect to transformations. A projective transformation  $\Pi$  of a form in  $S$  is fully determined, according to the fundamental theorem of projective geometry, by its effect on a finite set\* of elements of  $S$ . Since the fundamental theorem is also valid in  $S'$ , there is a unique projective transformation  $\Pi'$  which has the same effect on this set of elements as  $\Pi$ .

Specializing these remarks somewhat we have: A Euclidean plane  $\pi$  of  $S$  is a subset of the points of a certain Euclidean plane  $\pi'$  of  $S'$ . The line at infinity  $l_\infty$  associated with  $\pi$  is a subset of the line at infinity  $l'_\infty$  associated with  $\pi'$ . The absolute involution  $I$  on  $l_\infty$  determines an involution  $I'$  on  $l'_\infty$  in which all the pairs of  $I$  are paired. The involution  $I'$  has two imaginary double points, the circular points (§ 56), which shall be denoted by  $I_1$  and  $I_2$ . Since a circle in  $\pi$  is a conic having  $I$  as an involution of conjugate points, every circle in  $\pi$  is a subset of the points on a conic in  $\pi'$  which passes through  $I_1$  and  $I_2$ .

The problem of the intersection of a line and a circle, or indeed of a line and any ellipse, can now be discussed completely. In the proof of Theorem 25 the intersection of a line  $l$  and an ellipse  $E^2$  was seen to depend on finding the double points of a certain projectivity  $[L_1] \overline{\wedge} [L_2]$  on  $l$ . Any three points  $L'_1, L''_1, L'''_1$ , and their correspondents  $L'_2, L''_2, L'''_2$ , determine a projectivity on the complex line  $l'$  containing  $l$ , and, by the fundamental theorem of projective geometry, this projectivity is identical with  $[L_1] \overline{\wedge} [L_2]$  so far as real points are concerned. The double points of this projectivity are common to the complex

\*For example, in case of a one-dimensional form any three elements of the form are such a set.

line containing  $l$  and the complex conic containing  $E^2$ . These points are real if the hypothesis of Theorem 25 is satisfied; they are real and coincident if  $l$  is tangent to  $E^2$ ; otherwise they are imaginary.

A similar discussion will be made in the next section of the problem of the intersection of two circles, but first let us make certain definitions and conventions which will simplify our terminology.

According to the definitions in § 6, any point of  $S'$  is said to be *complex*, and a complex point is *real* or *imaginary* according as it is contained in  $S$  or not. In the case of lines, however, we have three things to distinguish: a line of the space  $S$ , a line of  $S'$  which contains a line of  $S$  as a subset, and a line of  $S'$  which contains no such subset. In current usage a line of the last sort is called *imaginary*, a line of either of the first two sorts is called *real*, and a line of either of the last two sorts is called *complex*. The current terminology therefore permits a confusion between a real line as a locus in  $S$  and a real line as a particular kind of a complex line.

In most cases, however, no misunderstanding need be caused by this ambiguity of language, and we shall in future usually employ the same notation for the real line  $l$  of  $S$  and the line  $l'$  of  $S'$  which contains  $l$ . The same remarks apply to conic sections and, indeed, to all one-dimensional forms.

**DEFINITION.** Any element (point, line, or plane) or set of elements of  $S'$  is said to be *complex*. Any element or set of elements of  $S$  is said to be *real*. A line or plane of  $S'$  which contains a line or plane, respectively, of  $S$  is said to be a *real line* or *real plane* of  $S'$ . A one-dimensional form of  $S'$ , a subset of whose elements are real elements of  $S'$  and contain all the elements of a one-dimensional form of  $S$ , is called a *real one-dimensional form* of  $S'$ . An element or one-dimensional form of  $S'$  which is not a real element or real one-dimensional form of  $S'$  is said to be *imaginary*.

**DEFINITION.** A projective transformation of a real form of  $S'$  is said to be *real* if it transforms each real element of  $S'$  into a real element of  $S'$ .

Strictly speaking, these definitions distinguish between the two senses of the word "real" by phrases such as "real line of  $S'$ ." But in practice we shall drop the "of  $S'$ ." The one-dimensional forms as thus far defined are all of the first or second degrees, but the definition can be extended without essential modification to forms of higher

degree and also to forms of more than one dimension. We shall take this extension for granted whenever we have occasion to use it.

In accordance with these conventions, the points  $I_1$  and  $I_2$  which are really the double points of  $I'$  will be referred to in future as the double points of the absolute involution  $I$ . In like manner, any line  $l$  and circle  $C^2$  which have no real points in common will be said to have in common the two points common to the complex line and the complex conic which contain  $l$  and  $C^2$  respectively.

The utility of these conventions will be understood by the reader if he will write out in full the discussion of pencils of circles in the following section, putting in explicitly, in notation and language, the distinction between elements of  $S$  and  $S'$ .

It is also convenient in many cases to extend the formulas for distance, area, etc. given in §§ 67–69 to imaginary elements. Thus, for example, in case  $(x_1, y_1)$  and  $(x_2, y_2)$  are imaginary points such that  $(x_1 - x_2)^2 + (y_1 - y_2)^2$  is a positive real number,  $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$  will be referred to as the distance from  $(x_1, y_1)$  to  $(x_2, y_2)$ . Extensions of terminology of this self-evident sort will be made when needed, without further explanation.

**71. Pencils of circles.** Consider two circles  $C_1^2$  and  $C_2^2$  in a real Euclidean plane. Let their centers be denoted by  $C_1$  and  $C_2$ , and in case  $C_1 \neq C_2$ , let  $b$  denote the line  $C_1C_2$ . By Theorem 25,  $b$  meets each circle in a pair of real points which we shall denote by  $P_1Q_1$  and  $P_2Q_2$  respectively. The two pairs may be entirely distinct, in which case let  $\Gamma$  denote the involution on  $b$  transforming each pair into itself; or they may have one point in common, in which case the line through this point perpendicular to  $b$  is a common tangent of the two circles. The two pairs cannot coincide, because the circles would then coincide. Thus four cases may be distinguished:

- (1) The circles have the same center.
- (2) The circles have a common tangent and point of contact.
- (3) The involution  $\Gamma$  is direct.
- (4) The involution  $\Gamma$  is opposite.

A circle is, by § 60, a real conic which, according to the terminology of the last section, contains the double points of the absolute involution. Let us denote these points (the circular points) by  $I_1$  and  $I_2$  and apply the results of § 47, Vol. I, on pencils of conics.

In the first case let  $O$  denote the common center of the two circles. The lines  $OI_1$  and  $OI_2$  are then tangent to both circles at  $I_1$  and  $I_2$  respectively. Hence, by reference to § 47, Vol. I, it is evident that the two circles belong to a pencil of circles of Type IV.

In the second case  $C_1^2$  and  $C_2^2$  have in common the points  $I_1$  and  $I_2$  as well as a common tangent and point of contact. Hence they belong to a pencil of Type II which contains all circles touching\* the given line at the given point.

In the third case, since the involution  $\Gamma$  is direct, the pairs  $P_1Q_1$  and  $P_2Q_2$  separate each other. Hence, by Theorem 29, the circles have two real points,  $A_1$  and  $A_2$ , in common. Hence they belong to a pencil of Type I consisting of all conics through  $A_1, A_2, I_1$  and  $I_2$ . This may also be seen as follows:

Since the involution  $\Gamma$  has no double points (§ 21), it has a center (§ 43) which we shall call  $O$ . Let  $a$  be the line perpendicular to  $b$  at  $O$ . Then by the argument used in the proof of Theorem 29,  $O$  is between  $P_1$  and  $Q_1$ . Hence  $a$  meets  $C_1^2$  in two real points  $A_1$  and  $A_2$  (fig. 52). The pencil of conics through  $A_1, A_2, I_1, I_2$  meets  $b$  in the pairs of an involution among which are  $P_1Q_1$  and  $O$  and the point at infinity of  $b$ . Hence  $C_2^2$  is a conic of the pencil, and hence  $a$  meets  $C_2^2$  in  $A_1$  and  $A_2$ . In this case, therefore, the two circles belong to a pencil of Type I.

In the fourth case the involution  $\Gamma$  cannot have a double point at infinity, because then the other double point would have to be the mid-point of  $P_1Q_1$  and also of  $P_2Q_2$ , and thus  $C_1^2$  and  $C_2^2$  would have a common center. Hence in this case also the center  $O$  of the involution  $\Gamma$  is an ordinary point. Let  $a$  denote

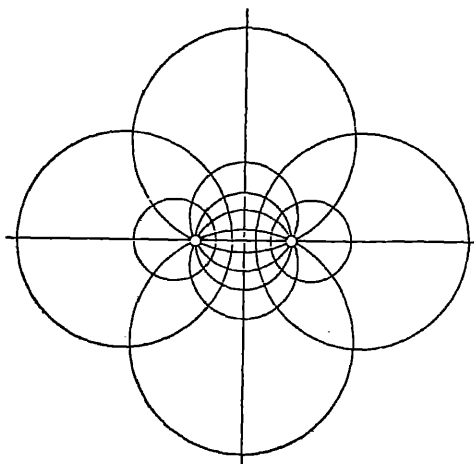


FIG. 56

\*A conic and one of its tangent lines are said to *touch* each other at the point of contact. Two conics touching a line at the same point are said to *touch* each other.

the perpendicular to  $b$  at  $O$ , and let  $A_1$  and  $A_2$  be the points in which  $a$  meets  $C_1^2$ . These points are imaginary; for otherwise, since they are interchanged by the orthogonal line reflection with  $b$  as axis,  $O$  would be between them, and hence, by Cor. 1, Theorem 25,  $O$  would be between  $P_1$  and  $Q_1$ , contrary to the hypothesis that  $\Gamma$  is opposite. Precisely as in the third case it follows that  $A_1$  and  $A_2$  are also on  $C_2^2$ . Hence in this case also  $C_1^2$  and  $C_2^2$  belong to a pencil of Type I.

In each case the facts established make it clear that the two circles could not both be members of more than one pencil of conics. Since any two circles fall under one of the four cases, we have

**THEOREM 30. DEFINITION.** *Any circle contains the real points of a certain conic in the complex plane. Two conics determined by circles are contained in a unique pencil of conics, which is of Type I, II, or IV. The set of circles which the conics of such a pencil have in common with the real plane is called a pencil of circles. If the pencil of conics is of Type IV, the pencil of circles is the set of all circles having a fixed point as center; if the pencil of conics is of Type II, the pencil of circles is the set of all circles tangent to a given line at a given point; if the pencil of conics is of Type I, the pencil of circles is the set of all circles having a given pair of distinct real points in common, or else the set of all circles with centers on a given line and meeting this line in the pairs of an involution with two ordinary double points.*

**DEFINITION.** The line  $a$  joining the centers of two nonconcentric circles is called *the line of centers* of the two circles or of the pencil of circles which contains them. If the circles have a common tangent and point of contact, this tangent is called *the radical axis* of the two circles or of the pencil of circles; if not, the line perpendicular to  $a$  at the center of the involution in which the circles of the pencil meet  $a$  is called *the radical axis*. The double points of this involution are called *the limiting points* of the pencil of circles. Any circle of the pencil is said to be *about* either one, or both, of the limiting points.

The discussion above has established

**THEOREM 31.** *The radical axis of two circles passes through all points common to them which are not on the line at infinity. The limiting points of the pencil which they determine are real if the circles meet only in imaginary points and imaginary if they meet in two real points.*

**THEOREM 32.** *The circular points, the limiting points of a pencil of circles of Type I, and the two points not at infinity in which the circles of the pencil intersect are the pairs of opposite vertices of a complete quadrilateral. The sides of the diagonal triangle of this quadrilateral are  $l_x$ , the radical axis, and the line of centers of the pencil.*

*Proof.* Let  $A_1$  and  $A_2$  (fig. 57\*) be the points other than  $I_1$  and  $I_2$  common to the circles of the pencil, and let  $B_1$  and  $B_2$  be the points of intersection of the pairs of lines  $I_1A_1, I_2A_2$  and  $I_1A_2, I_2A_1$  respectively. Whether  $A_1$  and  $A_2$  are real or imaginary, the line  $A_1A_2 = a$ , which is the radical axis, is real. Hence its point at infinity  $A_\infty$  is real; and hence the line  $B_1B_2$ , the polar of  $A_\infty$  with regard to any circle of the pencil, is real.

Since the line  $b = B_1B_2$  is the polar of  $A_\infty$ , it contains the centers of all conics through  $A_1, A_2, I_1, I_2$ . Hence  $b$  is the line of centers of the pencil of circles through  $A_1$  and  $A_2$ . The points  $B_1$  and  $B_2$  being diagonal points of the complete quadrangle  $A_1A_2I_1I_2$  are evidently the double points of the involution in which the pencil of circles meets  $b$ , and hence are the limiting points of the pencil.

Taking Theorems 31 and 32 together, we see that any pair of real points  $A_1, A_2$  determines a pair of imaginary points  $B_1, B_2$  such that either pair is the pair of limiting points of the pencil of circles through the other pair; that, conversely, any pair of imaginary points  $B_1, B_2$ , which are common to two circles, determines two real points  $A_1, A_2$  which are in the above relation to  $B_1, B_2$ ; and that the three pairs  $A_1A_2, B_1B_2, I_1I_2$  are pairs of opposite vertices of a complete quadrilateral. The relation between the two pencils of circles, the one

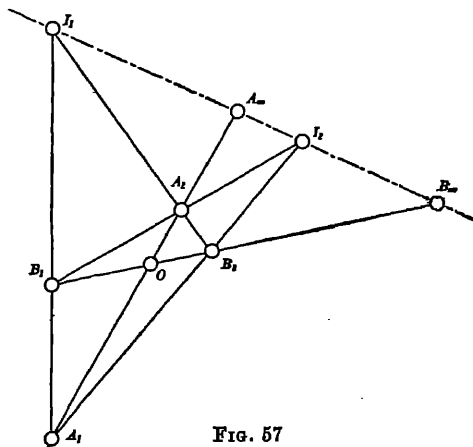


FIG. 57

\* Fig. 57 is, of course, a diagram in which certain imaginary elements are represented by real ones. On the use of figures in general, cf. p. 16, Vol. I.

through  $A_1$  and  $A_2$  and the other about  $A_1$  and  $A_2$ , is thus extremely symmetrical. It can be described in purely real terms by means of the following theorems and definition:

**THEOREM 33. DEFINITION.** *If two circles have a point in common such that the tangents to the two circles at this point are orthogonal, the two circles have another such point in common. Two circles so related are said to be orthogonal to each other.*

*Proof.* An orthogonal line reflection whose axis is the line of centers transforms each circle into itself and transforms the given point of intersection into another point of intersection. Since orthogonal lines are transformed to orthogonal lines, the tangents at the second point are also orthogonal.

**THEOREM 34.** *If a line through the center of a circle  $C^2$  meets the circle in a pair of points  $P_1Q_1$  and meets any orthogonal circle  $K^2$  in a pair of points  $P_2Q_2$ , the pairs  $P_1Q_1$  and  $P_2Q_2$  separate each other harmonically. Conversely, if  $P_1Q_1$  and  $P_2Q_2$  separate each other harmonically, any circle through  $P_2$  and  $Q_2$  is orthogonal to  $C^2$ .*

*Proof.* Let  $T$  be one of the points common to the two circles, and let  $t$  be the tangent to the circle  $TP_2Q_2$  at  $T$ . The pencil of circles tangent to  $t$  at  $T$  meets the line  $P_1P_2$  in the pairs of an involution  $\Gamma$ , and hence the first statement of the theorem will follow if we can prove that  $P_1$  and  $Q_1$  are the double points of this involution. The line perpendicular to  $t$  at  $T$  and the line perpendicular to  $P_1P_2$  at  $P_1$

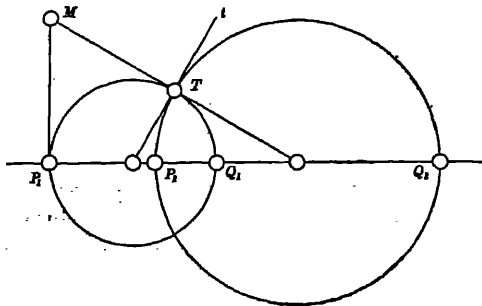


FIG. 58

are tangents to the circle  $TP_1Q_1$  at  $T$  and  $P_1$  respectively, and hence (Ex. 4, § 60) meet in a point  $M$  such that the pairs  $MP_1$  and  $MT$  are congruent. Hence the circle through  $T$  with  $M$  as center is tangent to  $t$  at  $T$  and to  $P_1P_2$  at  $P_1$ . Hence  $P_1$  is a double point of  $\Gamma$ . A similar argument shows that  $Q_1$  is also a double point.

To prove the converse proposition we observe that there is only one circle through  $P_2$  and  $T$  and orthogonal to  $C^2$ . One such circle, by

the argument above, passes through the point  $Q_2$ , which is harmonically separated from  $P_2$  by  $P_1$  and  $Q_1$ . Hence the circle  $P_2Q_2T$  is orthogonal to  $C^2$ .

As a corollary we have

**COROLLARY 1.** *The set of all circles orthogonal to a pencil of Type I is the pencil of circles through the limiting points of the first pencil.*

Another form in which this result may be stated is the following:

**COROLLARY 2.** *Let  $C^2$  be a circle,  $A_1$  any point not its center, and  $A_2$  the point on the line joining  $A_1$  to the center of  $C^2$  which is conjugate to  $A_1$  with regard to the conic  $C^2$ . Then all circles through  $A_1$  and orthogonal to  $C^2$  meet in  $A_2$ .*

**DEFINITION.** Two points are said to be *inverse* with respect to a circle if and only if they are conjugate with regard to the circle and collinear with its center. The transformation by which every point corresponds to its inverse is called an *inversion* or a *transformation by reciprocal radii*.

Thus the center of the circle is inverse to every real point at infinity. We shall return to the study of inversions in a later chapter.

### EXERCISES

1. In case the limiting points of a pencil of circles are real, the radical axis is their perpendicular bisector.

2. If  $O$  is any point of the plane of a circle, and a variable line through  $O$  meets the circle in two points  $X, Y$ , the product  $OX \cdot OY$  is constant, and equal to  $(OT)^2$  in case there is a line  $OT$  tangent to the circle at  $T$ . The product  $OX \cdot OY$  is called the *power* of  $O$  with respect to the circle.

3. The power of any point of the radical axis of a pencil of circles with respect to all circles of the pencil is a constant, and this constant is the same for all points of the radical axis.

4. If  $O$  is the center of a circle,  $C$  any point of the circle, and  $A_1$  and  $A_2$  any two points inverse with respect to it,

$$OA_1 \cdot OA_2 = (OC)^2.$$

5. Through two points not inverse relative to a given circle, there is one and but one circle orthogonal to it.

6. By a *center of similitude* of two circles is meant the center of a dilation (§ 47) or translation which transforms one of the circles into the other. If the circles are concentric, they have one center of similitude; if they are not concentric, they have two. The centers of similitude harmonically separate the centers of the two circles. The one which is between the centers of the two

circles is called the *interior*, and the other is called the *exterior*, center of similitude. The common tangents of two circles meet in the centers of similitude.

7. Three circles whose centers are not collinear determine by pairs six centers of similitude which are the vertices of a complete quadrilateral having the centers of the circles as vertices of its diagonal triangle. Generalize to the case of  $n$  circles.

8. If a circle  $K^2$  meets two circles  $C_1^2$  and  $C_2^2$  in four points at which the pairs of tangents are congruent or symmetric, the four points are collinear by pairs with the centers of similitude of  $C_1^2$  and  $C_2^2$ . Prove the converse proposition.

**72. Measure of line pairs.** The circular points  $I_1, I_2$  figure in a very important formula for the measure of a pair of lines.\* With the exception of these two points, and two lines  $i_1, i_2$  which pass through them, all the points and lines to which we shall refer in this section are real.

The center and the point at infinity of the axis of an orthogonal line reflection are harmonically conjugate with regard to  $I_1$  and  $I_2$ . Hence any orthogonal line reflection, regarded as a transformation of the complex space, interchanges  $I_1$  and  $I_2$ , and any displacement leaves  $I_1$  and  $I_2$  separately invariant. Moreover, there exists a displacement transforming any (real) point of  $l_\infty$  to any other (real) point of  $l_\infty$ . Hence a necessary and sufficient condition that a pair of points  $P, P'$  of  $l_\infty$  be transformable by a displacement to a pair  $Q, Q'$  of  $l_\infty$  is

$$(13) \quad R(PP', I_1 I_2) = R(QQ', I_1 I_2).$$

Now any pair of lines meeting  $l_\infty$  in  $P$  and  $P'$  can be transformed by a translation into any other pair of lines meeting it in  $P$  and  $P'$ , and any pair of lines meeting  $l_\infty$  in  $Q$  and  $Q'$  can be transformed by a translation into any other pair of lines meeting it in  $Q$  and  $Q'$ . Hence the necessary and sufficient condition that a pair of lines meeting  $l_\infty$  in  $P$  and  $P'$  be congruent to a pair of lines meeting it in  $Q$  and  $Q'$  is (13).

This suggests as a possible definition of the measure of a pair of nonparallel lines  $l_1, l_2$ ,

$$R(l_1 l_2, i_1 i_2),$$

where  $i_1$  and  $i_2$  are the lines joining the point of intersection of  $l_1$  and  $l_2$  to  $I_1$  and  $I_2$  respectively. It would satisfy the requirement of

\*This formula is due to A. Cayley. Cf. Encyclopädie der Math. Wiss. III AB 9, p. 901, footnotes 98 and 99.

being unaltered by displacements. In the case of measure of point pairs, however, we have

$$\text{Dist}(AB) + \text{Dist}(BC) = \text{Dist}(AC)$$

whenever  $\{ABC\}$ , and this condition is not satisfied by the cross ratio given above. We have, in fact,

$$(14) \quad \mathbf{R}(l_1 l_2, i_1 i_2) \cdot \mathbf{R}(l_2 l_3, i_1 i_2) = \mathbf{R}(l_1 l_3, i_1 i_2)$$

whenever  $l_1, l_2, l_3$  are concurrent. This is easily verified by substituting in the formula for cross ratio (§ 56, Vol. I).

From (14) it is obvious that if we define

$$(15) \quad m(l_1 l_2) = c \log \mathbf{R}(l_1 l_2, i_1 i_2),$$

the measure of line pairs will satisfy the condition

$$m(l_1 l_2) + m(l_2 l_3) = m(l_1 l_3)$$

whenever  $l_1, l_2, l_3$  are concurrent. Since the logarithm is a multiple-valued function, we must specify which value is chosen; and we must also determine the constant  $c$  conveniently.

Making use of the same coördinate system as in § 62, any point on  $l_\infty$  may be denoted by  $(0, \alpha, \beta)$ . In case  $\alpha/\beta$  is real,  $(\alpha/\beta)^2 > 0$ , and hence  $\alpha$  and  $\beta$  may be multiplied by a factor of proportionality so that

$$(16) \quad \alpha^2 + \beta^2 = 1.$$

Throughout the rest of this section we shall suppose  $\alpha$  and  $\beta$  subjected to this condition. This is equivalent to supposing that

$$\alpha = \cos(\theta + 2n\pi), \quad \beta = \sin(\theta + 2n\pi),$$

where  $0 \leq \theta \leq 2\pi$ , and  $n$  is an integer, positive, negative, or zero.

The double points of the absolute involution satisfy the condition (§ 62)

$$\alpha^2 + \beta^2 = 0,$$

and so may be written

$$I_1 = (0, 1, i) \quad \text{and} \quad I_2 = (0, 1, -i),$$

where  $i = \sqrt{-1}$ . Now if  $l_1$  and  $l_2$  meet  $l_\infty$  in  $(0, \alpha_1, \beta_1)$  and  $(0, \alpha_2, \beta_2)$  respectively, it follows that (§ 58, Vol. I)

$$\begin{aligned} \mathbf{R}(l_1 l_2, i_1 i_2) &= \frac{\alpha_1 - i\beta_1}{\alpha_1 + i\beta_1} + \frac{\alpha_2 - i\beta_2}{\alpha_2 + i\beta_2} \\ &= \frac{(\alpha_1 \alpha_2 + \beta_1 \beta_2) + i(\alpha_1 \beta_2 - \alpha_2 \beta_1)}{\alpha_1 \alpha_2 + \beta_1 \beta_2 - i(\alpha_1 \beta_2 - \alpha_2 \beta_1)}. \end{aligned}$$

The numbers  $\alpha = \alpha_1 \alpha_2 + \beta_1 \beta_2$  and  $\beta = \alpha_1 \beta_2 - \alpha_2 \beta_1$  satisfy the condition  $\alpha^2 + \beta^2 = 1$ . In fact, if  $\alpha_1 = \cos \theta_1$  and  $\alpha_2 = \cos \theta_2$ , then  $\alpha = \cos \theta$  and  $\beta = \sin \theta$ , where  $\theta = \theta_1 - \theta_2 + 2n\pi$ . Hence

$$\Re(l_1 l_2, i_1 i_2) = \alpha^2 - \beta^2 + 2i\alpha\beta.$$

Here again,  $\bar{\alpha} = \alpha^2 - \beta^2$  and  $\bar{\beta} = 2\alpha\beta$  satisfy the condition

$$\bar{\alpha}^2 + \bar{\beta}^2 = 1.$$

In fact,  $\bar{\alpha} = \cos 2\theta$ . Thus

$$\begin{aligned} (17) \quad \Re(l_1 l_2, i_1 i_2) &= \bar{\alpha} + i\bar{\beta} \\ &= \cos 2\theta + i \sin 2\theta \\ &= e^{2i\theta}. \end{aligned}$$

Hence

$$(18) \quad \log \Re(l_1 l_2, i_1 i_2) = 2i\theta,$$

where  $2\theta$  is real and may be chosen so that  $0 \leq 2\theta < 2\pi$ . Hence, choosing the constant  $c$  in (15) as  $\frac{-i}{2}$ , we have

$$(19) \quad m(l_1 l_2) = \frac{-i}{2} \log \Re(l_1 l_2, i_1 i_2) = \theta,$$

where  $\theta$  may be chosen so that  $0 \leq \theta < \pi$ .

The formula (19) is interesting in connection with the theorem that the sum of the angles of a triangle is equal to two right angles. This proposition can easily be established without the consideration of imaginaries, on the basis of the definitions in the last section. From our present point of view, however, it appears as follows: Let the three sides of a triangle be  $a$ ,  $b$ ,  $c$ , and let them meet the line at infinity in  $A_\infty$ ,  $B_\infty$ ,  $C_\infty$  respectively. It is easily verifiable that

$$\Re(A_\infty B_\infty, I_1 I_2) \cdot \Re(B_\infty C_\infty, I_1 I_2) \cdot \Re(C_\infty A_\infty, I_1 I_2) = 1,$$

from which it follows by (19) that

$$m(ab) + m(bc) + m(ca) = \pi.$$

Here we have a theorem on the line pairs rather than on the angles of a triangle. Indeed, (19) is necessarily a formula for the measure of a pair of lines and not of an angle, because of the fact that two opposite rays determine the same point at infinity.

The number  $m(ab)$  may also be defined as the smallest value between 0 and  $2\pi$ , inclusive, of the measures of the four angles  $\angle a_1 b_1$  which may be formed by a ray  $a_1$  of  $a$  and a ray  $b_1$  of  $b$ .

Following the common usage, we shall say that two pairs of lines which are congruent *make equal angles*, etc.

## EXERCISES

1. If  $A$  and  $B$  are any two points, the locus of a point  $P$  such that the rays  $PA$  and  $PB$  make a constant angle is a circle.

2. If in two projective flat pencils three lines of one make equal angles with the corresponding three lines of the other, the angle between any two lines of the one is the same as the angle between the corresponding lines of the other.

3. If  $OA, OB, OC, OD$  are four lines of a flat pencil,

$$\Re(OA, OB; OC, OD) = \frac{\sin \angle AOC}{\sin \angle AOD} + \frac{\sin \angle BOC}{\sin \angle BOD}.$$

In case the four lines form a harmonic set,

$$2 \cot \angle AOB = \cot \angle AOC + \cot \angle AOD.$$

4. If  $A_1, A_2, A_3, A_4$  are four points of a circle,

$$A_1A_3 \cdot A_2A_4 = A_1A_2 \cdot A_3A_4 + A_1A_4 \cdot A_2A_3,$$

where  $A_iA_j$  represents  $\text{Dist}(A_iA_j)$  or  $-\text{Dist}(A_iA_j)$  according as  $S(OA_iA_j) = S(OA_1A_2)$  or not,  $O$  being an arbitrary point of the circle and  $S(OA_iA_j)$  being a sense-class on the circle.

5. If  $a, b, c$  are the sides of a triangle and  $a_1a_2, b_1b_2, c_1c_2$  are pairs of lines through the vertices  $bc, ca, ab$  respectively, the six lines  $a_1, a_2, b_1, b_2, c_1, c_2$  are tangents of a conic if and only if

$$\frac{\sin(a_1b)}{\sin(a_1c)} \cdot \frac{\sin(a_2b)}{\sin(a_2c)} \cdot \frac{\sin(b_1c)}{\sin(b_1a)} \cdot \frac{\sin(b_2c)}{\sin(b_2a)} \cdot \frac{\sin(c_1a)}{\sin(c_1b)} \cdot \frac{\sin(c_2a)}{\sin(c_2b)} = 1.$$

6. The points of a ray having  $(x, y)$  as origin may be represented in the form

$$(x + \lambda\alpha, y + \lambda\beta),$$

where  $\alpha$  and  $\beta$  are fixed and  $\lambda > 0$ . There is a one-to-one reciprocal correspondence between the rays having  $(x, y)$  as origin and the ordered pairs of values of  $\alpha$  and  $\beta$  which satisfy the condition

$$\alpha^2 + \beta^2 = 1.$$

When  $\alpha$  and  $\beta$  satisfy this condition, the numerical value of  $\lambda$  is the distance between  $(x, y)$  and  $(x + \lambda\alpha, y + \lambda\beta)$ .

7. Two angles formed by the pairs of rays

$$(x_0 + \lambda\alpha, y_0 + \lambda\beta) \text{ and } (x_0 + \lambda\alpha', y_0 + \lambda\beta'),$$

$$(\bar{x}_0 + \lambda\bar{\alpha}, y_0 + \lambda\bar{\beta}) \text{ and } (\bar{x}_0 + \lambda\bar{\alpha}', y_0 + \lambda\bar{\beta}')$$

$$\lambda > 0$$

respectively are congruent if and only if

$$\alpha\alpha' + \beta\beta' = \bar{\alpha}\bar{\alpha}' + \bar{\beta}\bar{\beta}'.$$

8. Relative to the homogeneous coördinates employed above, the formula for the distance between  $(x_0, x_1, x_2)$  and  $(y_0, y_1, y_2)$  may be written

$$\frac{\sqrt{(x_1y_0 - x_0y_1)^2 + (x_2y_0 - x_0y_2)^2}}{x_0y_0} = \frac{1}{x_0y_0} \begin{vmatrix} x_0 & x_1 & x_2 \\ y_0 & y_1 & y_2 \\ 0 & 1 & i \end{vmatrix}^{\frac{1}{2}} \cdot \begin{vmatrix} x_0 & x_1 & x_2 \\ y_0 & y_1 & y_2 \\ 0 & 1 & -i \end{vmatrix}^{\frac{1}{2}}.$$

**73. Generalization by projection.** The relation established in § 66 between Euclidean and projective geometry furnishes a source of new theorems in each. A theorem which has been proved for projective geometry can be specialized into a theorem of Euclidean geometry, or a theorem of Euclidean geometry may be generalized so as to furnish a theorem of projective geometry.

The two processes, of generalization and of specialization, may often be combined in a happy way with the principle of duality or with other general methods of projective geometry. Thus a theorem proved for Euclidean geometry can be generalized into a theorem of projective geometry and the dual of the general theorem specialized into a new theorem of Euclidean geometry. As an example, let us take the theorem of Euclid:

*A. The perpendiculars from the vertices of a triangle to the opposite sides meet in a point (the orthocenter).*

The sides of the triangle meet the line at infinity in three points, and the three perpendiculars are lines from the vertices to the conjugates of these three points in the absolute involution. The Euclidean theorem is therefore a special case of the following projective theorem:

*B. The lines joining the vertices of a triangle to the conjugates, with respect to an arbitrary elliptic involution on a line  $l$ , of the points in which the opposite sides meet  $l$ , are concurrent.*

This is a portion of Theorem 27, Chap. IV, Vol. I, the orthocenter and the three vertices of the triangle being the vertices of a complete quadrangle. But though the Euclidean theorem is a special case, yet the general theorem for elliptic involutions in real geometry may easily be proved by means of it. For, given any elliptic involution whatever and any triangle, the involution can be projected into the absolute involution and the given triangle will go into a triangle of the Euclidean plane. Hence the general theorem, B, that certain three lines meet in a point could fail to be true only if the Euclidean theorem, A, failed.

It is to be noted that this proves the theorem only for a real space and an elliptic involution. In a complex space (§ 5) it might happen that any

transformation which carried the involution into the absolute involution would carry the triangle into one whose sides are not all real.

Now consider the plane dual of the projective theorem, B.

*B'. The points of intersection of the sides of a triangle with the conjugates in an arbitrary involution at a point  $L$ , of the lines joining the vertices to  $L$ , are collinear.*

If the involution at  $L$  is taken as the orthogonal involution we have the Euclidean theorem:

*A'. The three sides of a triangle are met in three collinear points by the perpendiculars from a fixed point to the lines joining this point to the opposite vertices.*

The second of the two processes which we are here emphasizing, namely the discovery of Euclidean theorems by specializing projective ones, is brilliantly illustrated in many of the textbooks on projective geometry. We may mention the following:

L. Cremona, *Elements of Projective Geometry*, Oxford, 1894.

T. Reye, *Geometrie der Lage*, Leipzig, 1907-1910.

R. Sturm, *Die Lehre von den Geometrischen Verwandtschaften*, Leipzig, 1909.

R. Böger, *Geometrie der Lage*, Leipzig, 1900.

H. Grassman, *Projective Geometrie der Ebene*, Leipzig, 1909.

J. J. Milne, *Cross-Ratio Geometry*, Cambridge, 1911.

J. L. S. Hatton, *Principles of Projective Geometry*, Cambridge, 1913.

The reader will find material for the illustration of the second process, namely the discovery of projective theorems by generalizing metric ones, in Euclid's *Elements*, and even more in such books as the following:

J. Casey, *A Sequel to the First Six Books of the Elements of Euclid*, Dublin, 1888.

C. Taylor, *Ancient and Modern Geometry of Conics*, Cambridge, 1881.

J. W. Russell, *Elementary Treatise on Pure Geometry*, Oxford, 1905.

The class of theorems which are here in question will be dealt with to some extent in the following chapter, and the methods available will be extended in Chap. VI by the study of inversions. But on account of the magnitude of the subject many important theorems will be found relegated to the exercises and many others omitted entirely. In nearly every such case, however, a good treatment can be found in one or another of the books on projective geometry referred to above.

The current textbooks do not often classify theorems on the basis of the geometries to which they belong (§ 34) and the assumptions which are necessary for their proof (§ 17). Some progress has been made on such a classification in the present book (cf. § 83 below), but more remains to be done.

Another criticism on current books is that they employ imaginary points in a rather shy and awkward manner. This is doubtless due to the fact that, previous to a logical treatment of the subject based on definite assumptions, the geometry of reals was regarded as having, somehow, a higher degree of validity than the complex geometry. The reader will often find it easy to abbreviate the proofs of theorems in the literature by a free use of imaginary elements (cf. § 78).

### EXERCISES

1. Generalize projectively the following theorems:

- (a) The medians of a triangle meet in a point.
- (b) The perpendiculars at the mid-points of the sides of a triangle meet in a point.
- (c) The diagonals of a parallelogram bisect each other.

2. Let  $A_1, B_1, C_1$  be the points in which the lines joining the vertices  $A, B, C$ , respectively, of a triangle to the orthocenter,  $O$ , meet the opposite sides. The circle through  $A_1, B_1$  and  $C_1$  contains the mid-points of the pairs  $AB, BC, CA$  and of the pairs  $OA, OB, OC$ . This circle is called the *nine point* or *Feuerbach circle* of the triangle. Cf. Ex. 7, § 41.

3. A hyperbola whose asymptotes are orthogonal is said to be *equilateral* or *rectangular*. Every hyperbola passing through four points of intersection of two equilateral hyperbolas is an equilateral hyperbola.

4. All equilateral hyperbolas circumscribed to a triangle pass through its orthocenter.

5. The centers of the equilateral hyperbolas circumscribed to a triangle lie on the nine-point circle.

## CHAPTER V \*

### ORDINAL AND METRIC PROPERTIES OF CONICS

**74. One-dimensional projectivities.** The general discussion of one-dimensional projectivities in Chap. VIII, Vol. I, has a great many points of contact with the ordinal and metric theorems of the last three chapters. For example, a rotation leaving a point  $O$  invariant transforms into itself any circle  $C^2$  with  $O$  as a center. The transformation effected on the circle by the rotation is a one-dimensional projectivity having the point  $O$  as center and the line at infinity as axis. The defining property of the axis of the projectivity in this case is that if a pair of points  $AB$  of the circle be rotated into a pair  $A'B'$  (i.e. if  $\angle AOB$  be congruent to  $\angle A'OB'$ ), then the line  $AB'$  is parallel to the line  $A'B$ , which is a well-known Euclidean theorem.

The proposition that any rotation is a product of two line reflections corresponds to the proposition that any projectivity is a product of two involutions. The point reflection with  $O$  as center is commutative with all the other rotations about  $O$  and hence effects on  $C^2$  an involution which (§ 79, Vol. I) belongs to all the projectivities effected on  $C^2$  by the rotations of this group. This involution is harmonic (§ 78, Vol. I) to the involution effected on  $C^2$  by any orthogonal line reflection whose axis contains  $O$ , and hence all the involutions of the latter sort form a pencil. Thus all the theorems of § 79, Vol. I, can be specialized so as to yield theorems about the group of rotations with  $O$  as center.

There are many other applications of the theorems in Chap. VI, Vol. I, to affine and Euclidean geometry (a few of them are indicated in the exercises below), but the main application which we are to consider at present is to the theory of order relations. Let us first recall some of the ordinal theorems which have already been established, and interpret them on the conic sections. Extending the definition of § 4, we shall say:

\* In the earlier chapters of this volume we have used only the first seven chapters of Vol. I. The present chapter may advantageously be read in connection with Chaps. VIII-X, Vol. I. Chap. IX is first used in § 77 and Chap. X in § 85.

DEFINITION. A projectivity of a one-dimensional form in any ordered space is *hyperbolic*, *parabolic*, or *elliptic* according as it has two, one, or no double points.

With regard to involutions, we have already established the following propositions (§ 21): *If an involution preserves sense, each pair separates every other pair. If an involution alters sense, no pair separates any other pair. An involution which does not alter sense is elliptic; that is to say, the pairs of a hyperbolic involution do not separate each other. The double points of a hyperbolic involution separate every pair of the involution.*

DEFINITION. If  $A, B, C, D$  are four distinct points of a conic, the point  $O$  of intersection of the lines  $AB$  and  $CD$  is called an *interior* point in case the pairs  $AB$  and  $CD$  separate each other\* and an *exterior* point in case these pairs do not separate each other. The set of all interior points is called the *interior* or *inside* of the conic, and the set of all exterior points is called the *exterior* or *outside* of the conic.

The pairs  $AB$  and  $CD$  are conjugate in the involution with  $O$  as center. Hence, if these two pairs separate each other, this involution preserves sense and is such that any two of its pairs separate each other. Hence any two lines through  $O$  which meet the conic meet it in pairs of points which separate each other. That is to say, the definition of an interior point is independent of the particular choice of the points  $A, B, C, D$ . A like argument applies in case  $O$  is exterior. In case the involution with  $O$  as center has double points, the lines joining  $O$  to these points are tangent to the conic. Hence the next to the last of the propositions about involutions stated above implies that there are no tangents through an *interior* point. These results may be stated as follows:

THEOREM 1. *The points coplanar with a conic fall into three mutually exclusive classes: the conic itself, its interior and its exterior. Each interior point is the center of an involution on the conic which preserves sense, and each exterior point of one which alters sense. All points of a tangent, except the point of contact, are exterior points of the conic.*

\* Cf. § 20, particularly the footnotes.

Now let  $O$  be any interior point. If  $O'$  is any point conjugate to  $O$  with regard to the conic, there exists (cf. fig. 59) a complete quadrangle  $ABCD$  whose vertices are points on the conic such that  $AB$  and  $CD$  meet in  $O$  and  $AD$  and  $CB$  meet in  $O'$ . But by Theorem 7,

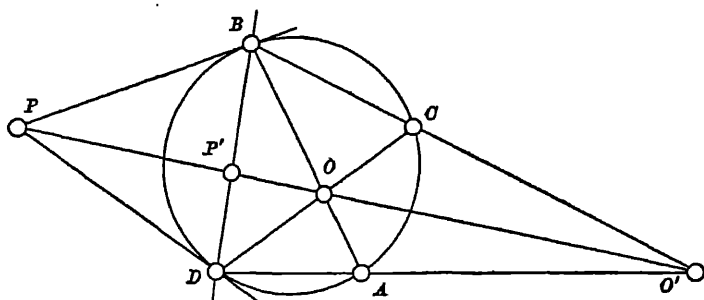


FIG. 59

Chap. II, if  $AB$  separates  $CD$ , then  $AD$  does not separate  $BC$ , and hence  $O'$  is an exterior point. Hence the polar line of any interior point consists entirely of exterior points. Hence

**THEOREM 2.** *All points conjugate to an interior point are exterior.*

Suppose, further, that the tangent to the conic at  $B$  meets the line  $OO'$  in a point  $P$  and the line  $BD$  meets  $OO'$  in a point  $P'$  (fig. 59). Then  $P$  and  $P'$  are conjugate points with regard to the conic. Moreover,

$$ABCD \overline{\wedge} OPO'P'.$$

Since  $A$  and  $C$  do not separate  $B$  and  $D$ , it follows that the pair  $OO'$  does not separate the pair  $PP'$ . That is,

**THEOREM 3.** *On a line containing an interior point of a conic the pairs of conjugate points with regard to the conic do not separate one another.*

By elementary propositions about poles and polar there follow at once:

**COROLLARY 1.** *The pole of a line which contains an interior point is an exterior point.*

**COROLLARY 2.** *The polar of an exterior point contains some interior points.*

In § 78, Vol. I, it was established that any projectivity is a product of two involutions one of which is hyperbolic. Since a hyperbolic involution is opposite, it follows that if the given projectivity is direct, it is a product of two opposite involutions; and if the given projectivity is opposite, it is a product of a direct and an opposite involution. But in the second case the direct involution is, by the argument just made, a product of two opposite involutions. Hence

**THEOREM 4.** *A direct projectivity is a product of two opposite involutions, and an opposite projectivity is a product of three opposite involutions. An opposite projectivity is also expressible as a product of a direct and an opposite involution.*

In the case of projectivities on a conic, the axis of the product of two involutions is the line joining their centers. Hence we have, as consequences of this theorem,

**COROLLARY 1.** *Any line in the plane of a conic contains points exterior to the conic.*

**COROLLARY 2.** *A projectivity whose center is an interior point, and whose axis therefore consists entirely of exterior points, is direct.*

In the fourth exercise, below, we need the following definition:

**DEFINITION.** The line perpendicular to a tangent to a conic and passing through its point of contact is called the *normal* to the conic at this point.

### EXERCISES

1. What transformations of the Euclidean group effect projectivities on  $l_\infty$  to which the absolute involution belongs? How are these distinguished from the remaining similarity transformations by their relation to the circular points? What transformations of the Euclidean group are harmonic on  $l_\infty$  to the absolute involution?

2. Show that the measure of a line pair as defined in § 72 is the logarithm of the characteristic cross ratio of a certain projectivity on  $l_\infty$ . Obtain an analogous formula for the measure of an angle in terms of the characteristic cross ratio of a projectivity on a circle.

3. Any noninvolutoric planar collineation which leaves invariant a conic and a line transforms the points of the line by a projectivity to which belongs the involution of conjugate points with regard to the conic.

4. If  $P$  is any fixed point of a conic and  $RQ$  a variable point pair such that  $\angle RPQ$  is a right angle, the lines  $RQ$  meet in a fixed point on the normal at  $P$ .

5. The lines joining homologous points in a noninvolutoric projectivity on a conic are the tangents of a second conic.

6. If  $P$  is any fixed point of a conic and  $RQ$  a variable pair of points such that  $\angle RPQ$  has constant measure, the lines  $RQ$  are the tangents to a second conic.

7. If a projectivity  $\Gamma$  on a line is a product of an involution having double points,  $A_1$  and  $B_1$ , followed by another involution, and if  $\Gamma^{-1}(A_1) = A_0 \neq A_1$  and  $\Gamma(A_1) = A_2$ , then  $A_1$  and  $B_1$  are harmonically conjugate with regard to  $A_0$  and  $A_2$  whenever  $A_0 \neq A_2$ ; and  $B_1 = A_0$  whenever  $A_0 = A_2$ .

8. If  $A_1$  and  $B_1$  are a pair of an involution  $I$  which is left invariant by a projectivity  $\Gamma$ , and if  $\Gamma^{-1}(A_1) = A_0 \neq A_1$  and  $\Gamma(A_1) = A_2 \neq A_0$ , then  $A_0$  and  $A_2$  are harmonically conjugate with regard to  $A_1$  and  $B_1$ .

9. Let  $A$  and  $A'$  be any pair of an involution  $I$ . If  $A \neq A'$ , any projectivity  $\Pi$  which transforms  $I$  into itself and leaves  $A$  invariant is either the involution, with  $A$  and  $A'$  as double points, or the identity.

10. Generalize § 80, Vol. I, so as to apply to the group of translations and the equiaffine group, using the fact that the transformations in each of these groups are products of pairs of involutoric projectivities.

## 75. Interior and exterior of a conic.

**THEOREM 5.** *Any two points of a conic are the ends of two linear segments one consisting entirely of interior points and the other entirely of exterior points.*

*Proof.* Let the given points be denoted by  $A$  and  $B$ , let  $C$  and  $D$  be any two other points of the conic which separate  $A$  and  $B$ , and let  $\sigma$  and  $\bar{\sigma}$  represent the segments  $\overline{ACB}$  and  $\overline{ADB}$  on the conic. By the definition of the order relations on the conic, the lines joining  $C$  to the points of  $\bar{\sigma}$  meet the line  $AB$  in the points of a segment  $\bar{\sigma}'$  whose ends are  $A$  and  $B$ , and these points satisfy the definition of interior points. In like manner the lines joining  $C$  to points of  $\sigma$  meet the line  $AB$  in a segment  $\sigma'$  which is complementary to  $\bar{\sigma}'$  and consists entirely of exterior points.

In a real plane the following theorem is a consequence of what we have just proved, but in order to have the result for any ordered plane we give a proof which is entirely general.

**THEOREM 6.** *Any two interior points of a conic are the ends of a segment consisting entirely of interior points.*

*Proof* (fig. 60). Let  $A$  and  $C$  be two interior points. Let  $A_1$  be any point of the conic not on the line  $AC$ . The lines  $A_1C$  and  $A_1A$  are not tangent to the conic, since (Theorem 1) the involutions at  $A$  and  $C$  are both elliptic. Let  $A_0$  and  $B_2$  respectively be the points, distinct from  $A_1$ , in which the lines  $A_1A$  and  $A_1C$  meet the conic. The two segments of the conic whose ends are  $A_0$  and  $B_2$  are projected by the lines through  $A_1$  into the two segments of the line  $AC$  which have  $A$  and  $C$  as their ends. We shall prove that the segment  $\sigma$  of the line  $AC$  which is the projection of the segment complementary to  $\overline{A_0A_1B_2}$  consists entirely of interior points.

Let  $B$  be any point of  $\sigma$ . The line  $A_1B$  then meets the conic in a point  $C_2$  which is separated from  $A_1$  by  $A_0$  and  $B_2$ . Let  $B_3A$  meet the conic in  $C_1$ , let  $C_1B$  meet it in  $A_3$ , and let  $A_2C$  meet it in  $B_1$ , so that  $A_1B_2C_1A_3B_1C_2$  form a Pascal hexagon whose pairs of opposite sides meet in  $A, B, C$ . Since  $A$  is an interior point, we have the

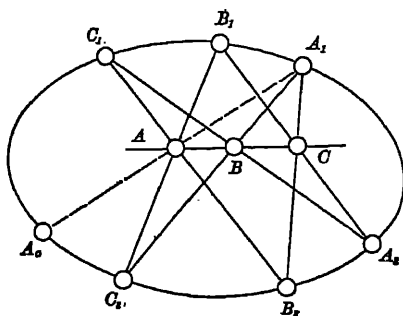


FIG. 60

order  $\{C_2A_0B_1A_1\}$ . Since  $B$  was chosen so that  $C_2$  and  $A_1$  are separated by  $B_2$  and  $A_0$ , we have  $\{B_2C_2A_0A_1\}$ . From these there follows  $\{B_2C_2A_0B_1A_1\}$ . Transforming this by the involution at  $A$  we have  $\{C_1B_1A_1C_2A_0\}$ . Hence we have  $\{B_2C_2A_0C_1B_1A_1\}$ . Since the involution with center at  $C$  is elliptic, we have  $\{B_3B_1A_1A_2\}$ . Hence we have  $\{B_2C_2A_0C_1B_1A_1A_2\}$ . Hence  $C_2$  and  $A_1$  separate  $A_2$  and  $C_1$ , and hence  $B$  is interior to the conic.

**THEOREM 7.** *Any two exterior points are ends of a segment consisting entirely of exterior points.*

*Proof.* Let the two exterior points be  $E_1$  and  $E_2$ . If the line  $E_1E_2$  is tangent, all points on it except the points of contact are exterior, since each of these points is the center of a hyperbolic involution on the conic. In this case the theorem is obvious. If the line  $E_1E_2$  meets the conic in two points, the theorem reduces to Theorem 5. If the line  $E_1E_2$  does not meet the conic, and both the segments with  $E_1$  and  $E_2$  as ends should contain interior points,  $I_1$  and  $I_2$  respectively, then neither of the segments whose ends are  $I_1$  and  $I_2$  could consist entirely of interior points, contrary to Theorem 6.

The theorems above are connected with the following algebraic considerations: Any involution can be written in the form

$$(1) \quad x' = \frac{ax + b}{cx - a}.$$

If we regard  $a, b, c$  as a set of homogeneous coördinates in a projective plane, then for every involution (1) there is one and only one point  $(a, b, c)$ ; and inversely for every point  $(a, b, c)$  there is a unique involution (1), provided that the point does not satisfy the condition

$$(2) \quad a^2 + bc = 0.$$

By § 18 the projectivities (1) for which

$$(3) \quad a^2 + bc > 0$$

are opposite, and those for which

$$(4) \quad a^2 + bc < 0$$

are direct.

The equation (2) represents a conic section of which the points satisfying (3) are the exterior and those satisfying (4) are the interior. This may be proved as follows:

The conic is given by the parametric representation (§ 82, Vol. I)

$$a : b : c = x : x^2 : -1,$$

and any involution on the conic is given by the transformation (1) of the parameter  $x$ . The center of the involution is the point of intersection of the lines containing pairs of the involution. The point  $(0, 0, 1)$  of the conic is given by the value 0 of the parameter  $x$  and thus is transformed to the point given by the value  $x = -b/a$ , namely, the point  $(-ab, b^2, -a^2)$ . The point  $(0, 1, 0)$  of the conic is given by  $x = \infty$  and thus is transformed to the point given by  $x = a/c$ , namely, the point  $(ac, a^2, -c^2)$ . The point of intersection of the lines joining  $(0, 0, 1)$  to  $(-ab, b^2, -a^2)$  and  $(0, 1, 0)$  to  $(ac, a^2, -c^2)$  is manifestly  $(-a, b, c)$ . Hence  $(-a, b, c)$  is the center of the involution (1), and therefore is interior to the conic if (4) is satisfied and the involution direct, and exterior to the conic if (3) is satisfied and the involution opposite.

### EXERCISES

1. Parabolic projectivities are direct.
2. Two of the three vertices of any self-polar triangle of a conic are exterior points.
3. The center of a hyperbola is an exterior point.
4. The center of a circle is an interior point.
5. In a Euclidean plane all points interior to a circle and all points on it (except the point of contact of the tangent in question) lie entirely on one side of any one of its tangents.

6. If a segment  $A_1B_1$  is contained in a segment  $A_2B_2$ , the circle the ends of whose diameter are  $A_1$  and  $B_1$  is composed of points interior to the circle the ends of whose diameter are  $A_2$  and  $B_2$ .

7. In a Euclidean plane all points interior to an ellipse lie entirely on one side of any line consisting entirely of exterior points.

8. Any two pairs of conjugate diameters of an ellipse separate each other. Two pairs of conjugate diameters of a hyperbola never separate each other.

9. If  $O$  is the center of a conic  $K^2$ , the polar reciprocal of a conic  $C^2$  with respect to  $K^2$  will be an ellipse, parabola, or hyperbola according as  $O$  is interior to, on, or exterior to  $C^2$ .

10. Consider a conic  $C^2$  in a planar net of rationality satisfying Assumption H. The points of the net exterior to the conic fall into two classes  $[E]$  and  $[F]$  such that two tangents to the conic can be drawn from any point  $E$  and no tangent can be drawn to the conic from any point  $F$ . On any line in which one  $E$  is conjugate to an  $F$  with regard to  $C^2$ , every  $E$  is conjugate to an  $F$ . On any line in which one  $E$  is conjugate to an  $E$ , every  $E$  is conjugate to an  $E$  and every  $F$  to an  $F$ . The interior points fall into two classes  $[I]$  and  $[J]$  such that the pairs of conjugate lines on a point  $I$  either both meet  $C^2$  or both do not meet  $C^2$ , whereas one member of any pair of conjugate lines on a point  $J$  meets  $C^2$  and the other member does not meet  $C^2$ .

11. Let the equation of a conic be  $f(x_0, x_1, x_2) = 0$  and let the determinant of the coefficients of  $f(x_0, x_1, x_2)$  be

$$A = \begin{vmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{vmatrix} \neq 0, \quad a_{ij} = a_{ji}.$$

A point  $(x'_0, x'_1, x'_2)$  is interior or exterior according as  $A \cdot f(x'_0, x'_1, x'_2)$  is greater or less than zero.

**76. Double points of projectivities.** The preceding theorems hold for any ordered space. On specializing to a real space we have the additional theorem that a projectivity which alters sense has two double points (§ 4). In the case of involutions this result combined with the theorem that a hyperbolic involution is always opposite gives

**THEOREM 8.** *The pairs of an elliptic involution always separate one another, and the pairs of a hyperbolic involution never separate one another.*

The last half of this theorem, combined with Theorem 3, gives the condition for the intersection of a line with a conic, a condition which has already been given in a more special form in § 64.

**THEOREM 9.** *On any line through an interior point of a conic the involution of conjugate points is hyperbolic, and the line meets the conic in the double points of this involution.*

By Cor. 2, Theorem 3, the polar of an exterior point is a line through an interior point. The lines joining the exterior point to the points of intersection of its polar with the conic are tangents. Hence

**COROLLARY 1.** *Through any exterior point there pass two tangents to a conic.*

**COROLLARY 2.** *Two involutions, one at least of which is elliptic, have one and only one common pair.*

*Proof.* The center of an elliptic involution represented on a conic is an interior point. The line joining this point to the center of any other involution meets the conic in two points which are pairs of both involutions. Since any pair of an involution is collinear with the center, the two points so constructed are the only pair common to the two involutions.

A special case of this corollary may be stated in the following form:

**COROLLARY 3.** *In a given one-dimensional form there is one and only one pair of elements which are conjugate with respect to a given elliptic involution and harmonically separated by a given pair of elements.*

Since a hyperbolic involution is determined by its double points, it is evident that any two hyperbolic involutions are equivalent under the group of all projectivities of a one-dimensional form. The corresponding theorem for elliptic involutions is best seen by representing the involutions on a conic. The two centers  $I_1, I_2$  are interior points, and the line joining them meets the conic in two points  $C_1, C_2$  which do not separate them (Theorem 5). Let  $O_1$  and  $O_2$  be the double points (Theorem 8) of the involution in which  $I_1 I_2$  and  $C_1 C_2$  are pairs. An involution with either of the points  $O_1$  or  $O_2$  as center will evidently transform the one with  $I_1$  as center into the one with  $I_2$  as center. Hence

**COROLLARY 4.** *Any two elliptic involutions in the same real one-dimensional form are conjugate under the projective group of that form.*

## EXERCISES

1. All involutions which are harmonic to (i.e. commutative with and distinct from) an elliptic involution are hyperbolic.

2. If two points  $A, B$  of a line separate each point  $P$  ( $P \neq A, P \neq B$ ) of the line from its conjugate point in a given elliptic involution,  $A$  and  $B$  are conjugate in this involution.

3. A hyperbolic projectivity is opposite or direct according as a pair of homologous points does or does not separate the double points.

4. Elliptic projectivities are direct.

5. The center of an ellipse is an interior point.

6. The involution determined on the line at infinity of a Euclidean plane by an ellipse is elliptic, by a hyperbola, hyperbolic.

7. Any two ellipses are conjugate under the affine group.\*

8. An involution in a flat pencil is either such that every pair of conjugate lines is orthogonal or there is one and only one orthogonal pair of conjugate lines.

9. A conic having two pairs of perpendicular conjugate diameters is a circle.

10. If  $A_1$  and  $A_2$  are the real limiting points of a pencil of circles, each circle of the pencil either contains  $A_1$  and is on the opposite side of the radical axis from  $A_2$ , or contains  $A_2$  and is on the opposite side of the radical axis from  $A_1$ .

11. Of two circles of a pencil, both containing the same limiting point, one is entirely interior to the other.

12. For any angle,  $\angle ABC$ , there is one and only one pair  $l, l'$  of orthogonal lines through  $B$  which separate the lines  $BA$  and  $BC$  harmonically. One line,  $l$ , of the pair contains points  $P$  interior to  $\angle ABC$ , and  $\angle ABP$  is congruent to  $\angle PBC$ . The line  $l$  is called the *interior bisector*, and the line  $l'$  the *exterior bisector*, of the angle  $\angle ABC$ .

13. The asymptotes of an equilateral hyperbola bisect any pair of conjugate diameters.

14. The bisectors of the angles of a triangle  $ABC$  meet in four points, one in each of the four regions determined by  $ABC$  according to § 26. These four points are the centers of four circles inscribed in  $ABC$  and are the vertices of a complete quadrilateral of which  $ABC$  is the diagonal triangle. The mid-point of the pair  $BC$  is the mid-point of the points of contact of either pair of inscribed circles whose centers are collinear with  $A$ .

15. Let  $V$  and  $V'$  be the vanishing points (§ 43) of a projectivity on a line, the notation being so assigned that the point at infinity is transformed to  $V'$ . There exist two points  $A, B$  which are transformed to two points  $A', B'$  such that

$$AV = VB = A'V' = V'B'.$$

\* Cf. § 37, Exs. 14 and 15.

**77. Ruler-and-compass constructions.** The discussion in Chap. IX, Vol. I, reduces any quadratic problem to the problem of finding the points of intersection of an arbitrary line with a fixed conic. According to Theorems 5 and 9 the necessary and sufficient condition that a line coplanar with a conic meet it in two points is that the line pass through an interior point of the conic. Hence this condition will serve to determine the solvability of any problem of the second degree in a real space. Thus the discussion of linear and quadratic constructions, under the projective meaning of these terms, may be regarded as complete.

When we adopt the Euclidean point of view, the fixed conic may be taken as a circle; and therefore every problem of the second degree is reduced to the problem of determining the points of intersection of an arbitrary line with a fixed circle (cf. § 86, Vol. I).

The constructions of elementary Euclidean geometry which are known as ruler-and-compass constructions involve the determination of the points of intersection (whenever existent) of two arbitrary lines, or of an arbitrary line with an arbitrary circle, or of two arbitrary circles. The last of these problems has been shown in § 65 to be reducible to the first and second. Hence any ruler-and-compass construction may be reduced to the problem of finding the intersection of an arbitrary line with a fixed circle.

On account of the special character of the line at infinity, there is not a perfect correspondence between the linear constructions of projective geometry and the Euclidean constructions by means of a ruler. The operations involved in the linear constructions of projective geometry are

- (a) to join two points by a (projective) line;
- (b) to take the point of intersection of any two lines.

These are evidently equivalent to the following Euclidean operations:

- (1) to join two ordinary points by a line;
- (2) to take the point of intersection of two nonparallel lines;
- (3') to draw a line through a given point parallel to a given line.

The first of these operations corresponds to the proposition that two points are on a unique line, the second to the proposition that two nonparallel lines determine a unique point. These operations

may be thought of as carried out with a straightedge or ruler whose length is not limited.

The operation (3') can be effected by means of (1) and (2), together with the following operation:

(3) to find on any ray through a point  $A$ , a point  $C$  such that the point pair  $AC$  is congruent to a preassigned point pair  $AB$ .\*

For let  $A$  be the given point and let  $BC$  be the given line. Let  $O$  be a point on the line  $AB$  in the order  $\{ABO\}$  such that  $BA$  is congruent to  $BO$ . Let  $\bar{A}$  be the point of the line  $OC$  in the order  $O\bar{C}\bar{A}$  such that  $CO$  is congruent to  $\bar{C}\bar{A}$ . Then  $A\bar{A}$  is evidently parallel to  $BC$ .

Thus (1), (2), and (3) serve as a basis for all linear operations in the projective sense. They obviously yield also a certain class of quadratic constructions; but they do not suffice for all quadratic constructions. The latter may be provided for, as explained above, by adjoining the operation of taking the point of intersection with a fixed circle of an arbitrary line through an arbitrary interior point.

For the proof that (3') is not a consequence of (1) and (2), and that (1), (2), (3) do not provide for all quadratic constructions, the reader is referred to Hilbert, *Grundlagen der Geometrie*, Chap. VII (4th edition, 1913).

### EXERCISES

1. Given three collinear points  $A, B, C$  such that  $AB$  is congruent to  $BC$ , show how to construct a parallel to the line  $AB$  through an arbitrary point  $P$  by means of the operations (1) and (2) alone.

2. Given two parallel lines, show how to find the mid-point of any pair of points on either of the lines by means of (1) and (2) alone.

3. Given a parallelogram and a point  $P$  and a line  $l$  in its plane. Through  $P$  draw a line parallel to  $l$ , making use of the ruler only.

\* It is important to notice that the pairs  $AB$  and  $AC$  have the point  $A$  in common. Thus (3) provides merely for drawing a circle through a given point and with a given other point as center. The drawing instrument to which this corresponds is a pair of compasses which snaps together when lifted from the paper, so that it cannot be used to transfer a point pair  $AB$  to a point pair  $A'B'$  unless  $A = A'$ . This will be understood by anyone reading the second proposition in Euclid's *Elements*, which shows how to lay off a point pair congruent to a given point pair on a given ray. The operation (3) may be replaced by the operation of finding on any ray  $AB$  a point  $C$  such that the point pair  $AC$  is congruent to a fixed point pair  $OP$ . The instrument for this operation may be thought of as a measuring rod of fixed length (say unit length) without subdivisions. (Cf. the reference to Hilbert, below.)

4. Given a point pair  $AC$  and its mid-point  $B$ , using the ruler alone, construct the point pair  $AD$  such that

$$\frac{AC}{AD} = n.$$

5. Given four collinear points  $A, A', B, B'$ , construct the fixed point of the parabolic projectivity carrying  $A$  to  $A'$  and  $B$  to  $B'$ .

6. Given a projectivity on a line, find a pair of corresponding points  $A$  and  $A'$  such that a given point  $M$  is the mid-point of the segment  $AA'$ .

7. Inscribe in a given triangle a rectangle of given area.

8. Given four tangents of a parabola, construct a tangent parallel to a given line.

9. Given three points of a hyperbola and a line parallel to each asymptote, find the point of intersection of the hyperbola with a line parallel to one of the asymptotes.

10. Construct by ruler and compass any number of tangents to a conic given by five of its points; also any number of points of a conic given by five of its tangents.

11. Construct any number of points of a parabola through four given points.

12. Construct any number of points of a parabola touching three given lines and passing through a given point.

13. Through a given point construct an orthogonal pair of lines conjugate with regard to a conic. (If the point is exterior to the conic, these lines are the bisectors of the angles formed by the tangents to the conic from this point.)

**78. Conjugate imaginary elements.** It has been shown in § 6 that a real projective space  $S$  can be regarded as immersed in a complex projective space  $S'$  in such a way that every line of  $S$  is a subset of a unique line of  $S'$ . Certain additional definitions and conventions have been introduced in § 70. But in both these places little use was made of the properties of imaginary elements beyond their existence and the fact that  $S'$  satisfies Assumptions A, E, P. We shall now prove some of the most elementary theorems about the relation between elements of  $S$  and  $S'$ .

**DEFINITION.** Two imaginary points, lines, or planes are said to be *conjugate relative to a real one-dimensional form* of the first or second degree if and only if they are the double elements of an involution in the real form.

As an example consider a real conic  $C^2$  and a line  $l$  exterior to it. The conic and the line have in common the double points of an elliptic involution on  $l$ . But these points are also the double points of

the involution on  $C^2$  whose axis is  $l$ . Hence the points common to  $C^2$  and  $l$  are conjugate imaginaries both with respect to  $C^2$  and to  $l$ . Since any one-dimensional form of the first or second degree whose elements are points is a line or a point conic, and since the double points of any involution on a conic are the intersections of the axis of the involution with the conic, we have

**THEOREM 10.** *Any two conjugate imaginary points are on a real line.*

By duality we have that any two conjugate imaginary planes are on a real line.

Two conjugate imaginary lines are by definition on a real point, line conic, cone of lines, or regulus. If they are on a real line conic, the plane dual of the argument above shows that they are on a real point. By dualizing in space we obtain the same result for conjugate imaginary lines of a cone of lines. Hence we have

**THEOREM 11.** *Any two conjugate imaginary coplanar lines are on a real point and any two conjugate imaginary concurrent lines are on a real plane.*

Conjugate imaginary lines on a regulus will be considered in a later chapter.

**THEOREM 12.** *The lines joining a real point to two conjugate imaginary points not collinear with it are conjugate imaginary lines.*

*Proof.* The conjugate imaginary points are double points of an elliptic involution on a real line. From any point not on this line this involution is projected into an involution of lines whose double lines are the projections of the given points.

**THEOREM 13.** *If  $A_1A_2$  and  $B_1B_2$  are two pairs of conjugate imaginary points on different lines, the lines  $A_1B_1$  and  $A_2B_2$  meet in a real point and are conjugate imaginary lines.*

*Proof.* By hypothesis the lines  $A_1A_2$  and  $B_1B_2$  are real and hence they meet in a real point  $C$ . Let  $B$  be the conjugate of  $C$  in the elliptic involution with  $A_1$  and  $A_2$  as double points. By Corollary 3, Theorem 9, there are two real points  $P$  and  $Q$  which are paired in this involution and separate  $B$  and  $C$  harmonically. Let  $A$  be the conjugate of  $C$  in the elliptic involution with  $B_1$  and  $B_2$  as double points, and let  $R$  and  $S$  be

the two real points which are paired in this involution and separate  $A$  and  $C$  harmonically. Since any two harmonic sets are projective,

$$CBPQ \stackrel{\Delta}{=} CARS \quad \text{and} \quad CBPQ \stackrel{\Delta}{=} CASR.$$

The centers of these two perspectivities are two real points  $C_1$  and  $C_2$ , and since each perspectivity transforms two pairs of the elliptic involution on the line  $A_1A_2$  into two pairs of the elliptic involution on the line  $B_1B_2$ , it transforms  $A_1$  and  $A_2$  to  $B_1$  and  $B_2$ . Hence one of the points  $C_1$  and  $C_2$  is the intersection of the lines  $A_1B_1$  and  $A_2B_2$  and the other that of the lines  $A_1B_2$  and  $A_2B_1$ . By Theorem 12 each of these pairs of lines is a pair of conjugate imaginaries.

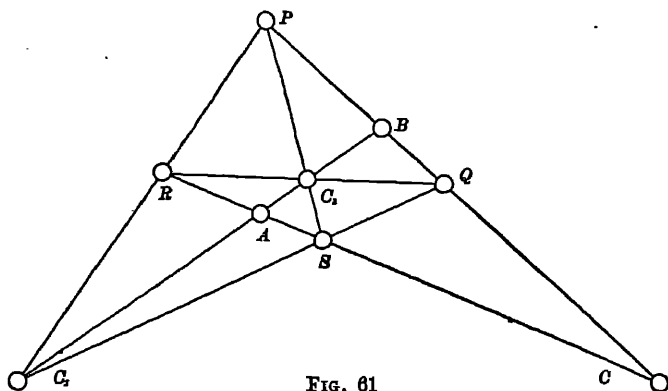


FIG. 61

The complete quadrilateral whose pairs of opposite vertices are  $A_1A_2$ ,  $B_1B_2$ , and  $C_1C_2$  is analogous to the quadrilateral considered in § 71 whose vertices were  $I_1I_2$  and the limiting points of two orthogonal pencils of circles (cf. fig. 57). With regard to the existence of such quadrilaterals we have

**THEOREM 14.** *Let  $A_1A_2$ ,  $B_1B_2$ ,  $C_1C_2$  be the pairs of opposite vertices of a complete quadrilateral. If  $A_1A_2$  and  $B_1B_2$  are pairs of conjugate imaginary points, then  $C_1$  and  $C_2$  are real and the diagonal triangle of the complete quadrilateral is real. If  $A_1$  and  $A_2$  are real and  $B_1$  and  $B_2$  are conjugate imaginaries, then  $C_1$  and  $C_2$  are conjugate imaginaries and the diagonal triangle is real.*

*Proof.* In the first case  $C_1$  and  $C_2$  are determined as in the proof of the last theorem and hence are real. The diagonal triangle has for its sides the three real lines  $A_1A_2$ ,  $B_1B_2$ ,  $C_1C_2$ .

In the second case let  $a$  be the line through  $A_2$  which is harmonically conjugate to  $A_2A_1$  with respect to the pair of lines  $A_2B_1$  and  $A_2B_2$ . Since the latter two lines are conjugate imaginaries and  $A_2A_1$  is real,  $a$  is real. The harmonic homology with  $A_1$  as center and  $a$  as axis transforms  $B_1$  and  $B_2$  to  $C_1$  and  $C_2$ . Hence  $C_1$  and  $C_2$  are conjugate imaginaries and the line  $C_1C_2$  is real.

Relatively to a real frame of reference a real involution is represented by a bilinear equation with real coefficients (§ 58, Vol. I), and its double points appear as the roots of a quadratic equation with real coefficients. Hence the coördinates of a pair of conjugate imaginary points are expressible in the form

$$(x_0 + iy_0, x_1 + iy_1, x_2 + iy_2, x_3 + iy_3)$$

and

$$(x_0 - iy_0, x_1 - iy_1, x_2 - iy_2, x_3 - iy_3),$$

where  $x_0, x_1, x_2, x_3, y_0, y_1, y_2, y_3$  are real. Like remarks can be made with regard to the coördinates of a plane or a line, and Theorems 10–14 can easily be proved analytically on this basis. The following theorem appears to be easier to prove analytically than synthetically:

**THEOREM 15.** *A complex line on a real plane contains at least one real point.*

*Proof.* Let the equation of the line be

$$u_0x_0 + u_1x_1 + u_2x_2 = 0.$$

This may be expressed in the form

$$(u'_0 + iu''_0)x_0 + (u'_1 + iu''_1)x_1 + (u'_2 + iu''_2)x_2 = 0,$$

where  $u'_0, u''_0$ , etc. are real. This equation is equivalent, if  $x_0, x_1, x_2$  are required to be real, to

$$u'_0x_0 + u'_1x_1 + u'_2x_2 = 0,$$

$$u''_0x_0 + u''_1x_1 + u''_2x_2 = 0,$$

two equations which are satisfied by at least one real point.

### EXERCISES

1. A conic section through three real and two conjugate imaginary points is real.
2. A pair of conjugate imaginary points cannot be harmonically conjugate with regard to another pair of conjugate imaginary points.
3. An imaginary point is on one and only one real line and has one and only one conjugate imaginary point.

**79. Projective, affine, and Euclidean classification of conics.** Let us regard a real plane  $\pi$  as immersed in a complex plane  $\pi'$ , and consider all conics in  $\pi'$  with respect to which the polar of a real point is always a real line.\*

Throughout the rest of this chapter the word "conic" shall be used in this sense. The involution of conjugate points with regard to such a conic is one in which real points are paired with real points. Hence, if a conic contains one real point, every real nontangent line through this point contains another point of the conic, and the conic is real. The conics under consideration therefore fall into two classes, the real conics† and those containing no real point.

By § 76, Vol. I, any two real conics are equivalent under the group of projective collineations. The same proposition holds also for any two conics of the other class, as we shall now prove. Let two such conics be denoted by  $C_1^2$  and  $C_2^2$ . On an arbitrary real line  $l$  they each determine an elliptic involution of conjugate points. By Cor. 4, Theorem 9, there is a projectivity of the line  $l$  carrying the involution determined by  $C_2^2$  into that determined by  $C_1^2$ . Any projectivity of the real plane which effects this transformation on  $l$  will carry  $C_2^2$  into a conic  $C_3^2$  which has the two conjugate imaginary points  $A_1, A_2$  on  $l$  in common with  $C_1^2$ . A collineation leaving  $l$  invariant will now carry the pole of  $l$  with regard to  $C_3^2$  to the pole of  $l$  with regard to  $C_1^2$ ; and therefore carries  $C_3^2$  to a conic  $C_4^2$  which has  $A_1, A_2$  and the tangents at these points in common with  $C_1^2$ . Let  $L$  be the pole of  $l$  with regard to  $C_1^2$  and  $L_1$  be any real point of  $l$ . By Cor. 3, Theorem 9, there is a pair of points  $MM_1$  which are conjugate with respect to  $C_1^2$  and harmonically separate  $L$  and  $L_1$  and also a pair  $M'M'_1$  conjugate with respect to  $C_4^2$  and harmonically separating  $L$  and  $L_1$ . The homology with  $l$  as axis,  $L$  as center, and carrying  $M'$  to  $M$  carries  $C_4^2$  to  $C_1^2$ . Hence we have

**THEOREM 16.** *Any two real conics or any two imaginary conics with real polar systems are conjugate under the group of real projective collineations.*

\* In § 85 this condition is seen to be equivalent to the condition that the equation of the conic relative to a frame of reference in  $\pi$  shall be expressible with real coefficients. For the present discussion, however, we do not need the general theory of correlation which is used in § 85.

† According to some usage any complex locus which has a real equation is called real. Cf. Pascal's Repertorium der Höheren Mathematik, Vol. II (1910), Chap. XIII (Berzolari). According to this definition both of the above classes of conics would be called real.

If the line  $l$  be taken as the line at infinity of a Euclidean plane the argument above shows that any two imaginary conics are also conjugate under the affine group. Since these conics do not meet any real line in real points, they are analogous to ellipses no matter how the line at infinity is chosen. Hence we make the definition:

DEFINITION. An imaginary conic with a real polar system is called an *imaginary ellipse*.

The results just established, together with those stated in Ex. 7, § 76, and Exs. 14 and 15, § 37, may be summarized as follows:

THEOREM 17. *Under the affine group the conics with real polar systems fall into four classes, parabolas, hyperbolas, real ellipses, imaginary ellipses. Any two conics of the same class are equivalent.*

Under the Euclidean group conics must be characterized by their relations to the circular points  $I_1, I_2$ . Since a real conic which does not meet  $l_\infty$  in real points meets it in conjugate imaginary points, any real conic through  $I_1$  also contains  $I_2$  and is therefore a circle. For the same reason the imaginary conic determined by an elliptic polar system must contain  $I_2$  if it contains  $I_1$ .

DEFINITION. An imaginary ellipse with respect to which the pairs of conjugate points on  $l_\infty$  are pairs of the absolute involution is called an *imaginary circle*.

THEOREM 18. *Any two real circles or any two imaginary circles are similar.*

*Proof.* Let the centers, necessarily real, of two circles  $C^2$  and  $K^2$  be  $O_1$  and  $O_2$  respectively. The center  $O_1$  may be transformed to  $O_2$  by a translation  $T_1$ . This carries  $C^2$  to a circle  $C_1^2$ . Any real line  $l$  through  $O_2$  meets  $C_1^2$  in two points  $C_1$  and  $C_2$  and  $K^2$  in two points  $K_1$  and  $K_2$ . Since each of these pairs is harmonically conjugate with respect to  $O_2$  and the point at infinity  $O_\infty$  of  $l$ , the homology  $T_2$  with  $O_2$  as center and  $l_\infty$  as axis which carries  $C_1$  to  $K_1$  also carries  $C_2$  to  $K_2$ . This homology evidently carries all real points to real points if  $C_1, C_2, K_1, K_2$  are real. If  $C_1, C_2$  and  $K_1, K_2$  are pairs of conjugate imaginary points, consider (§ 77) the real pair of points  $PP'$  harmonically conjugate with regard to  $C_1, C_2$  and  $OO_\infty$  and the real pair  $QQ'$  harmonically conjugate with regard to  $K_1, K_2$  and  $OO_\infty$ . The homology  $T_2$  must carry  $P$  and  $P'$  to  $Q$  and  $Q'$  and therefore carries all real points to real points in this case.

Now the conic  $C_1^2$  is fully determined by its points  $I_1, I_2, C_1, C_2$  and its center  $O_2$  and  $K^2$  is fully determined by  $I_1, I_2, K_1, K_2$  and  $O_2$ . Hence  $T_2$  carries  $C_1^2$  to  $K^2$ . The product  $T_2T_1$  carries  $C^2$  to  $K^2$ .

**THEOREM 19.** *Any two parabolas are similar.*

*Proof.* Let  $C^2$  and  $K^2$  be two parabolas and let  $C_\infty$  and  $K_\infty$  be their points of contact with  $l_\infty$ . Let  $T_1$  be any rotation carrying  $C_\infty$  to  $K_\infty$  and let  $T_1(C^2) = C_1^2$ . Let  $\bar{K}_\infty$  be the conjugate of  $K_\infty$  in the absolute involution and let  $c$  be the ordinary line through  $\bar{K}_\infty$  tangent to  $C_1^2$  and  $C$  its point of contact; also let  $k$  be the ordinary line through  $\bar{K}_\infty$  tangent to  $K^2$ , and  $K$  its point of contact. The translation  $T_2$

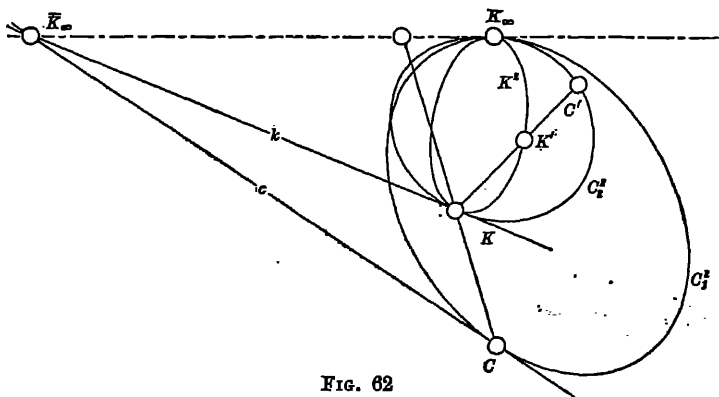


FIG. 62

carrying  $C$  to  $K$  carries  $c$  to  $k$  and  $C_1^2$  to a conic  $C_2^2$  touching  $l_\infty$  at  $K_\infty$ . Any line  $l$  through  $K$ , not containing  $K_\infty$  or  $\bar{K}_\infty$ , meets  $C_2^2$  in a point  $C'$  and  $K^2$  in a point  $K'$ . The homology  $T_3$  with  $K$  as center,  $l_\infty$  as axis, and carrying  $C'$  to  $K'$  carries  $C_2^2$  to  $K^2$ . The product  $T_3T_2T_1$  is a similarity transformation carrying  $C^2$  to  $K^2$ .

No theorem analogous to the last two holds for ellipses and hyperbolas. Suppose an ellipse or a hyperbola  $C^2$  meets  $l_\infty$  in  $C_1$  and  $C_2$  and another ellipse or hyperbola  $K^2$  meets it in  $K_1$  and  $K_2$ . In case a similarity transformation carries  $C_1$  and  $C_2$  into  $K_1$  and  $K_2$ ,

$$(5) \quad \mathbb{R}(I_1I_2, C_1C_2) = \mathbb{R}(I_1I_2, K_1K_2).$$

Conversely, if  $C^2$  and  $K^2$  satisfy the condition (5) there evidently exists a rotation carrying  $C_1$  and  $C_2$  to  $K_1$  and  $K_2$ . This rotation carries  $C^2$  to a conic  $C_1^2$  which passes through  $K_1$  and  $K_2$ . By an argument

analogous to the proof of Theorem 18 it can be shown that if  $C_1^2$  and  $K^2$  are both real ellipses, or both imaginary ellipses, or both hyperbolas, there is a similarity transformation carrying  $C_1^2$  to  $K^2$ . Hence

**THEOREM 20.** *Two real ellipses or two imaginary ellipses or two hyperbolas which meet  $l_\infty$  in pairs of points  $C_1C_2$  and  $K_1K_2$  are similar if and only if  $B(I_1I_2, C_1C_2) = B(I_1I_2, K_1K_2)$ .*

### EXERCISE

A hyperbola for which  $B(I_1I_2, K_1K_2) = -1$  is rectangular (Ex. 3, § 73).

**80. Foci of the ellipse and hyperbola.** Let  $C^2$  be any hyperbola or real or imaginary ellipse, and let  $l_1, l_2$  be the tangents to  $C^2$  through  $I_1$  and  $l_3, l_4$  the tangents to  $C^2$  through  $I_2$ . The circular points  $I_1, I_2$  are one pair of opposite vertices of the complete quadrilateral  $l_1l_2l_3l_4$ . Let the other two pairs of opposite vertices be  $F_1F_2$  and  $F'_1F'_2$  respectively (fig. 63), let  $a$  be the line  $F_1F_2$ ,  $b$  the line  $F'_1F'_2$ , and  $O$  the point of intersection of  $a$  and  $b$ . Also let  $A_\infty$  and  $B_\infty$  be the points at infinity of the lines  $a$  and  $b$  respectively. The triangle  $OA_\infty B_\infty$  is self-polar with respect to  $C^2$ . Hence  $O$  is the center of  $C^2$  and is therefore real.

Let  $X$  be any real point not on  $l_1, l_2, l_3, l_4$  or  $C^2$ . By the dual of the Desargues theorem on conics (§ 46, Vol. I) the tangents to  $C^2$  through  $X$  are paired in the same involution with  $XI_1, XI_2$  and  $XF_1, XF_2$  and  $XF'_1, XF'_2$ . The double lines  $x_1, x_2$  of this involution are harmonically conjugate with regard to  $XI_1, XI_2$  and to the tangents to  $C^2$ . Hence they are paired both in the involution of orthogonal lines at  $X$  and the involution of lines conjugate with respect to  $C^2$  at  $X$ . Hence by Cor. 2, Theorem 9,  $x_1$  and  $x_2$  are real, and are the unique pair of orthogonal lines on  $X$  which are conjugate with regard to  $C^2$ .

In particular, if  $X = O$  it follows that  $a$  and  $b$  are real and are the only pair of orthogonal and conjugate diameters of  $C^2$ . Hence  $A_\infty$  and  $B_\infty$  are also real. If  $X$  is not on  $a, b$ , or  $l_\infty$ , the lines  $x_1$  and  $x_2$  meet  $a$  in a pair of real points  $X_1, X_2$  distinct from  $A_\infty$  and  $O$ . Since  $F_1$  and  $F_2$  are harmonically conjugate with respect to the real pairs  $X_1X_2$  and  $A_\infty O$ , they are either real or conjugate imaginaries. But since  $I_1$  and  $I_2$  are conjugate imaginaries, by Theorem 14 if one of the pairs  $F_1F_2$  and  $F'_1F'_2$  is a pair of real points, the other is a pair of conjugate imaginaries, and conversely. Hence the notation may be so assigned that  $F_1$  and  $F_2$  are real and  $F'_1$  and  $F'_2$  are conjugate imaginaries.

Let  $A_1$  and  $A_2$  be the points in which  $a$  meets  $C^2$  and  $B_1$  and  $B_2$  the points in which  $b$  meets  $C^2$ . By construction neither of the lines  $a$  and  $b$  can be tangent to  $C^2$  so that each of the pairs  $A_1A_2$  and  $B_1B_2$  is either real or a pair of conjugate imaginaries.

In case  $C^2$  is an imaginary ellipse, both  $A_1A_2$  and  $B_1B_2$  are necessarily pairs of conjugate imaginaries. In case  $C^2$  is a real ellipse, the line  $l_\infty$  does not meet it in any real point, and hence  $O$ , the pole of  $l_\infty$ , is an interior point. Hence both  $a$  and  $b$  meet  $C^2$  in real points. Hence if  $C^2$  is an ellipse,  $A_1, A_2, B_1, B_2$  are all real. Whether  $C^2$  is an ellipse or a hyperbola, the tangents to  $C^2$  from  $F_1$  are conjugate imaginary lines since they join the real point  $F_1$  to the conjugate

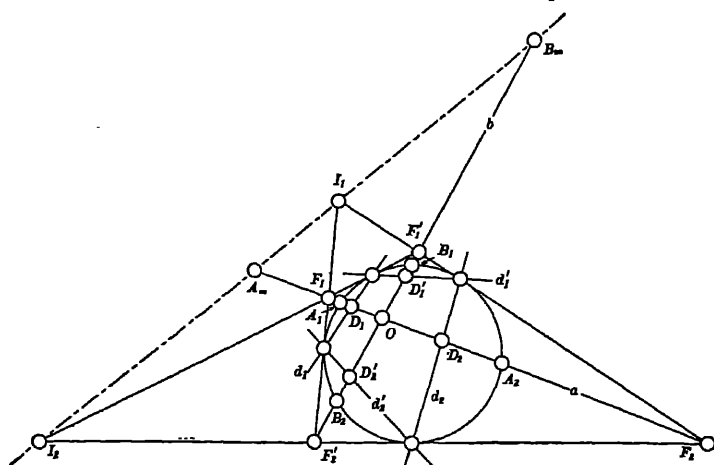


FIG. 68

imaginary points  $I_1$  and  $I_2$ . Hence  $F_1$  is interior to  $C^2$ , as is also  $F_2$  by a like argument. Hence the line  $F_1F_2$  meets  $C^2$  in real points. Hence if  $C^2$  is a hyperbola,  $A_1$  and  $A_2$  are real. But if  $C^2$  is a hyperbola,  $O$  is an exterior point, and hence  $A_\infty$ , which is harmonically separated from  $O$  by  $A_1$  and  $A_2$ , must be an interior point. Hence  $b$ , the pole of  $A_\infty$ , does not meet  $C^2$  in real points, and consequently  $B_1$  and  $B_2$  are conjugate imaginaries.

Let the polars of  $F_1, F_2, F'_1, F'_2$  relative to  $C^2$  be denoted by  $d_1, d_2, d'_1, d'_2$  respectively. Then  $d_1$  and  $d_2$  being the polars of real points are real; and since their point of intersection is polar to  $a$ , it is  $B_\infty$ , and hence they are parallel to  $b$ . In like manner  $d'_1$  and  $d'_2$  pass through  $A_\infty$  and are conjugate imaginaries.

DEFINITION. The lines  $a$  and  $b$  defined above are called the *axes* of the conic  $C^2$ ,  $a$  being called the *major*, or *principal*, axis and  $b$  the *minor*, or *secondary*, axis. Each of the points  $F_1, F_2, F'_1, F'_2$  is called a *focus*, and each of the points  $A_1, A_2, B_1, B_2$  a *vertex*, of the conic  $C^2$ . Each of the lines  $d_1, d_2, d'_1, d'_2$  is called a *directrix* of  $C^2$ .

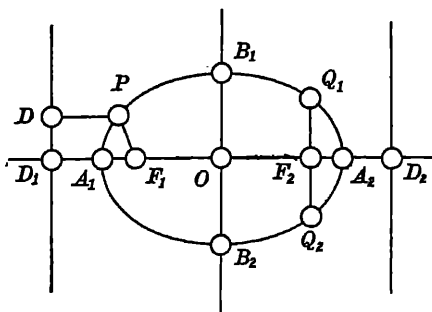


FIG. 64

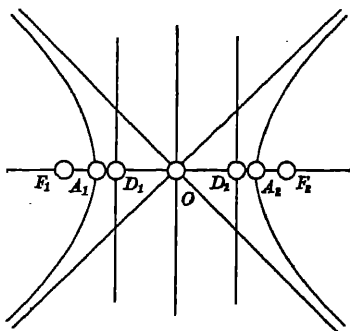


FIG. 65

In the course of the discussion of the complete quadrilateral  $l_1 l_2 l_3 l_4$  we have established the following propositions:

THEOREM 21. *If  $C^2$  is a hyperbola or a real or imaginary ellipse which is not a circle, its axes are the unique pair of conjugate diameters which are mutually perpendicular. Two of the foci and two of the directrices are real. The real foci lie on the major axis and the real directrices are perpendicular to it. The other two foci are conjugate imaginaries and lie on the minor axis. If  $C^2$  is real, the real foci are interior points and the real directrices are exterior lines. If  $C^2$  is a real ellipse, all four of the vertices are real; if  $C^2$  is a hyperbola, the two vertices on the major axis are real and those on the minor axis are conjugate imaginaries.*

The two tangents to  $C^2$  through  $F_1$  pass also through  $I_1$  and  $I_2$ . Pairs of conjugate lines at  $F_1$  are separated harmonically by these two tangents and hence meet  $l_\infty$  in pairs of the involution whose double points are  $I_1$  and  $I_2$ . If we limit attention to real elements, this may be expressed by saying that the pairs of conjugate lines with respect to  $C^2$  which pass through a focus are orthogonal. Conversely, if the pairs of orthogonal lines at any point  $P$  are conjugate with respect to  $C^2$ , the double lines of the involution of orthogonal lines at

$P$  would have to coincide with the double lines of the involution of conjugate lines, and hence  $P$  would be a focus. Hence

**THEOREM 22.** *The real foci of a hyperbola or a real or imaginary ellipse are the unique pair of real points at which all pairs of conjugate lines are orthogonal.*

The set of all conics tangent to the four minimal lines  $l_1, l_2, l_3, l_4$  form a range (§ 47, Vol. I). Hence the pairs of tangents to these conics through any point  $P$  not on the sides of the diagonal triangle  $OA_\infty B_\infty$  form an involution among the pairs of which are the pairs of lines  $PI_1, PI_2$ ;  $PF_1, PF_2$ ; and  $PF'_1, PF'_2$ . Now if  $P$  is on  $C^2$ , there is only one tangent to  $C^2$  at  $P$ , and this tangent is therefore a double line of the involution. This and the other double line have to be harmonically conjugate with respect to  $PI_1$  and  $PI_2$ ; that is, if  $C^2$  and  $P$  are real, the two double lines have to be orthogonal. These double lines must be harmonically conjugate also with respect to  $PF_1$  and  $PF_2$ . Thus we have a result which may be expressed as follows (cf. Ex. 12, § 76):

**THEOREM 23.** *The tangent and the normal to a real ellipse or hyperbola at any real point are the bisectors of the pair of lines joining this point to the real foci.*

In the proof of this proposition we have excepted the vertices of the conic, but the validity of the proposition for these points is self-evident. Another proposition which follows directly from the discussion above is the following, in which we make use of the fact that the pair of real foci determines the pair of imaginary ones, and vice versa.

**THEOREM 24. DEFINITION.** *The system of all conics having two real or two imaginary foci in common is a range of conics of Type I. The two conics of the set which pass through any real point have orthogonal tangents at this point. Such a range of conics is called a system of confocal conics or of confocals.*

The construction for the foci which has been considered in this section, when applied to a circle, reduces to a very simple one. The tangents to the circle at  $I_1$  and  $I_2$  meet in the center of the circle. The center of the circle is therefore sometimes referred to as the *focus* and the line at infinity as the *directrix*.

The term "focus" is derived from the property stated in Theorem 23, in consequence of which, if the conic be regarded as a reflecting surface, all rays of light diverging from one focus will be reflected back to the other focus.

In the rest of the chapter the foci, center, directrices, and axes of an ellipse or a hyperbola will be denoted by the same letters as in this section. The notation has been assigned so that for an ellipse the points are in the order

$$\{D_1 A_1 F_1 O F_2 A_2 D_2\},$$

and for the hyperbola in the order

$$\{F_1 A_1 D_1 O D_2 A_2 F_2\},$$

where  $D_1$  and  $D_2$  denote the points of intersection of the principal axis with the directrices  $d_1$  and  $d_2$  respectively.

**81. Focus and axis of a parabola.** Let  $C^2$  be any parabola. Since it is tangent to  $l_\infty$ , there are two ordinary tangents to it through  $I_1$  and  $I_2$  respectively; let these be denoted by  $l_1$  and  $l_2$  respectively. Let their point of intersection be denoted by  $F$ , their points of contact with  $C^2$  by  $L_1$  and  $L_2$  respectively, and the line  $L_1 L_2$  by  $d$ . Also let the point of contact of  $C^2$  with  $l_\infty$  be denoted by  $A_\infty$ , the line  $A_\infty F$  by  $a$ , and the point, other than  $A_\infty$ , in which  $a$  meets  $C^2$ , by  $A$ .

**DEFINITION.** The point  $F$  is called the *focus*, the line  $d$  the *directrix*, the line  $a$  the *axis*, the point  $A$  the *vertex*, of the parabola  $C^2$ .

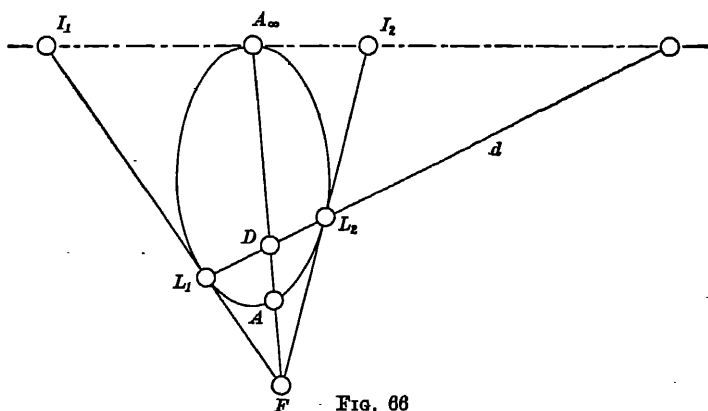


FIG. 66

That the focus, directrix, etc. of a parabola are real may be proved as follows: The transformation from pole to polar with regard to  $C^2$  transforms the absolute involution to an involution of the lines through  $A_\infty$  and transforms  $I_1$  and  $I_2$  into  $A_\infty L_1$  and  $A_\infty L_2$  respectively. The involution in the lines at  $A_\infty$  is perspective with an involution among the points of  $C^2$  which has  $L_1$  and  $L_2$  as double points. Hence  $L_1$  and  $L_2$  are conjugate imaginary points. Hence by Theorem 10 the

line  $d$  is real. Hence its pole,  $F$ , is real. Hence the line  $a$  joining  $F$  to  $A_\infty$  is real, and also the point  $A$ .

Since the two tangents to  $C^2$  through  $F$  pass through  $I_1$  and  $I_2$ , any two conjugate lines through  $F$  are perpendicular. Conversely, if the pairs of conjugate lines at any point are orthogonal, the tangents through this point must contain  $I_1$  and  $I_2$  respectively. Hence  $F$  is the only such point. Since the tangents through  $F$  are imaginary,  $F$  is interior to  $C^2$ , and hence all real points on  $d$  are exterior.

The tangent at  $A$  is parallel to  $d$ , and hence by the construction of  $d$  perpendicular to  $a$ . Since the tangent at any other ordinary point of  $C^2$  is not parallel to  $d$ , it follows that the line  $a$  is the only diameter of  $C^2$  which is perpendicular to its conjugate lines. These and other obvious consequences of the definition may be summarized as follows:

**THEOREM 25.** *The axis of a parabola is real and is the only diameter perpendicular to all its conjugate lines. The focus of a parabola is real and lies on the axis. The focus is the unique point at which all pairs of conjugate lines are orthogonal. It is interior to the parabola. The directrix is real, is the polar of the focus, and is perpendicular to the axis. All real points of the directrix are exterior to the parabola. The vertex is real and is the mid-point of the focus and the point in which the directrix meets the axis.*

The system of all conics tangent to  $l_1$  and  $l_2$  and to  $l_\infty$  at  $A_\infty$  forms a range of Type II (§ 47, Vol. I) which consists of all parabolas having  $F$  as focus and  $a$  as axis. The pairs of tangents to these conics through any real point  $P$  of the plane are by the dual of Theorem 20, Chap. V, Vol. I, the pairs of an involution in which  $PI_1$  is paired with  $PI_2$  and  $PF$  with  $PA_\infty$ . The tangents to the two conics of the range which pass through  $P$  are the double lines of this involution and hence separate  $PI_1$  and  $PI_2$  harmonically. Thus we have

**THEOREM 26.** *The parabolas with a fixed focus and axis form a range of Type II. The two parabolas of the range which pass through a given point have orthogonal tangents at this point.*

The tangent to either parabola through  $P$  is therefore normal to the other. Since these two lines separate  $PF$  and  $PA_\infty$  harmonically, we have

**THEOREM 27.** *The tangent and the normal to a parabola at any point are the bisectors of the pair of lines through this point of which one passes through the focus and the other is a diameter.*

## EXERCISES

1. If  $P$  is any point of an ellipse, the normal at  $P$  is the interior bisector of  $\angle F_1PF_2$ . If  $P$  is any point of a hyperbola, the tangent at  $P$  is the interior bisector of  $\angle F_1PF_2$ .

2. At any nonfocal point in the plane of a conic there is a unique pair of orthogonal lines which are conjugate with regard to the conic. In case of an ellipse or a hyperbola these lines harmonically separate the real foci. In case of a parabola they meet the axis in a pair of points of which the focus is the mid-point.

3. For any point  $P$  of an axis of a conic there is a unique point  $P'$  on the same axis such that any line through  $P$  is orthogonal to its conjugate line through  $P'$ . The pairs of points  $P$  and  $P'$  are pairs of an involution (called a *focal involution*) whose double points are the foci of the conic, or, in case of a parabola, the focus and the point at infinity of the axis. If  $P$  and  $P'$  are on the minor axis,  $\angle PPF_1P'$  is a right angle. If the conic is a parabola,  $F$  is the mid-point of the pair  $PP'$ .

4. Of two confocal central conics having a real point in common, one is an ellipse and the other a hyperbola.

5. The tangents at the points in which a conic is met by a line through a focus meet on the corresponding directrix.

6. If two conics have a focus in common, the poles with regard to the two conics of any line through this focus are collinear with the focus.

7. Let  $P$  be any point of a conic, and  $Q$  the point in which the tangent at  $P$  meets a directrix. If  $F$  is the corresponding focus,  $\angle PFQ$  is a right angle.

8. If a circle passes through the two real foci and a point  $P$  of a conic, it will have the two points in which the tangent and normal at  $P$  cut the other axis as extremities of a diameter.

9. If a variable tangent meets two fixed tangents in points  $P$  and  $Q$  respectively, and  $F$  is a focus, the measure of  $\angle PFQ$  is constant.

10. Let  $t_1$  and  $t_2$  be two tangents of a central conic meeting in a point  $T$ ; the pair of lines  $t_1, TF_1$  is congruent to the pair  $TF_2, t_2$ .

11. The line joining the focus to the point of intersection of two tangents to a parabola makes with either tangent the same angle that the other tangent makes with the axis.

12. Let  $p$  be a variable tangent of a parabola, and  $P$  a point of  $p$  such that the line  $PF$  makes a constant angle with  $p$ . The locus of  $P$  is a tangent to the parabola.

13. The foci of all parabolas inscribed in a triangle lie on a circle.

14. A circle circumscribed to a triangle which is circumscribed to a parabola passes through the focus.

15. The circles circumscribing four triangles whose sides form a complete quadrilateral pass through a point which is the focus of the parabola having the sides of the quadrilateral as tangents.

16. Let  $P$  be any point coplanar with, but not on an axis of, a conic  $C^2$ . The lines which are at once perpendicular to and conjugate with regard to  $C^2$  to the lines through  $P$  are the tangents of a parabola (the *Steiner parabola*). The axes of  $C^2$  are tangents of this parabola.

17. If  $P$  and  $P'$  are a pair of one focal involution of a central conic, and  $Q$  and  $Q'$  a pair of the other,  $P, P', Q, Q'$  are on an equilateral hyperbola, which may degenerate into a pair of orthogonal lines.

18. Given five points of a conic, construct by ruler and compass the center, the axes, the vertices, the foci, and the directrices. Construct the same elements when five tangents are given.

82. **Eccentricity of a conic.** Let  $F$  be a real focus, and  $d$  the corresponding directrix, of a conic  $C^2$  which is not a circle. Let  $a$  be the major axis of  $C^2$ , and  $h$  the line parallel to  $d$  such that if  $a$  meets  $d$  in a point  $D$ , and  $h$  in a point  $H$ ,  $D$  is the mid-point of the pair  $FH$ . Then  $d$  is the vanishing line (§ 43) of the harmonic homology  $\Gamma$  with  $F$  as center and  $h$  as axis.

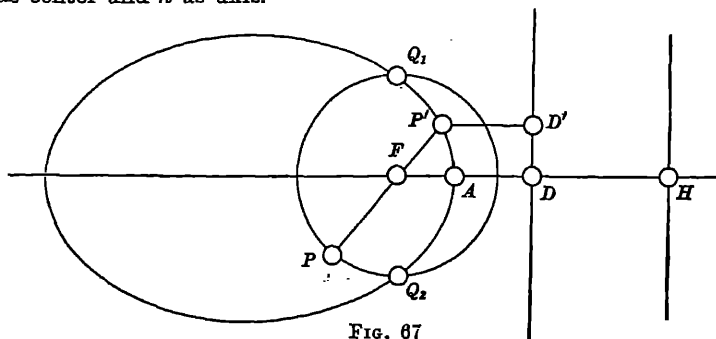


FIG. 67

Since  $F$  is a focus, the tangents to  $C^2$  through  $F$  pass also through the circular points. Hence the transformation  $\Gamma$  changes  $C^2$  into a circle  $K^2$  with  $F$  as center. Now if  $P$  is any point of the circle,  $P'$  the point of  $C^2$  to which  $P$  is transformed by  $\Gamma$ , and  $D'$  the point in which the line through  $P'$  parallel to  $FD$  meets  $d$ , it follows by Cor. 2, Theorem 21, Chap. III, that

$$\frac{\text{Dist}(P'F)}{\text{Dist}(P'D')} = \frac{\text{Dist}(PF)}{\text{Dist}(FD)}.$$

Since  $\text{Dist}(PF)$  and  $\text{Dist}(FD)$  are constants, it follows that

**THEOREM 28. DEFINITION.** *The ratio of the distances of a point of a conic to a focus and to the corresponding directrix is a constant called the eccentricity.*

The conic  $C^2$  is a parabola if and only if the circle  $K^2$  is tangent to  $d$ , the vanishing line of  $\Gamma$ . In this case

$$\text{Dist}(FD) = \text{Dist}(PF),$$

and hence the eccentricity is unity. The conic  $C^2$  is a hyperbola if and only if  $K^2$  meets  $d$  in real points. In this case

$$\text{Dist}(FD) < \text{Dist}(PF),$$

and hence the eccentricity is greater than one. Applying a like remark to the ellipse we have

**THEOREM 29.** *A conic section is an ellipse, hyperbola, or parabola according as its eccentricity is less than, greater than, or equal to unity.*

A circle is said to have eccentricity zero, because if  $P$  and  $F$  be held constant, and  $D$  be moved so as to increase  $FD$  without limit, the ratio  $\text{Dist}(PF)/\text{Dist}(FD)$  approaches zero.

The eccentricity of a hyperbola or an ellipse is evidently the same relatively to either of its real foci, because the two foci and the corresponding directrices are interchangeable by an orthogonal line reflection whose axis is the minor axis of the conic.

As an immediate corollary of the definition of eccentricity we have

**THEOREM 30.** *Two real conics are similar if and only if they have the same eccentricity.*

On comparing this theorem with Theorem 20, it is evident that the eccentricity is a function of the cross ratio of the double points of the absolute involution and the points in which the conic meets  $l_\infty$ . As an example of this relation we have (by comparison with § 72) the theorem that any two hyperbolas whose asymptotes make equal angles have the same eccentricity. The formula connecting the eccentricity of a hyperbola with the angular measure of its asymptotes is given in Ex. 7, below, and the formula for the eccentricity in terms of the cross ratio referred to in Theorem 20 is given in Ex. 9.

Since a real focus of any conic is an interior point, the line through a real focus (e.g.  $F_2$ , fig. 64) perpendicular to the principal axis meets the conic in two points,  $Q_1Q_2$ . The number  $\text{Dist}(Q_1Q_2)$  is evidently the same for both foci of an ellipse or hyperbola, and hence is a fixed number for any conic  $C^2$ .

DEFINITION. The number  $p = \text{Dist}(Q_1 Q_2)$  is called the *parameter*, or *latus rectum*, of the conic  $C^2$ .

In the following exercises  $e$  will denote the eccentricity and  $p$  the parameter of any conic. For an ellipse or hyperbola  $a$  denotes  $\text{Dist}(OA_1)$  and  $c$  denotes  $\text{Dist}(OF_1)$ . For an ellipse  $b$  denotes  $\text{Dist}(OB_1)$ . For a hyperbola  $b$  denotes  $\sqrt{c^2 - a^2}$ .

In all cases a radical sign indicates a *positive* square root.

### EXERCISES

1. If  $P$  is any point of an ellipse,  $\text{Dist}(F_1 P) + \text{Dist}(F_2 P) = 2a$ .
2. If  $P$  is any point of a hyperbola,  $\text{Dist}(F_1 P) - \text{Dist}(F_2 P) = \pm 2a$ .
3. In an ellipse  $\text{Dist}(B_1 F_1) = a$  and  $a^2 = b^2 + c^2$ .
4.  $\text{Dist}(A_1 F_1) \cdot \text{Dist}(F_1 A_2) = b^2$ .
5. In an ellipse or hyperbola  $e = \frac{c}{a}$  and  $p = \frac{2b^2}{a}$ .
6. In a parabola  $\text{Dist}(AF) = p/4$ .
7. The measure  $\theta$  (§ 87) of the pair of asymptotes of a hyperbola is determined by the equation

$$\cos \theta = 1 - \frac{2}{e^2}.$$

8. For an equilateral hyperbola  $e = \sqrt{2}$ .

9. The cross ratio  $R(C_1 C_2, I_1 I_2) = k^2$  referred to in Theorem 20 is connected with the eccentricity by the relation

$$e^2 = \frac{4k}{1 + 2k + k^2}$$

in case of an ellipse, and by  $e^2 = \frac{-4k}{1 - 2k + k^2}$

in case of a hyperbola.

10. Let  $A^2$  and  $B^2$  be the circles with  $O$  as center and passing through the vertices  $A_1$  and  $B_1$ , respectively, of an ellipse, and let a variable ray making an angle of measure  $\theta$  with the ray  $OA$  meet these circles in  $X$  and  $Y$  respectively. Then the line through  $Y$  parallel to  $OA_1$  meets the line through  $X$  parallel to  $OB_1$  in a point  $P$  of the conic. If  $x$  and  $y$  are the coördinates of  $P$  relative to the axes of the conic,

$$x = a \cos \theta, \quad y = b \sin \theta.$$

$\theta$  is called the *eccentric anomaly* of the point  $P$ .

11. Relative to a nonhomogeneous coördinate system in which the principal axis of a conic is the  $x$ -axis, and the tangent at a vertex the  $y$ -axis, the equation of a parabola, ellipse, and hyperbola, respectively, can be put in the form

$$y^2 = px,$$

$$y^2 = px - \frac{p}{2a} x^2,$$

$$y^2 = px + \frac{p}{2a} x^2.$$

12. Relative to the asymptotes as axes, the equation of a hyperbola may be written

$$xy = \frac{a^2 + b^2}{4}.$$

13. Relative to any pair of conjugate diameters as axes, an ellipse has the equation

$$\frac{x^2}{a'^2} + \frac{y^2}{b'^2} = 1,$$

and a hyperbola,

$$\frac{x^2}{a'^2} - \frac{y^2}{b'^2} = 1.$$

If  $A'$  is a point in which the  $x$ -axis meets the conic,  $\text{Dist}(OA') = a'$ . In the case of an ellipse, if  $B'$  is one of the points in which the  $y$ -axis meets the conic,  $\text{Dist}(OB') = b'$ .

14. The measure of the ordered point triads  $OA'B'$  is a constant.

15. The numbers  $a'$  and  $b'$  satisfy the conditions  $a'^2 + b'^2 = a^2 + b^2$  in case of an ellipse and  $a'^2 - b'^2 = a^2 - b^2$  in case of a hyperbola.

16. The equation of a system of confocal central conics relative to a system of nonhomogeneous point coordinates in which the axes of the conics are  $x = 0$  and  $y = 0$  is

$$\frac{x^2}{a^2 - \lambda} + \frac{y^2}{b^2 - \lambda} = 1,$$

where  $\lambda$  is a parameter. In the homogeneous line coordinates such that  $u_1x + u_2y + u_0 = 0$  gives the condition that the point  $(x, y)$  be on the line  $[u_0, u_1, u_2]$ , the equation of a system of confocals is  $u_0^2 = (a^2 - \lambda)u_1^2 + (b^2 - \lambda)u_2^2$ .

17. Relative to point coordinates in which the origin is the focus,  $y = 0$  the axis of the parabolas, and  $x = 0$  perpendicular to the axis, the equation of a system of confocal parabolas is

$$y^2 - 2(p - \lambda)x + \lambda(p - \lambda) = 0.$$

In the corresponding homogeneous line coordinates this is (cf. Ex. 16)

$$pu_2^2 - 2u_1u_0 - \lambda(u_1^2 + u_2^2) = 0.$$

**83. Synoptic remarks on conic sections.** An inspection of the literature will convince one that it would not be practical to include a complete list of the known metric theorems on conic sections in a book like this one. The theorems which we have derived, however, are sufficient to indicate how the rest may be obtained either directly as special cases of projective theorems or as consequences of the focal and affine theorems given in this chapter and Chap. III.

The theorems on conic sections have been classified according to the geometries to which they belong. The most general and elementary which we have considered are those which belong to the proper projective geometry (§ 17), the geometry corresponding to the projective group in any space satisfying Assumptions A, E, P. Theorems

of this class are given in Vol. I, particularly in Chaps. V, VIII, X. A second large class contains those theorems which belong to the affine geometry in any proper projective space. These are treated somewhat fully in Chap. III.

The theorems of the class considered in §§ 74, 75 of this chapter belong to the projective geometry of an ordered space. The theorems of § 76 belong to the projective geometry of a real space. Finally, in §§ 80-82 we have been considering theorems of the Euclidean geometry of a real space.

It is quite feasible to make a much finer classification of theorems on conics. This would mean, for example, distinguishing those properties of foci which hold in a parabolic metric geometry in a general space, then those which hold in an ordered space, and then those which are peculiar to the real space.

The theorems which have been under discussion in the remarks above refer in general to figures composed of one conic section and a finite number of points and lines. Theorems regarding more than one conic at a time have not been considered in any considerable number, and the theory of families of conics has not been carried beyond pencils and ranges. For an outline of this subject the reader is referred to the *Encyclopädie der Math. Wiss.*, III C1, §§ 56-90.

### EXERCISES

1. The diagonals of the rectangle formed by the tangents at the vertices of an ellipse are conjugate diameters for which  $a' = b'$ . The angle between this pair of conjugate diameters is less than that between any other pair of conjugate diameters. For this pair of conjugate diameters  $a' + b'$  is a maximum. It is a minimum for  $a' = a$ ,  $b' = b$ .

2. If two orthogonal diameters of a conic meet it in  $P$  and  $Q$ ,

$$\frac{1}{OP^2} + \frac{1}{OQ^2} = \frac{1}{a^2} + \frac{1}{b^2}$$

for an ellipse, and

$$\frac{1}{OP^2} - \frac{1}{OQ^2} = \pm \left( \frac{1}{a^2} - \frac{1}{b^2} \right)$$

for a hyperbola.

3. The locus of a point from which the two tangents to a conic  $C^2$  are orthogonal is a real circle in case  $C^2$  is an ellipse or a hyperbola for which  $a > b$ ; is a pair of conjugate imaginary lines through the center and the circular points in case  $C^2$  is a hyperbola for which  $a = b$ ; is an imaginary circle in case  $C^2$  is a hyperbola for which  $a < b$ ; is the directrix in case  $C^2$  is a parabola. The circle thus defined is called the *director circle* of  $C^2$ . Construct it by ruler and compass.

4. A variable tangent to a central conic is met by the lines through a focus which make a fixed angle with it in the points of a circle. In particular, the locus of the foot of a perpendicular from a focus to a tangent is a circle.

5. If  $t$  is a variable tangent of a central conic,  $\text{Dist } (F_1 t) \cdot \text{Dist } (F_2 t) = b^2$ . If  $t'$  is the other tangent parallel to  $t$ ,  $\text{Dist } (F_1 t') \cdot \text{Dist } (F_2 t') = b^2$ .

6. If  $F$  is a focus of a conic and  $P_1, P_2$  the points of intersection of an arbitrary line through  $F$  with the conic,

$$\frac{1}{P_1 F} + \frac{1}{F P_2}$$

is a constant.

7. If the tangent to a conic at a variable point  $P$  meets the axes in two points  $T_1$  and  $T_2$ , and the normal at  $P$  meets them in  $N_1$  and  $N_2$ , then

$$\begin{aligned} \text{Dist } (P T_1) \cdot \text{Dist } (P T_2) &= \text{Dist } (P N_1) \cdot \text{Dist } (P N_2) \\ &= \text{Dist } (P F_1) \cdot \text{Dist } (P F_2). \end{aligned}$$

8. There is a unique circle which osculates\* a given conic at a given point  $P$ . This is called the *circle of curvature* at  $P$ . Its center is called the *center of curvature* for  $P$  and lies on the normal at  $P$ .

9. Construct by ruler and compass the center of the circle of curvature at an arbitrary point of a given conic.

10. The circle of curvature of a conic  $C^2$  at a point  $P$  meets  $C^2$  in one and only one other point,  $Q$ . The line  $PQ$  is the axis and the point  $P$  the center of an elation which transforms  $K^2$  into  $C^2$ . The center of curvature is transformed by this elation into the center of the involution on  $C^2$  in which the pairs of orthogonal lines at  $P$  meet  $C^2$ .

11. The tangent and normal at any point  $P$  of a conic  $C^2$  are both tangent to the Steiner parabola (Ex. 16, § 81) determined by this point. The point of contact of the normal with the parabola is the center of the circle of curvature of  $C^2$  at  $P$ , and the point of contact of the tangent with the parabola is the pole of the normal with respect to  $C^2$ . (For further properties of the circle of curvature, cf. Encyclopädie der Math. Wiss., III C1, § 36.)

12. The polar reciprocal of a circle with respect to a circle having a point  $O$  as center is a conic having  $O$  as a focus. (A set of theorems related to this one will be found in Chap. VIII of the book by J. W. Russell referred to in § 73.)

**84. Focal properties of collineations.** The focal properties of conic sections are closely related to a set of theorems on collineations some of which are given in the exercises below. A good treatment of the subject is to be found in the Collected Papers of H. J. S. Smith, Vol. I, p. 545, and further references in the Encyclopädie der Math. Wiss., III AB 5, § 9.

\* Cf. § 47, Vol. I.

Let  $\Pi$  be any real projective collineation which does not leave  $l_\infty$  invariant, and let  $p$  and  $q$  be its vanishing lines; so that  $\Pi(p) = l_\infty$  and  $\Pi(l_\infty) = q$ . If  $I_1$  and  $I_2$  are the circular points, let  $\Pi^{-1}(I_1) = P_1$ ,  $\Pi^{-1}(I_2) = P_2$ ,  $\Pi(I_1) = Q_1$ ,  $\Pi(I_2) = Q_2$ . By the theorems of § 78 the lines  $P_1I_1$  and  $P_2I_2$  meet in a real point  $A_1$ , and  $P_1I_2$  and  $P_2I_1$  meet in a real point  $A_2$ . If  $\Pi(A_1) = B_1$  and  $\Pi(A_2) = B_2$ , it is clear that the complete quadrilateral whose pairs of opposite vertices are  $I_1I_2$ ,  $P_1P_2$ ,  $A_1A_2$  is transformed into one whose pairs of opposite vertices are  $Q_1Q_2$ ,  $I_1I_2$ ,  $B_1B_2$ . The following propositions are now easily verifiable, and are stated as exercises.

### EXERCISES

1.  $A_1$  is such that any ordered pair of lines meeting at  $A_1$  is transformed by  $\Pi$  into a congruent pair of lines.  $A_2$  is such that any two lines meeting in  $A_2$  are transformed by  $\Pi$  into a symmetric pair of lines. No other points have either of these properties.

2. Every conic having a focus at  $A_1$  or  $A_2$  goes to a conic with a focus at  $B_1$  or  $B_2$  respectively.

3. The range of conics having  $A_1$  and  $A_2$  as foci is transformed by  $\Pi$  into the range of conics with  $B_1$  and  $B_2$  as foci; and this is the only system of confocals which goes into a system of confocals.

4. The pencil of circles with  $A_1, A_2$  as limiting points is transformed by  $\Pi$  into that having  $B_1, B_2$  as limiting points; and these are the only two pencils of circles homologous under  $\Pi$ . The radical axes of the two pencils are the two vanishing lines.

5. If  $P$  is any point and  $\Pi(P) = P'$ , then the ordered point triad  $A_1PA_2$  is similar (but not directly similar) to the ordered point triad  $B_1P'B_2$ .

6. At a point of a Euclidean plane there is in general one and only one pair of perpendicular lines which is transformed into a pair of perpendicular lines by a given affine collineation.

7. In any two projective pencils of lines there is a pair of corresponding orthogonal pairs of lines. The line pairs which are homologous with congruent line pairs form an involution.

8. Any projective collineation which does not leave  $l_\infty$  invariant is expressible as a product of a displacement and a homology.

85. **Homogeneous quadratic equations in three variables.** Reversing the process which is common in analytic geometry, it is possible to derive certain classes of algebraic theorems from the theory of conic sections. We shall illustrate this process in a few important cases and leave the development of further algebraic applications to the reader.

The general homogeneous equation of the second degree can be written in the form

$$(6) \quad \begin{aligned} & a_{00}x_0^2 + a_{01}x_0x_1 + a_{02}x_0x_2 \\ & + a_{10}x_1x_0 + a_{11}x_1^2 + a_{12}x_1x_2 \\ & + a_{20}x_2x_0 + a_{21}x_2x_1 + a_{22}x_2^2 = 0, \end{aligned}$$

where  $a_{ij} = a_{ji}$ . Let us first suppose that

$$(7) \quad A \equiv \begin{vmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{vmatrix} \neq 0.$$

In § 98, Vol. I, it has been shown, from the point of view of general projective geometry, that every projective polarity is represented by a bilinear equation of the form

$$(8) \quad \begin{aligned} & a_{00}x_0x'_0 + a_{01}x_0x'_1 + a_{02}x_0x'_2 \\ & + a_{10}x_1x'_0 + a_{11}x_1x'_1 + a_{12}x_1x'_2 \\ & + a_{20}x_2x'_0 + a_{21}x_2x'_1 + a_{22}x_2x'_2 = 0, \end{aligned}$$

where  $a_{ij} = a_{ji}$ , and where  $A \neq 0$ .

It was also shown that every bilinear equation of this form, subject to the condition  $A \neq 0$ , represents a polarity; that the equation in point coördinates of the fundamental conic of the polarity is (6), which is obtained from (8) by setting  $x'_i = x_i$ ; and that the equation of this conic in line coördinates is

$$(9) \quad A_{ij}u_iu_j = 0,$$

where  $A_{ij}$  is the cofactor of  $a_{ij}$  in  $A$ .

The coefficients  $a_{ij}$  are elements of the geometric number system. Therefore in the case of the real plane they are real numbers, and we have

**THEOREM 31.** *Every equation of the form (6) with real coefficients such that  $a_{ij} = a_{ji}$  and  $A \neq 0$  represents a conic whose polar system transforms real points into real lines. Conversely, every conic with regard to which real points have real polars has an equation of the form (6) with real coefficients such that  $a_{ij} = a_{ji}$  and  $A \neq 0$ .*

In § 79 we have seen that any conic having a real polar system is in one of two classes, and that any two conics of the same class are projectively equivalent. Now it is obvious that

$$(10) \quad x_0^2 + x_1^2 + x_2^2 = 0$$

is the equation of an imaginary conic, and that

$$(11) \quad x_0^2 + x_1^2 - x_2^2 = 0$$

is the equation of a real conic. Hence we have

**THEOREM 32.** *Any quadratic equation in three homogeneous variables whose discriminant  $\Delta$  does not vanish is reducible by real linear homogeneous transformation of the variables to the form (10) or to the form (11).*

Algebraic criteria to determine whether a given conic  $C^2$  whose equation is in the form (6) belongs to one or the other of these classes may easily be determined by the aid of simple geometric considerations. In case  $C^2$  contains no real points, the line  $x_0 = 0$  has no real point in common with it, and the point  $u_1 = 0$  (which is on the line  $x_0 = 0$ ) is on no real tangent to it. On the other hand, if the line  $x_0 = 0$  contained no real point of  $C^2$ , and  $C^2$  were real, this line would consist entirely of exterior points, and hence there would be a tangent to  $C^2$  through the point  $u_1 = 0$ . Hence a pair of necessary and sufficient conditions that  $C^2$  contain no real points are (1)  $x_0 = 0$  is on no point of  $C^2$  and (2)  $u_1 = 0$  is on no tangent of  $C^2$ .

Substituting  $x_0 = 0$  and  $x'_0 = 0$  in (8), we have the equation of an involution

$$(12) \quad \begin{aligned} & a_{11}x_1x'_1 + a_{12}x_1x'_2 \\ & + a_{21}x_2x'_1 + a_{22}x_2x'_2 = 0, \end{aligned}$$

which, by § 4, is elliptic if and only if  $A_{00} > 0$ . By a dual argument applied to (9), the necessary and sufficient condition that there be no real tangents to  $C^2$  through the point  $u_1 = 0$  is

$$(13) \quad \begin{vmatrix} A_{00} & A_{02} \\ A_{20} & A_{22} \end{vmatrix} > 0.$$

By a well-known theorem on determinants (or a simple computation) this reduces to

$$a_{11} \cdot \Delta > 0.$$

Hence we have

THEOREM 33. *The imaginary conics are those for which*

$$A_{00} > 0 \text{ and } a_{11} \cdot A > 0,$$

*and the real ones are those for which not both of these conditions are satisfied and for which  $A \neq 0$ .*

In these conditions it is obvious that  $A_{00}$  and  $a_{11}$  may be replaced by  $A_{ii}$  and  $a_{jj}$ , where  $i, j = 0, 1, 2$ , provided that  $i \neq j$ .

Let us now investigate the cases where  $A = 0$ , and first the case in which not all the cofactors  $A_{00}, A_{11}, A_{22}$  are zero. To fix the notation, suppose that  $A_{00} \neq 0$ . Then the bilinear equation (8) is satisfied by  $x_0 = A_{00}, x_1 = A_{01}, x_2 = A_{02}$ , no matter what values are taken by  $x'_0, x'_1, x'_2$ . Hence in this case (8) determines a transformation,  $\Gamma$ , of all the points  $(x'_0, x'_1, x'_2)$  distinct from  $(A_{00}, A_{01}, A_{02})$  into lines through  $(A_{00}, A_{01}, A_{02})$ . A collineation which transforms  $(A_{00}, A_{01}, A_{02})$  to  $(1, 0, 0)$  must reduce (8) to

$$(14) \quad \begin{aligned} & b_{11}x_1x'_1 + b_{12}x_1x'_2 \\ & + b_{21}x_2x'_1 + b_{22}x_2x'_2 = 0 \end{aligned} \quad b_{12} = b_{21}.$$

It is to be noted that

$$\begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} \neq 0,$$

because if this determinant vanished,  $\Gamma$  would transform all points  $(x'_0, x'_1, x'_2)$  into a single line, and hence  $A_{00}$  would vanish. Hence  $\Gamma$  transforms any point  $(x'_0, x'_1, x'_2)$  into the line paired in a certain involution with the line joining  $(x'_0, x'_1, x'_2)$  to  $(A_{00}, A_{01}, A_{02})$ . The double lines of the involution must satisfy the quadratic equation (6).

Comparing with the definitions in § 45, Vol. I, we have that when  $A = 0$  and not all the cofactors  $A_{00}, A_{11}, A_{22}$  are zero, (6) represents a degenerate conic consisting of two distinct lines and that (8) represents the polar system of the conic. Since the lines represented by (6) are the double lines of a real involution, they are either real or a pair of conjugate imaginaries. In the first case (6) can evidently be transformed by a collineation to

$$(15) \quad x_1^2 - x_2^2 = 0,$$

and in the second case to

$$(16) \quad x_1^2 + x_2^2 = 0.$$

The criteria to distinguish the two cases may be found by considering the intersection with (6) of a line  $x_i = 0$ . This yields imaginary points (just as in the nondegenerate case) if and only if  $A_{ii} > 0$ , and real points if and only if  $A_{ii} \leq 0$ . Hence the case where (6) represents a pair of real lines occurs if and only if  $A_{ii} \leq 0$ , for  $i = 0, 1, 2$ .

Finally, suppose that  $A_{00} = A_{11} = A_{22} = A = 0$ . In view of the identity,

$$(17) \quad A_{ii}A_{jj} - A_{ij}^2 \equiv a_{kk} \cdot A, \quad (i \neq j \neq k \neq i)$$

this implies that all the cofactors  $A_{ij}$  are zero, and hence that (8) represents the same line, no matter what values are substituted for  $x'_0, x'_1, x'_2$ . Hence (6) represents a single real line (i.e. two coincident real lines), and the polar system (8) transforms all points not on this line into this line. If this line be transformed to  $x_1 = 0$ , (6) obviously becomes

$$(18) \quad x_1^2 = 0.$$

A degenerate point conic is two distinct or coincident lines. These may always be represented by a quadratic equation which is a product of two linear ones. For such a quadratic  $A = 0$ , because if  $A \neq 0$ , the equation has been seen to represent a nondegenerate conic. Hence the theory of degenerate point conics is equivalent to that of homogeneous quadratic equations for which  $A = 0$ .

The complete projective classification of conics, degenerate or not, may now be stated as an algebraic theorem in the form:

**THEOREM 34.** *Any homogeneous quadratic equation in three variables may be reduced by a real linear homogeneous transformation,*

$$(19) \quad x'_i = \sum_{j=0}^2 \alpha_{ij} x_j, \quad (i = 0, 1, 2), \quad |\alpha_{ij}| \neq 0$$

*to one of the normal forms (10), (11), (16), (15), (18). The criteria which determine to which one of these forms an equation (6) is reducible may be summarized in the following table:*

$A \neq 0$		$A = 0$		
IMAGINARY CONIC	REAL CONIC	IMAGINARY LINE PAIR	REAL LINE PAIR	COINCIDENT REAL LINE PAIR
$a_{11}A > 0$	$a_{11}A \leq 0$	$A_{00} > 0$	$A_{00} < 0$	$A_{00} = 0$
		OR $A_{11} > 0$	OR $A_{11} < 0$	$A_{11} = 0$
$A_{00} > 0$	OR $A_{00} \leq 0$	OR $A_{22} > 0$	OR $A_{22} < 0$	$A_{22} = 0$

Since the algebraic expressions in the above criteria determine conditions on the conic which are independent of the choice of coördinates and thus are invariant under the projective group, it is natural to inquire whether they are algebraic invariants in the sense of § 90, Vol. I. A direct substitution will readily verify that  $A$  is a relative invariant of (6).

Suppose we regard the coefficients of (6) as homogeneous coördinates  $(a_{00}, a_{11}, a_{22}, a_{01}, a_{10}, a_{12})$  of a point in a five-dimensional space. Then  $A = 0$  determines a certain cubic locus in this space the points on which represent degenerate conics. Now if there were any other invariant of (6) under the projective group, say  $\phi(a_{ij})$ , the equation  $\phi(a_{ij}) = 0$  would represent a locus in this five-dimensional space. But since each nondegenerate conic is projectively equivalent to every other nondegenerate conic, this locus would have to be contained in the locus of  $A = 0$ . From this it can be proved, by the general theory of loci represented by algebraic equations, that the locus of  $\phi(a_{ij}) = 0$  coincides with that of  $A = 0$ , and that hence  $\phi(a_{ij})$  is rationally expressible in terms of  $A$ . Thus  $A$  is essentially the only invariant of (6) under the projective group.

The question, however, arises whether there are not other rational functions of the coefficients of (6) which are invariant whenever  $A = 0$ . If there were such a function, say  $\phi(a_{ij})$ , the conics for which  $\phi(a_{ij}) = 0$  would be a subclass of the degenerate conics which is transformed into itself by all complex projective collineations. The only class of this sort consists of the coincident line pairs which are given by two conditions,  $A_{00} = 0$ ,  $A_{11} = 0$ . In view of the theorem that a locus represented by two independent algebraic equations cannot be the complete locus of a single algebraic equation, this shows that there is no other invariant of (6) even for the cases in which  $A = 0$ .

This reasoning could be expressed still more briefly by saying that, while the set of all conics is a five-parameter family, and the set of degenerate conics a four-parameter family given by one condition, the only invariant subset of the degenerate conics is the two-parameter set of coincident line pairs which have to be given by two conditions and so cannot correspond to a single invariant in addition to  $A$ .

### EXERCISES

1. In case  $A = 0$ , the lines represented by (6) intersect in the point  $(\sqrt{A_{00}}, \sqrt{A_{11}}, \sqrt{A_{22}})$ , unless the three cofactors  $A_{ii}$  vanish, in which case (6) represents the coincident line pair

$$(\sqrt{a_{00}}x_0 + \sqrt{a_{11}}x_1 + \sqrt{a_{22}}x_2)^2 = 0.$$

2. In case (6) represents a pair of distinct lines, (9) represents their point of intersection counted twice. In case (6) represents a pair of coincident lines,  $A_{ij} = 0$  ( $i, j = 0, 1, 2$ ).

86. Nonhomogeneous quadratic equations in two variables. The affine theory of point conics corresponds to the theory of

$$(20) \quad \begin{aligned} & a_{00} + a_{01}x + a_{02}y \\ & + a_{10}x + a_{11}x^2 + a_{12}xy \\ & + a_{20}y + a_{21}yx + a_{22}y^2 = 0, \end{aligned}$$

where the  $a_{ij}$ 's satisfy the same conditions as in the last section. The theorem that any nondegenerate conic is an imaginary ellipse, real ellipse, hyperbola, or parabola, and that any two conics of the same class are equivalent under the affine group, translates into the following: Any quadratic equation in two variables, for which  $A \neq 0$ , is transformable by a transformation of the form

$$(21) \quad \begin{aligned} x' &= a_1x + b_1y + c_1, \\ y' &= a_2x + b_2y + c_2, \end{aligned} \quad \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0,$$

into one of the following four forms:

$$(22) \quad x^2 + y^2 + 1 = 0,$$

$$(23) \quad x^2 + y^2 - 1 = 0,$$

$$(24) \quad x^2 - y^2 - 1 = 0,$$

$$(25) \quad x^2 + y = 0.$$

To know this it is merely necessary to observe that these equations represent conics of the four types respectively.

The criteria to determine in which class a given conic  $C^2$  belongs may be inferred from the discussion in the last section if we set  $x = x_1/x_0$  and  $y = x_2/x_0$ . It is then evident that  $A_{00} > 0$  for an ellipse,  $A_{00} = 0$  for a parabola, and  $A_{00} < 0$  for a hyperbola. Hence the affine classification of cases where  $A \neq 0$  may be summarized in the following table:

$$A \neq 0$$

IMAGINARY ELLIPSE	REAL ELLIPSE	HYPERBOLA	PARABOLA
$A_{00} > 0$ $a_{11}A > 0$	$A_{00} > 0$ $a_{11}A \leq 0$	$A_{00} < 0$	$A_{00} = 0$

The cases where  $A = 0$  correspond, as in the last section, to degenerate conics. Geometrically the types of figures are obvious, and to obtain the algebraic criteria we need only combine with considerations already adduced, the observation that when  $A_{00} = 0$  and either  $a_{11} = 0$  or  $a_{22} = 0$ , then  $a_{12} = a_{21} = 0$ .

$$\Delta = 0$$

CONJUGATE IMAGINARY LINES		DISTINCT REAL LINES			COINCIDENT REAL LINES	
Concurrent at ordinary point	Parallel pair	Concurrent at ordinary point	Parallel pair	One at infinity	Ordinary	At infinity
$\Delta_{00} > 0$	$\Delta_{00} = 0,$ $\Delta_{11} > 0$ or $\Delta_{22} > 0$	$\Delta_{00} < 0$	$\Delta_{00} = 0,$ $\Delta_{11} < 0$ or $\Delta_{22} < 0;$ $a_{11} \neq 0$ or $a_{22} \neq 0$	$a_{11} = a_{22} = 0$	$\Delta_{00} = \Delta_{11} = \Delta_{22} = 0;$ $a_{11} \neq 0$ or $a_{22} \neq 0$	$a_{11} = a_{22} = 0$

As normal forms for the first six cases we may take

$$(26) \quad x^2 + y^2 = 0,$$

$$(27) \quad x^2 + 1 = 0,$$

$$(28) \quad x^2 - y^2 = 0,$$

$$(29) \quad x^2 - 1 = 0,$$

$$(30) \quad x = 0,$$

$$(31) \quad x^2 = 0.$$

The case of coincident real lines at infinity does not correspond to any equation in nonhomogeneous coördinates.

Summarizing these results we have the following algebraic theorem:

**THEOREM 35.** *Any quadratic equation in two variables may be reduced to one and only one of the normal forms (22)–(31) by a transformation of the form (21). The normal form to which it is reducible is determined by the criteria in the two tables above.*

The question of invariants of (20) under the affine group may be investigated in the manner indicated for the corresponding projective problem in the fine print at the end of the last section. The results of such an investigation are given in the exercises below.

There are no absolute invariants of conics under the projective and affine groups, because two conics would fail to be equivalent under the one group or the other if they determined different values of an absolute invariant, and this would contradict the fact that there are only a finite number of conics distinct under the affine group.

## EXERCISES

1.  $A$  and  $A_{00}$  are invariants of (20) under the affine group.
2. In case  $A = A_{00} = 0$ ,  $A_{11}/a_{22}$  and  $A_{22}/a_{11}$  are invariants of (20) under the affine group.
3. The homogeneous coördinates of the center of (20) are  $(A_{00}, A_{01}, A_{02})$ .
4. If  $A_{00} \neq 0$ , the translation  $\bar{x} = x - \frac{A_{10}}{A_{00}}$ ,  $\bar{y} = y - \frac{A_{20}}{A_{00}}$  transforms (20) into

$$a_{11}\bar{x}^2 + 2a_{12}\bar{x}\bar{y} + a_{22}\bar{y}^2 + \frac{A}{A_{00}} = 0.$$

5. If  $A \neq 0$  and  $A_{00} \neq 0$ , the asymptotes of (20) are given by the equation

$$a_{11}\bar{x}^2 + 2a_{12}\bar{x}\bar{y} + a_{22}\bar{y}^2 = 0.$$

6. Any diameter of a parabola is parallel to  $a_{11}x + a_{12}y = 0$  and to  $a_{12}x + a_{22}y = 0$ .

**87. Euclidean classification of point conics.** With respect to a non-homogeneous coördinate system in which the pair of lines  $x = 0$  and  $y = 0$  is orthogonal and bisected by the lines  $x = y$  and  $x = -y$ , the transformations of the Euclidean group take the form (21) subject to the conditions

$$(32) \quad a_1^2 + a_2^2 = b_1^2 + b_2^2, \quad a_1b_1 + a_2b_2 = 0,$$

and the displacements are subject to the additional condition

$$(33) \quad \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 1.$$

Since any ellipse or hyperbola is congruent to one whose principal axes are  $x = 0$  and  $y = 0$ , and since any parabola is congruent to a parabola with the origin as vertex and  $y = 0$  as its principal axis, it follows that any conic is congruent to a conic having one of the following equations:

$$(34) \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} + 1 = 0,$$

$$(35) \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0,$$

$$(36) \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0,$$

$$(37) \quad y^2 - px = 0.$$

The normal forms to which degenerate point conics can be reduced by displacements are evident when one recalls that two pairs of non-parallel lines are congruent when they have the same cross ratio with

the circular points and that two pairs of parallel lines are congruent if the lines of each pair are the same distance apart.\* By comparison with the second table ( $A = 0$ ) in § 86 we find

$$(38) \quad \frac{x^2}{a^2} + y^2 = 0,$$

$$(39) \quad x^2 + c^2 = 0,$$

$$(40) \quad \frac{x^2}{a^2} - y^2 = 0,$$

$$(41) \quad x^2 - c^2 = 0,$$

$$(42) \quad x = 0,$$

$$(43) \quad x^2 = 0.$$

The group of displacements is extended to the group of similarity transformations by adjoining transformations of the form

$$(44) \quad \begin{aligned} x' &= kx, \\ y' &= ky, \end{aligned} \quad k \neq 0.$$

Transformations of this sort will reduce the equations (34)–(43) to normal forms in which  $b$ ,  $c$ , and  $p$  are all unity.

The criteria for determining to which of these normal forms a conic is reducible under the group of displacements or that of similarity transformations are the same as those already found for the affine group. Two conics whose equations can be reduced to the same normal form are evidently equivalent under the group of displacements if and only if they determine the same values for  $a$  and  $b$  or  $c$  or  $p$ , and under the Euclidean group if they determine the same value for  $a$ . The numbers  $a$ ,  $b$ ,  $c$ ,  $p$  are evidently absolute invariants of the corresponding conics under the group of displacements, and  $a$  in (38) and (40) also under the Euclidean group.

The problem of determining  $a$ ,  $b$ ,  $c$ ,  $p$  in terms of the coefficients of (20) presents no special difficulty, and will be left to the reader to be considered in connection with the exercises below and those at the end of the next section.

When  $b$ ,  $c$ ,  $p$  are all unity,  $a$  is a function of the eccentricity given by the equations in Exs. 7 and 9, § 82. The same reference gives the connection between the eccentricity and the invariant  $\sqrt{-A_{00}}/(a_{11} + a_{22})$ .

\* The distance apart is the distance of an arbitrary point on one of the parallel lines from the other line. The formula for distance is applied to the case of a pair of conjugate imaginary lines as explained in § 70.

## EXERCISES

1. If  $A \neq 0$  and  $A_{00} \neq 0$ , the angular measure of the asymptotes is  $\theta$ , where

$$\tan \theta = \frac{2\sqrt{-A_{00}}}{a_{11} + a_{22}}.$$

Moreover,

$$\theta = -\frac{i}{2} \log R(C_1 C_2, I_1 I_2),$$

where  $C_1$  and  $C_2$  are the points in which the conic meets  $l_\infty$ , and  $I_1$  and  $I_2$  are the circular points. If  $A = 0$  and  $A_{00} \neq 0$ , these formulas give the angular measure of the lines represented by (20). Derive from this the formula for  $a$  in (38) and (40) in terms of the coefficients of (20).

2.  $A_{00}$  and  $a_{11} + a_{22}$  are absolute invariants of (20) under the group of displacements, and  $\sqrt{-A_{00}}/(a_{11} + a_{22})$  under the Euclidean group. If  $A \neq 0$  and  $a_{11} + a_{22} = 0$ , (20) represents an equilateral hyperbola; if  $A = 0$  and  $a_{11} + a_{22} = 0$ , it represents a pair of orthogonal lines or  $l_\infty$  and an ordinary line.

3. If  $A \neq 0$  and  $A_{00} \neq 0$ , the axes of (20) are

$$a_{12}(\bar{x}^2 + \bar{y}^2) + (a_{22} - a_{11})\bar{x}\bar{y} = 0,$$

where  $\bar{x}$  and  $\bar{y}$  are defined as in Ex. 4, § 86.

4. For an ellipse the constants  $a$  and  $b$  are  $\sqrt{\frac{-A}{A_{00}\lambda_1}}$  and  $\sqrt{\frac{-A}{A_{00}\lambda_2}}$ , where  $\lambda_1$  and  $\lambda_2$  are the roots of

$$(45) \quad \lambda^2 - (a_{11} + a_{22})\lambda + A_{00} = 0;$$

and for a hyperbola  $a$  and  $ib$  are  $\sqrt{\frac{-A}{A_{00}\lambda_1}}$  and  $\sqrt{\frac{-A}{A_{00}\lambda_2}}$ . The discriminant of (45) is  $(a_{11} - a_{22})^2 + 4A_{00}^2$ .

5. If  $A \neq 0$  and  $A_{00} = 0$ , the parabola (20) touches  $l_\infty$  at  $(0, a_{12}, -a_{11})$ , which is the same as  $(0, a_{22}, -a_{12})$ . The axis is

$$(46) \quad a_{11}x + a_{12}y + \frac{a_{01}a_{11} + a_{02}a_{12}}{a_{11} + a_{22}} = 0.$$

**88. Classification of line conics.** The projective classification of line conics is entirely dual to that of point conics and so need not be considered separately. The affine classification, however, corresponds to a new algebraic problem. If the line coördinates are chosen so that

$$u_0x_0 + u_1x_1 + u_2x_2 = 0$$

is the condition that the point  $(x_0, x_1, x_2)$  be on the line  $[u_0, u_1, u_2]$ , the point coördinates being the same as already used, we have the problem of reducing equations of the form (9) to normal forms by means of transformations of the form

$$(47) \quad \begin{aligned} u'_0 &= d_0u_0 + d_1u_1 + d_2u_2 \\ u'_1 &= \quad + b_2u_1 - a_2u_2 \\ u'_2 &= \quad - b_1u_1 + a_1u_2, \end{aligned} \quad \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0.$$

These are the transformations which leave the line  $[1, 0, 0]$  invariant. If

$$d_0 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, \quad d_1 = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}, \quad \text{and} \quad d_2 = \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix},$$

(47) is the same collineation as (21).

The affine classification of nondegenerate line conics is of course the same as that of nondegenerate point conics. To express the criteria in terms of the equation (9) regarded as given primarily,\* let us write

$$(48) \quad \alpha \equiv \begin{vmatrix} A_{00} & A_{01} & A_{02} \\ A_{10} & A_{11} & A_{12} \\ A_{20} & A_{21} & A_{22} \end{vmatrix},$$

where the  $A_{ij}$ 's are the coefficients of (9), and let  $\alpha_{ij}$  denote the cofactor of  $A_{ij}$  in  $\alpha$ . The point conic associated with (9) must have the equation

$$(49) \quad \sum \alpha_{ij} x_i x_j = 0.$$

By the criteria already worked out, this is an ellipse, hyperbola, or parabola according as the value of

$$\begin{vmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{vmatrix} \equiv A_{00} \cdot \alpha$$

is greater than, less than, or equal to zero; and, in the case of an ellipse, real or imaginary according as  $\alpha_{11} > 0$  or  $\alpha_{11} \leq 0$ . Thus we have

$$\alpha \neq 0$$

IMAGINARY ELLIPSE	REAL ELLIPSE	HYPERBOLA	PARABOLA
$\alpha \cdot A_{00} > 0$ $\alpha_{11} > 0$	$\alpha \cdot A_{00} > 0$ $\alpha_{11} \leq 0$	$\alpha \cdot A_{00} < 0$	$A_{00} = 0$

The normal forms for these four classes are respectively

$$(50) \quad u_0^2 + u_1^2 + u_2^2 = 0,$$

$$(51) \quad u_0^2 - u_1^2 - u_2^2 = 0,$$

$$(52) \quad u_0^2 - u_1^2 + u_2^2 = 0,$$

$$(53) \quad u_1^2 - u_2 = 0.$$

The projective classification of degenerate line conics is dual to that of degenerate point conics, and therefore yields the following three cases: (1) two distinct real points,  $\alpha = 0$ ,  $\alpha_{11} \leq 0$ , one at least

\*Instead of in terms of the coefficients of (6).

of  $\alpha_{00}$ ,  $\alpha_{11}$ ,  $\alpha_{22}$  being different from zero; (2) coincident real points,  $\alpha = \alpha_{11} = \alpha_{22} = \alpha_{33} = 0$ ; (3) conjugate imaginary points,  $\alpha = 0$ ,  $\alpha_{ii} > 0$  for at least one value of  $i$ .

For the affine classification let us observe that since  $[1, 0, 0]$  is the line at infinity, the condition that at least one factor of (9) represent a point at infinity is  $A_{00} = 0$ . The following criteria are now evident.

$$\alpha = 0$$

CONJUGATE IMAGINARY POINTS		DISTINCT REAL POINTS			COINCIDENT REAL POINTS	
Ordinary	At infinity	Both ordinary	One ordinary	Both at infinity	Ordinary	At infinity
$\alpha_{11} > 0$ or $\alpha_{22} > 0$	$\alpha_{00} > 0$  $\alpha_{11} = \alpha_{22} = 0$	$\alpha_{11} < 0$ or $\alpha_{22} < 0$ $A_{00} \neq 0$	$A_{00} = 0$	$\alpha_{00} < 0$ $\alpha_{11} = 0$ $\alpha_{22} = 0$	$\alpha_{00} = \alpha_{11} = \alpha_{22} = 0$  $A_{00} \neq 0$	$A_{00} = 0$

The normal forms for these cases are respectively

$$(54) \quad u_0^2 + u_1^2 = 0,$$

$$(55) \quad u_1^2 + u_2^2 = 0,$$

$$(56) \quad u_0^2 - u_1^2 = 0,$$

$$(57) \quad u_0 u_1 = 0,$$

$$(58) \quad u_1 u_2 = 0,$$

$$(59) \quad u_0^2 = 0,$$

$$(60) \quad u_1^2 = 0.$$

### EXERCISES

1. The two pairs of foci of (9) are the degenerate conics of the range

$$(61) \quad \begin{aligned} & A_{00}u_0^2 + A_{01}u_0u_1 + A_{02}u_0u_2 \\ & + A_{10}u_1u_0 + (A_{11} - \rho)u_1^2 + A_{12}u_1u_2 \\ & + A_{20}u_2u_0 + A_{21}u_2u_1 + (A_{22} - \rho)u_2^2 = 0, \end{aligned}$$

which are given by the values of  $\rho$  satisfying

$$(62) \quad A_{00}\rho^2 - (a_{11} + a_{22})\rho + a = 0.$$

The discriminant of this quadratic is  $(a_{11} - a_{22})^2 + 4a_{12}^2$ .

2. In case  $a = 0$  and  $A_{00} \neq 0$ , the distance between the points represented by (9) is

$$\frac{2\sqrt{-(a_{11} + a_{22})}}{A_{00}}.$$

3. The normal forms for line conics under the group of displacements are

$$(63) \quad u_0^2 + a^2 u_1^2 + b^2 u_2^2 = 0,$$

$$(64) \quad u_0^2 - a^2 u_1^2 - b^2 u_2^2 = 0,$$

$$(65) \quad u_0^2 - a^2 u_1^2 + b^2 u_2^2 = 0,$$

$$(66) \quad 4u_0 u_1 + p u_2^2 = 0,$$

$$(67) \quad u_0^2 + k^2 u_1^2 = 0,$$

$$(68) \quad u_1^2 + c^2 u_2^2 = 0,$$

$$(69) \quad u_0^2 - k^2 u_1^2 = 0,$$

$$(70) \quad u_0 u_1 = 0,$$

$$(71) \quad u_1^2 - c^2 u_2^2 = 0,$$

$$(72) \quad u_0^2 = 0,$$

$$(73) \quad u_1^2 = 0.$$

Here  $a, b, p$  have the same significance as in (34)-(37);  $2ki$  is the distance between the two points represented by (67);  $2k$  is the distance between the two points represented by (69);  $c$  is expressible in terms of the cross ratio of the circular points and the two points represented by (68) or (71).

**\* 89. Polar systems.** The theorems on the classification of conics (§ 79) may be regarded as completing the discussion of projective polar systems in a real plane. There is, however, a certain amount of interest in making the discussion of polar systems without the intervention of complex elements, and basing it entirely on the most elementary theorems about order relations. This treatment will hold good for a projective space satisfying Assumptions A, E, S, P.

**THEOREM 36.** *In any projective polar system in an ordered plane the involutions of conjugate points on the sides of a self-polar triangle are all direct, or else one involution is direct and the other two opposite.*

*Proof.* Let  $ABC$  be the self-polar triangle (fig. 68), and let  $PP'$  be a pair of points on the side  $BC$  and  $QQ'$  a pair on the side  $CA$ . Let  $R$  be the point of intersection of the lines  $PQ$  and  $AB$ ,  $O$  that of  $AP'$  and  $BQ'$ , and  $R'$  that of  $CO$  and  $AB$ . Then  $AP'$  is the polar of  $P$ ,  $BQ'$  of  $Q$ ,  $PQ$  of  $O$ , and  $CO$  of  $R$ . Hence  $R$  and  $R'$  are paired in the involution of conjugate points on  $AB$ . Let  $R''$  be the point in which  $P'Q'$  meets  $AB$ ;  $R''$  is the harmonic conjugate of  $R'$  with respect to  $A$  and  $B$ .

If the involutions on  $BC$  and  $CA$  are direct,  $P$  and  $P'$  separate  $B$  and  $C$ , and  $Q$  and  $Q'$  separate  $C$  and  $A$ . It follows by Theorem 19, Chap. II, that  $R$  and  $R''$  do not separate  $B$  and  $A$ . Hence by Theorems 7 and 8, Chap. II,  $R'$  is separated from  $R$  by  $A$  and  $B$ , and hence the involution on the line  $AB$  is direct.

On the other hand, if the involutions on  $BC$  and  $CA$  are not direct,  $P$  and  $P'$  do not separate  $B$  and  $C$ , and  $Q$  and  $Q'$  do not separate  $C$  and  $A$ . Hence  $R$  and  $R''$  do not, and therefore  $R$  and  $R'$  do, separate  $A$  and  $B$ . Hence again the third involution is direct.

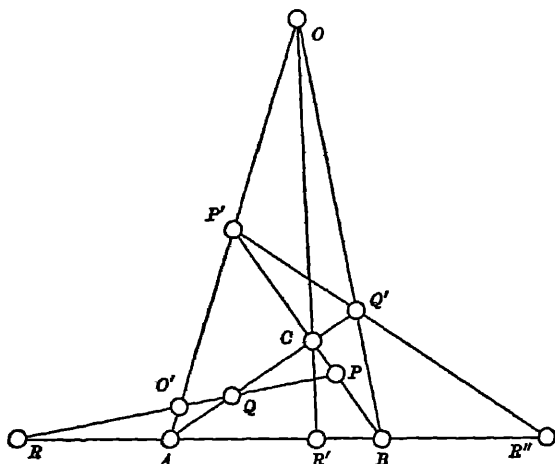


FIG. 88

We have thus shown that at least one of the three involutions is direct; and that if two are direct, so is also the third. From this the statement in the theorem follows.

The reasoning above is valid in any ordered projective space. Specializing to the real space, we have

**COROLLARY 1.** *The involutions on the sides of a self-polar triangle of a projective polar system in a real plane are all three elliptic, or else two are hyperbolic and the third is elliptic.*

**THEOREM 37.** *If the involutions of conjugate points on the sides of one self-polar triangle of a projective polar system in an ordered plane are direct, the involution of conjugate points on any line is direct.*

*Proof.* Let the given self-polar triangle on the sides of which the involutions of conjugate points are direct be  $ABC$ . The theorem will

follow if we can prove that the involution of conjugate points on any line through a vertex of such a triangle is direct. For any line  $l$  meets  $BC$  in a point  $M$  which has a conjugate point  $N$  on  $BC$ . By the proposition which we are supposing proved, the involutions on the sides of the self-polar triangle,  $AMN$ , are direct; and by a second application of the same proposition, the involution of conjugate points on  $l$  is direct. Thus the proof of the theorem reduces to the proof that the involution of conjugate points on any line through  $A$  is direct.

Let such a line meet  $BC$  in a point  $P'$ , and let  $P$  be the conjugate of  $P'$  in the involution on  $BC$ . Let  $Q$  and  $Q'$  be a conjugate pair distinct from  $A$  and  $C$  on the line  $AC$ , and let  $O, R, R', R''$  have the same meaning as in the proof of the last theorem (fig. 68). Also let  $O'$  be the conjugate of  $O$  on the line  $AP'$ , i.e. let  $O'$  be the intersection of  $AP'$  with  $PQ$ . Applying Theorem 19, Chap. II, to the triangle  $ABP'$  and the lines  $O'R$  and  $OR'$ , it follows that, since  $C$  and  $P$  do not separate  $B$  and  $P'$ , and  $R$  and  $R'$  do separate  $A$  and  $B$ ,  $O$  and  $O'$  are separated by  $A$  and  $P'$ . Hence the involution of conjugate points on the line  $AP'$  is direct.

**COROLLARY 1.** *If the involutions on two sides of a self-polar triangle of a polar system in an ordered plane are opposite, then two of the involutions on the sides of any self-polar triangle are opposite and the third is direct.*

*Proof.* If there were any self-polar triangle not satisfying the conclusion of the theorem, this would, by Theorem 36, be one for which all three involutions were direct. By Theorem 37 it would follow that the involutions on all lines were direct, contrary to hypothesis.

The propositions stated in the last two theorems and in the last corollary may evidently be condensed into the following:

**COROLLARY 2.** *Any projective polar system in an ordered plane is either such that the involution of conjugate points on any line is direct, or such that on the sides of any self-polar triangle two of the involutions are opposite and the third direct.*

Applying this result in a real plane, we have that every projective polar system is either such that all involutions of conjugate points are elliptic, or such that on the sides of any self-polar triangle two involutions are hyperbolic and the third elliptic. In the latter case let  $ABC$  be a self-polar triangle,  $AB$  and  $AC$  being the sides upon

which the involutions are hyperbolic. Let the double points of the involution on  $AB$  be  $C_1$  and  $C_2$ , and those of the involution on  $AC$  be  $B_1$  and  $B_2$ . The polar of  $C_1$  is then the line  $C_1C$ . The conic section  $K^2$  through  $C_1, C_2, B_1, B_2$  and tangent to the line  $C_1C$  at  $C_1$  has a polar system in which  $ABC$  is a self-polar triangle, and in which the given involutions are involutions of conjugate points. By § 93, Vol. I, these conditions are sufficient to determine a polarity. Hence the given polarity is the polar system of  $K^2$ . Thus we have

**THEOREM 38. DEFINITION.** *A projective polar system in a real plane is either the polar system of a real conic, or such that the involution of conjugate points on any line is elliptic. A polar system of the latter type is said to be elliptic.*

The existence of elliptic polar systems is easily seen as follows: Let  $ABC$  be any triangle,  $O$  any point not on a side of this triangle,  $P'$  the point of intersection of  $OA$  with  $BC$ ,  $Q'$  the point of intersection of  $OB$  with  $CA$ , and  $P$  and  $Q$  any two points separated from  $P'$  and  $Q'$  by the pairs  $BC$  and  $CA$  respectively. By the theorems in § 93, Vol. I, there exists a polar system in which the triangle  $ABC$  is self-polar and the point  $O$  is the pole of the line  $PQ$ , and by the theorems in the present section this polar system is elliptic.

## CHAPTER VI

### INVERSION GEOMETRY AND RELATED TOPICS \*

**90. Vectors and complex numbers.** The properties of the addition of vectors have been derived in § 42 from those of the group of translations. If the operation of multiplication is to satisfy the distributive law,

$$a(b + c) = ab + ac,$$

multiplication by a vector,  $a$ , must effect a transformation on the vector field such that  $b + c$  is carried into the vector which is the sum of those to which  $b$  and  $c$  are carried. Since the group of translations is a self-conjugate subgroup of the Euclidean group, any similarity transformation of the vector field satisfies this condition.

Let us then consider the transformations effected on a vector field by the Euclidean group. Any similarity transformation is a product of a translation by a similarity transformation leaving an arbitrary point  $O$  invariant. But a translation carries every vector into itself. Hence any similarity transformation has the same effect on the field of vectors as a similarity transformation leaving  $O$  invariant. Hence the totality of transformations effected on the vector field by the Euclidean group is identical with the totality of transformations effected on it by the similarity transformations leaving  $O$  invariant. Since no such transformation changes every vector into itself, any two of them effect different transformations of the field of vectors. Hence we have

**THEOREM 1.** *The group of transformations effected by the Euclidean group in a plane upon the field of vectors is isomorphic with the group of similarity transformations leaving an arbitrary point invariant.*

To obtain a definition of multiplication we restrict attention to the group of direct similarity transformations and make use of the fact that if  $OA$  and  $OB$  are any two nonzero vectors, there is one and but

\* The main part of Chap. VII is independent of this chapter. The two chapters may therefore be taken up in reverse order if the reader so desires.

one transformation of this group carrying the points  $O$  and  $A$  to  $O$  and  $B$  respectively.

DEFINITION. Relative to an arbitrary vector  $OA$ , which is called the *unit vector*, the *product* of two vectors  $OX$  (where  $X \neq O$ ) and  $OY$  is the vector  $OZ$  to which  $OY$  is carried by the direct similarity transformation carrying  $OA$  to  $OX$ , and is denoted by  $OX \cdot OY$ . In case  $X = O$ ,  $OX \cdot OY$  denotes the zero vector.

As obvious corollaries of this definition we have the following two theorems:

THEOREM 2. *The triad of points  $OAY$  is directly similar to the triad  $OZX$  if and only if*

$$OZ = OX \cdot OY.$$

THEOREM 3. *The equation*

$$OZ = OX \cdot OY$$

*is satisfied if and only if  $\angle AOX + \angle AOY = \angle AOZ$  and  $\text{Dist}(OZ) = \text{Dist}(OX) \cdot \text{Dist}(OY)$ , the unit of distance being  $OA$ .*

Since the direct similarity transformations leaving a point  $O$  invariant form a group, the operation of multiplication must be associative, i.e.

$$OX \cdot (OY \cdot OZ) = (OX \cdot OY) \cdot OZ,$$

and also such that there is a unique inverse for every vector  $OB$  for which  $O \neq B$ , i.e. there must be a vector  $OY$  such that

$$OB \cdot OY = OA.$$

The group of direct similarity transformations leaving  $O$  invariant is commutative because it consists of the rotations about  $O$  (which form a commutative group by § 58) combined with dilations with  $O$  as center. Hence the operation of multiplication is commutative, i.e.

$$OX \cdot OY = OY \cdot OX.$$

The fact that the group of translations is self-conjugate under the group of displacements translates into the distributive law,

$$OX \cdot (OY + OZ) = OX \cdot OY + OX \cdot OZ.$$

Recalling the definition of a number system given in Chap. VI, Vol. I, we may summarize these results by saying,

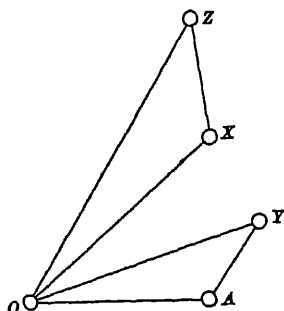


FIG. 69

**THEOREM 4.** *With respect to the operation of addition described in § 42 and of multiplication defined in this section, a planar vector field is a commutative number system.*

In proving this theorem we have made use of no properties of the Euclidean group except such as hold for any parabolic metric geometry for which the absolute involution is elliptic. In case the absolute involution were hyperbolic, exceptions would have to be made corresponding to properties of the minimal lines.

The definition of multiplication of vectors as given here does not conflict with the notion of the ratio of collinear vectors as developed in Chap. III. For the quotient of two collinear vectors is a vector collinear with the unit vector  $OA$ , and the system of vectors collinear with  $OA$  constitutes a number system isomorphic with the real number system. Thus, if we denote the unit vector by  $1$ , any vector  $OX$  collinear with it may be denoted by

$$x1,$$

where, according to the definition of § 43,  $x$  is a real number and where, according to our present definition,  $x$  denotes  $OX$  itself.

Let us denote a vector  $OB$  such that the line  $OB$  is perpendicular to the line  $OA$  and such that  $\text{Dist}(OB) = \text{Dist}(OA)$ , by  $i$ . Then by the definition of multiplication,

$$i^2 = -1.$$

Any vector collinear with  $i$  is expressible in the form  $xi$ , where  $x$  is a vector parallel to  $1$ , and by Theorem 8, Chap. III, any vector whatever is expressible uniquely in the form

$$a1 + bi$$

The product of two vectors may be reduced by the associative, distributive, and commutative laws as follows:

$$\begin{aligned}(a1 + bi)(c1 + di) &= (a1 + bi)c1 + (a1 + bi)di \\ &= (ac - bd)1 + (bc + ad)i\end{aligned}$$

By comparison with §§ 3 and 14 this shows that

**THEOREM 5.** *A planar field of vectors is a number system isomorphic with the complex number system, i.e. the geometric number system of a complex line.*

The isomorphism in question is that by which the complex number  $a + bi$  corresponds to the vector  $a\mathbf{i} + b\mathbf{j}$ . Supposing that the fundamental points of the scale on the complex line are  $P_0, P_1, P_\infty$ , this means that there is a correspondence between the complex line and the Euclidean plane in which  $P_0$  corresponds to  $O$ ,  $P_1$  to  $A$ , and every point whose coördinate relative to the scale  $P_0, P_1, P_\infty$  is

$$a + bi$$

corresponds to the point  $Q$  of the Euclidean plane such that

$$OQ = a\mathbf{i} + b\mathbf{j}$$

One obvious property of this correspondence which we shall have to use later is that the points of the complex line which have real coördinates relative to the scale  $P_0, P_1, P_\infty$  correspond to the points of the line  $OA$ , or, in other words, that *the points of the chain\*  $C(P_0P_1P_\infty)$ , other than  $P_\infty$ , correspond to the points on the real line  $OA$ .*

Theorem 5 may be made the basis of a method for the investigation of theorems of Euclidean geometry, particularly those relating to  $n$ -lines and circles. The complex numbers may be regarded as the coördinates of the points of the Euclidean plane and many interesting theorems obtained by interpreting simple algebraic equations. Compare the articles by F. Morley, Transactions of the American Mathematical Society, Vol. I, p. 97; Vol. IV, p. 1; Vol. V, p. 467; Vol. VIII, p. 14.

The whole subject is closely related to certain elementary parts of the theory of functions of a complex variable. Cf. an article by F. N. Cole, Annals of Mathematics, 1st Series, Vol. V (1890), p. 121.

**91. Correspondence between the complex line and the real Euclidean plane.** The operation of addition of vectors has been so defined that

$$OX' = OX + OP,$$

where  $O$  and  $P$  are fixed and  $X$  and  $X'$  variable points, may be taken as representing a translation carrying  $X$  to  $X'$ . The operation of multiplication has been defined so that

$$OX' = OP \cdot OX$$

may be taken to represent a direct similarity transformation carrying  $O$  into itself and  $X$  to  $X'$ . Thus the general direct similarity transformation may be written

$$OX' = OP \cdot OX + OQ.$$

\* Cf. § 11. The reader who has omitted the starred sections in Chap. I may take a chain  $C(P_0P_1P_\infty)$  as by definition consisting of those points of a complex line which have real coördinates relative to the scale  $P_0, P_1, P_\infty$ .

The last theorem may therefore be stated in the following form :

**THEOREM 6.** *Let  $Q_0, Q_1, Q_\infty$  be three arbitrary points of a complex projective line  $l$ , and let  $P_0$  and  $P_1$  be two arbitrary points of a Euclidean plane  $\pi$  in whose line at infinity  $l_\infty$  an elliptic absolute involution is given. There exists a one-to-one and reciprocal correspondence  $\Gamma$  in which  $P_0$  corresponds to  $Q_0, P_1$  to  $Q_1, l_\infty$  to  $Q_\infty$ , and every ordinary point of  $\pi$  to a point of  $l$  distinct from  $Q_\infty$ . This correspondence is such that to every projective transformation of  $l$  leaving  $Q_\infty$  invariant, i.e. to every transformation of the form*

$$(1) \quad x' = ax + b, \quad a \neq 0,$$

*there corresponds a direct similarity transformation of  $\pi$ , and conversely.*

The question immediately arises, What group of transformations of  $\pi$  corresponds to the general projective group on  $l$ , i.e. to the set of transformations

$$(2) \quad x' = \frac{ax + b}{cx + d}, \quad \begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0?$$

The transformation of  $\pi$  corresponding to

$$(3) \quad x' = 1/x$$

must change any point  $P$  to a point  $P'$  such that

$$P_0P' \cdot P_0P = P_0P_1.$$

Hence, by Theorem 3,  $\triangle PP_0P_1$  is congruent to  $\triangle P_1P_0P'$ . Therefore the orthogonal line reflection with  $P_0P_1$  as axis must carry  $P$  to a point  $P''$  of the line  $P_0P'$ . If  $P$  be regarded as a variable point of a line through  $P_0$ , it follows that the correspondence between  $P'$  and  $P''$  is projective. In this correspondence  $P_0$  corresponds to the point at infinity of the line  $P_0P'$ , and each of the points in which this line meets the circle through  $P_1$  with  $P_0$  as center corresponds to itself. Hence the correspondence between  $P'$  and  $P''$  on a given line through  $P_0$  is an involution, and  $P'$  and  $P''$  are conjugate points with respect to the circle. Hence (§ 71), if  $P$  be a variable point of the plane, the correspondence between  $P'$  and  $P''$  is an inversion. Hence

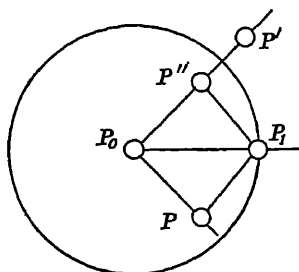


FIG. 70

the transformation of  $\pi$  corresponding to  $x' = 1/x$  is the product of the orthogonal line reflection with  $P_0P_1$  as axis and the inversion with respect to the circle through  $P_1$  with  $P_0$  as center.

Now any transformation (2) is evidently (cf. § 54, Vol. I) a product of transformations of the forms (1) and (3). But the transformation (1) has been seen to correspond to a direct similarity transformation, i.e. to a product of a dilation and a displacement. A displacement has been proved in Chap. IV to be a product of two orthogonal line reflections; and a dilation will now be shown to be a product of two or four inversions and orthogonal line reflections.

For consider a dilation  $\Delta$  with a point  $O$  as center and carrying a point  $A$  to a point  $B$ . If  $O$  is not between  $A$  and  $B$ , there exists (Theorem 8, Chap. V) a pair of points  $C_1C_2$  which separate  $A$  and  $B$  harmonically and have  $O$  as mid-point. Let  $I_1$  be the inversion with respect to the circle with  $O$  as center and passing through  $C_1$ . The transformation  $I_1\Delta$  leaves invariant all points of the circle through  $A$  with  $O$  as center, and effects a projectivity on each line through  $O$  which interchanges  $O$  and the point at infinity. The projectivity on each line through  $O$  is therefore the involution carrying each point to a conjugate point with regard to the circle through  $A$  with  $O$  as center. Hence  $I_1\Delta$  is an inversion,  $I_2$ , with respect to this circle. From  $I_1\Delta = I_2$  follows  $\Delta = I_1I_2$ . If  $O$  is between  $A$  and  $B$ , let  $\Lambda$  be the point reflection with  $O$  as center. The product  $\Lambda\Delta$  is a dilation such that  $O$  is not between  $A$  and  $\Lambda\Delta(A)$ . Hence  $\Lambda\Delta$  is a product of two inversions  $I_1, I_2$  and  $\Delta = \Lambda I_2 I_1$ . Since  $\Lambda$  is a product of two orthogonal line reflections,  $\Delta$  is a product of four inversions and orthogonal line reflections.

*Hence any projective transformation of a complex line  $l$  corresponds under  $\Gamma$  to a transformation of a real Euclidean plane  $\pi$  which is a product of an even number of inversions and orthogonal line reflections.*

The converse of this proposition is also valid. In order to prove it we need only verify ( $\alpha$ ) that the product of two orthogonal line reflections in  $\pi$  corresponds to a projectivity of  $l$ , ( $\beta$ ) that the product of an orthogonal line reflection  $\Lambda$  and an inversion  $P$  of  $\pi$  corresponds to a projectivity of  $l$ , and ( $\gamma$ ) that the product of two inversions  $P_1P_2$  of  $\pi$  corresponds to a projectivity of  $l$ . The first of these statements is a corollary of Theorem 6.

To prove ( $\beta$ ) let us first consider the case where the axis of  $\Lambda$  passes through the center  $O$  of  $P$ . Let  $O_1$  be one of the points in which the axis of  $\Lambda$  meets the invariant circle of  $P$ ,  $X$  be any point of  $\pi$ , and  $X' = \Lambda P(X)$ . The considerations given above in connection with the transformation ( $\beta$ ) show that

$$OX' = \frac{OO_1}{OX},$$

and hence that  $\Lambda P$  corresponds to a transformation of  $l$  of the same type as ( $\beta$ ), i.e. to an involution. Moreover,  $\Lambda P$  is obviously the same as  $PA$ . In case the axis of  $\Lambda$  does not pass through the center of  $P$ , let  $\Lambda'$  be an orthogonal line reflection whose axis passes through the center of  $P$ . Then

$$\Lambda P = \Lambda \Lambda' \cdot \Lambda' P \quad \text{and} \quad PA = P \Lambda' \cdot \Lambda' \Lambda.$$

The products  $\Lambda \Lambda'$  and  $\Lambda' \Lambda$  correspond to projectivities by Theorem 6, and  $PA' = \Lambda' P$  corresponds to an involution by what has just been proved. Hence  $\Lambda P$  and  $PA$  correspond to projectivities.

To prove ( $\gamma$ ) let  $\Lambda$  be an orthogonal line reflection whose axis contains the centers of  $P_1$  and  $P_2$ . Then

$$P_1 P_2 = P_1 \Lambda \cdot \Lambda P_2.$$

The products  $P_1 \Lambda$  and  $\Lambda P_2$  correspond to projectivities by ( $\beta$ ). Hence  $P_1 P_2$  corresponds to a projectivity. Thus we have the important result:

**THEOREM 7.** *A projective transformation on a complex line corresponds under  $\Gamma$  to a transformation of the real Euclidean plane which is a product of an even number of inversions and orthogonal line reflections, and, conversely, any transformation of the real Euclidean plane of this type corresponds to a projectivity of the complex line.*

## 92. The inversion group in the real Euclidean plane.

**DEFINITION.** The transformations of a Euclidean plane and its line at infinity which are products of orthogonal line reflections and inversions are called *circular transformations*, and any circular transformation which is a product of an even number of inversions and orthogonal line reflections is said to be *direct*.

**THEOREM 8. DEFINITION.** *The set of all circular transformations of a Euclidean plane and its line at infinity in which an absolute*

*involution is given constitute a group which is called the inversion group. The set of direct circular transformations form a subgroup of the inversion group, which, if the Euclidean plane is real, is isomorphic with the projective group of a complex line.*

The first part of this theorem is an obvious consequence of the definition, and the second is equivalent to Theorem 7. That not all circular transformations are direct is shown by the special case of an inversion. An inversion is not a direct circular transformation, because it leaves invariant all points of a circle and hence cannot correspond under  $\Gamma$  to a projectivity. Combining Theorems 8 and 6 we have

**COROLLARY.** *In a real Euclidean plane the group of circular transformations leaving  $l_\infty$  invariant is the Euclidean group, and the direct circular transformations leaving  $l_\infty$  invariant are the direct similarity transformations.*

The isomorphism between the group of direct circular transformations and the projective group on the line may be used as a source of theorems about the former. Thus the fundamental theorem of projective geometry (Assumption P) translates into the following theorem about the real Euclidean plane:

**THEOREM 9.** *A direct circular transformation which leaves three ordinary points, or two ordinary points and  $l_\infty$ , invariant is the identity. There exists a direct circular transformation carrying any three distinct ordinary points  $A, B, C$  respectively into three distinct points  $A', B', C'$  respectively, or into  $A', B'$ , and  $l_\infty$  respectively.*

Now consider a circular transformation  $\Pi$  which is not direct and which leaves three distinct points  $A, B, C$  invariant. By definition

$$\Pi = \Lambda_{2n+1} \cdot \Lambda_{2n} \cdots \Lambda_2 \cdot \Lambda_1,$$

where  $\Lambda_i$  ( $i = 1, 2, \dots, 2n+1$ ) is an inversion or an orthogonal line reflection. Let  $\Lambda$  be an orthogonal line reflection whose axis contains  $A, B, C$ , if these points are collinear, or an inversion with respect to the circle containing them in case they are not collinear. Then  $\Lambda\Pi$  is a direct circular transformation leaving  $A, B, C$  invariant. Hence

$$\Lambda\Pi = 1.$$

Since  $\Lambda$  is of period two, this implies

$$\Pi = \Lambda.$$

The same argument applies in case one of the points  $A, B, C$  is replaced by  $l_{\infty}$ . Hence we have

**THEOREM 10.** *A circular transformation which is not direct and leaves invariant three distinct ordinary points  $A, B, C$ , or two ordinary points  $A, B$ , and  $l_{\infty}$ , is an orthogonal line reflection or an inversion according as the invariant points are collinear or not.*

**THEOREM 11.** *If  $\Pi$  is a circular transformation and  $\Lambda$  an inversion or orthogonal line reflection,  $\Pi\Lambda\Pi^{-1}$  is an inversion or orthogonal line reflection.*

*Proof.* Let  $A, B, C$  be three of the invariant points of  $\Lambda$ ; then  $\Pi\Lambda\Pi^{-1}$  leaves  $\Pi(A), \Pi(B), \Pi(C)$  invariant. If

$$\Pi = \Lambda_1 \Lambda_2 \cdots \Lambda_n,$$

where  $\Lambda_1, \dots, \Lambda_n$  are orthogonal line reflections or inversions, then

$$\Pi\Lambda\Pi^{-1} = \Lambda_1 \Lambda_2 \cdots \Lambda_n \Lambda \Lambda_n \cdots \Lambda_2 \Lambda_1,$$

and is thus a product of an odd number of orthogonal line reflections or inversions. Hence by the last theorem it is an orthogonal line reflection or an inversion.

The invariant elements of  $\Pi\Lambda\Pi^{-1}$  are those to which the invariant elements of  $\Lambda$  are carried by  $\Pi$ . Since  $\Pi\Lambda\Pi^{-1}$  is an inversion or an orthogonal line reflection, we have

**COROLLARY 1.** *Any circular transformation carries any circle into a circle or into the set of points on an ordinary line and on  $l_{\infty}$ . It carries the set of points on  $l_{\infty}$  and an ordinary line into a set of this sort or into a circle.*

**COROLLARY 2.** *If  $C^2$  and  $K^2$  are any two circles and  $l$  any line, there exists a direct circular transformation carrying  $C^2$  to  $K^2$  and one carrying  $C^2$  to the set of all points on  $l$  and  $l_{\infty}$ .*

*Proof.* Let  $A, B, C$  be any three points of  $C^2$ , let  $A', B', C'$  be any three points of  $K^2$ , and let  $A', B'$  be any two points of  $l$ . By Theorem 9, there exist direct circular transformations  $\Pi$  and  $\Pi'$  such that

$$\Pi(ABC) = A'B'C' \quad \text{and} \quad \Pi'(ABC) = A'B'l_{\infty}.$$

Since  $A', B', C'$  are not collinear, the set of points into which  $\Pi$  carries  $C^2$  must be a circle; and since there is only one circle containing  $A', B', C'$ , this circle is  $K^2$ . Since there is no circle containing  $A', B'$ , and  $l_{\infty}$ , the set of points into which  $\Pi'$  carries  $C^2$  must be the set of

points on  $l_\infty$  and an ordinary line. Since the ordinary line contains  $A'$  and  $B'$ , it must be  $l$ .

An inversion (§ 71) transforms all lines through its center into themselves and interchanges the center with  $l_\infty$ . Hence, by the last two corollaries, we have at once

**COROLLARY 3.** *An inversion carries a circle through its center into the set of points on  $l_\infty$  and a line not passing through the center.*

**COROLLARY 4.** *A pair of circles which touch each other is carried by an inversion into a pair of circles which touch each other, or into a circle and a tangent line together with  $l_\infty$ , or into two parallel lines and  $l_\infty$ .*

*Proof.* Let  $C^2$  and  $K^2$  be two circles which touch each other. Since an inversion is a one-to-one reciprocal correspondence except for the origin and  $l_\infty$ , if neither  $C^2$  nor  $K^2$  passes through the origin, they must be carried into two circles having only one point in common and which therefore touch each other. If  $C^2$  passes through the origin and  $K^2$  does not,  $C^2$  is carried into  $l_\infty$  and an ordinary line  $l$ , while  $K^2$  is carried into a circle  $K_1^2$  which has one and only one point in common with the line pair  $l_\infty l$ . Since  $l_\infty$  cannot meet  $K_1^2$  in a real point,  $l$  meets it in a single point and therefore is tangent. If  $C^2$  and  $K^2$  both pass through the center of inversion, they are transformed into  $l_\infty$  and a pair of ordinary lines  $l, m$ . Since  $C^2$  and  $K^2$  have only the center of inversion in common and this is transformed into  $l_\infty$ , the lines  $l$  and  $m$  can have no ordinary point in common. Hence  $l$  and  $m$  are parallel.

It was remarked in § 90 (just before the fine print at the end) that the correspondence  $\Gamma$  between the complex line and the real Euclidean plane is such that the points of a certain chain  $C(P_0P_1P_\infty)$ , with the exception of  $P_\infty$ , correspond to the points of a certain Euclidean line  $l$ . Since  $P_\infty$  corresponds to  $l_\infty$ , the chain  $C(P_0P_1P_\infty)$  corresponds to the line pair  $l_\infty l$ . Under the projective group on a line any two chains are equivalent; and under the group of direct circular transformations any circle is equivalent to any circle or any line pair  $l_\infty l$  (Cor. 2). Hence we have

**THEOREM 12.** *The correspondence  $\Gamma$  is such that chains in the complex line correspond to real circles or to line pairs  $l_\infty l$ , where  $l$  is ordinary and  $l_\infty$  the line at infinity of the Euclidean plane.*

The theory of chains on a complex line is therefore equivalent to the theory of the real circles and lines of a Euclidean plane. In view of this equivalence we shall freely transform the terminology of the complex line to the Euclidean plane, and vice versa. Thus we shall speak of the cross ratio of four points in the Euclidean plane and of pencils of chains in the complex line. The exercises below contain a number of important theorems some of which can be obtained directly from the definitions in § 71 and some of which can be proved most simply by translating projective theorems on the complex line into the terminology of the Euclidean plane.

DEFINITION. An *imaginary circle* is an imaginary conic through the circular points such that its polar system transforms real points into real lines.

The definition of an inversion given in § 71 applies without change to the case of imaginary circles.

On the geometry of circles in general the reader is referred to the papers by Möbius in Vol. II of his collected works; to those by Steiner in Vol. I (especially pp. 16-83, 461-527) of his collected works; to Vol. II, Chaps. II, III, of the textbook by Doehlemann referred to in Ex. 4; and to the forthcoming book by J. L. Coolidge, *A Treatise on the Circle and the Sphere*, Oxford, 1916.

### EXERCISES

1. An inversion with respect to an imaginary circle is a product of an inversion with respect to a real circle and a point reflection having the same center as the circle.

2. The inverse points on any line through the center  $O$  of a circle  $C^2$  are the pairs of an involution having  $O$  as center. If  $A_1$  and  $A_2$  are any two inverse points,  $OA_1 \cdot OA_2$  is a constant, which in case of a real circle is equal to  $(OC)^2$ ,  $C$  being a point of  $C^2$ .

3. Two pairs of points  $AA'$  and  $BB'$  are inverse with respect to a circle with  $O$  as center if and only if (1)  $O$  is collinear with the pairs  $AA'$  and  $BB'$ , and (2) the ordered triads  $OAB$  and  $OB'A'$  are similar, but not directly similar.

4. A linkage which consists of a set of six bars  $OA, OC, AB, BC, CD, DA$ , jointed movably at the points  $O, A, B, C, D$ , and such that  $\text{Dist}(OA) = \text{Dist}(OC)$  and  $ABCD$  is a rhombus, is called a "Peaucellier invensor." If  $O$  is held fixed and  $B$  varies, the locus of  $D$  is inverse to that of  $B$  with respect to a circle with  $O$  as center. If  $B$  be constrained, say by an additional link, to move on a circle through  $O$ ,  $D$  describes a line. On the general

subject of linkages, cf. K. Doehlemann, *Geometrische Transformationen*, Vol. II, p. 90, Leipzig, 1908, and A. Emch, *Projective Geometry*, §§ 62-67, New York, 1905.

5. If  $A, B, C, D$  are four points of a Euclidean plane,

$$R(AB, CD) = ke^{i\theta},$$

where 
$$k = \frac{\text{Dist}(AC)}{\text{Dist}(AD)} + \frac{\text{Dist}(BC)}{\text{Dist}(BD)} \quad \text{and} \quad \theta = \alpha - \beta,$$

where  $\alpha$  and  $\beta$  are the measures of  $\angle CAD$  and  $\angle CBD$  respectively. The number  $k$  is invariant under the inversion group, and  $\theta$  under the group of direct circular transformations. The four points are on a circle or collinear if  $\theta = 0$ .

6. Construct a point having with three given points a given cross ratio.

7. If  $\Pi$  is any circular transformation, the points  $O = \Pi^{-1}(l_{\infty})$  and  $O' = \Pi(l_{\infty})$  are called its vanishing points. The lines through  $O$  are transformed by  $\Pi$  into the lines through  $O'$ . If  $X$  is any point of the plane, and  $X' = \Pi(X)$ , then  $\text{Dist}(OX) \cdot \text{Dist}(O'X')$  is a constant, called the *power* of the transformation (cf. § 43).

8. Let  $A$  and  $B$  be two points not collinear with  $O$  and let  $\Pi(A) = A'$ ,  $\Pi(B) = B'$ . The ordered point triads  $OAB$  and  $O'B'A'$  are directly similar if  $\Pi$  is direct, and similar, but not directly so, if  $\Pi$  is not direct.

9. The equations of an inversion relative to rectangular nonhomogeneous coördinates, having the center of inversion as origin, are

$$x' = \frac{kx}{x^2 + y^2}, \quad y' = \frac{ky}{x^2 + y^2}.$$

The circle of inversion is real or imaginary according as  $k > 0$  or  $k < 0$ .

10. The coördinate system for the real Euclidean plane obtained by means of the isomorphism of the Euclidean group with the projective group leaving a point invariant on a complex line is such that the coördinate  $z$  of any point is  $x + iy$ , where  $x$  and  $y$  are the coördinates in a system of rectangular nonhomogeneous coördinates and  $i^2 = -1$ . The points  $z$  of a circle satisfy the condition

$$z = \frac{at + b}{ct + d}, \quad \begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0,$$

where  $t$  is real and variable and  $a, b, c, d$  are complex and fixed. If  $c = 0$ , this circle reduces to a line.

11. The circles orthogonal to  $z = \frac{at + b}{ct + d}$  are

$$z = \frac{(a + b\beta)u + b + \alpha\alpha}{(c + d\beta)u + d + c\alpha},$$

where  $\alpha$  and  $\beta$  are real.

12. The circles through two points  $z_1, z_2$  are given by

$$z = \frac{atz_1 + z_2}{at + 1}.$$

13. A circle with  $z_1$  as center is given by

$$z - z_1 = ke^{i\theta},$$

where  $0 \leq \theta < 2\pi$  and  $k$  is a real constant.

14. The centers of the circles circumscribing the four triangles formed by the sides of a complete quadrilateral are on a circle. This circle is called the *center circle* of the complete quadrilateral. The centers of the center circles of the five complete quadrilaterals formed by the sides of a complete five-line are on a circle called the *center circle* of the five-line. Generalize this result.

93. **Generalization by inversion.** By the corollary of Theorem 8 the set of direct circular transformations leaving  $l_\infty$  invariant is the group of direct similarity transformations, and the set of all circular transformations leaving  $l_\infty$  invariant is the Euclidean group. This is the basis of a method of *generalization by inversion* entirely analogous to the *generalization by projection* employed in § 73.

In case a figure  $F_1$  which is under investigation can be transformed by one or more inversions into a known figure  $F_2$ , then such of the relations among the elements of  $F_2$  as are invariant under circular transformations must hold good among the corresponding elements of  $F_1$ .

In order to apply this method it is necessary to know relations which are left invariant by the circular transformations. The most elementary of these are given in the last section, but perhaps the most important property of an inversion for this purpose is that of *isogonality*, or "preservation of angles."

DEFINITION. If  $C_1^2$  and  $C_2^2$  are two circles having a point  $Q$  in common, and  $m_1$  and  $m_2$  are the tangents to  $C_1^2$  and  $C_2^2$  respectively at  $Q$ , the measure (according to § 72) of the ordered line pair  $m_1 m_2$  is called the *angular measure* of the ordered pair of circles at  $Q$ , or simply the *angle* between the two circles at  $Q$ . If  $C_1^2$  is any circle,  $m_2$  a line meeting it in a point  $Q$ , and  $m_1$  the tangent to  $C_1^2$  at  $Q$ , the measure of the ordered line pair  $m_2 m_1$  is called the *angle* between  $m_2$  and  $C_1^2$ , and the measure of  $m_1 m_2$  is called the *angle* between  $C_1^2$  and  $m_2$ . The measure of a line pair  $m_1 m_2$  is called the *angle\** between  $m_1$  and  $m_2$ .

THEOREM 13. *An angle  $a$  between two circles or a circle and a line or between two lines is changed into  $\pi - a$  by an inversion or*

\* In accordance with common usage, we are here using the term "angle" to denote a number, in spite of the fact that we use it in § 28 to denote a geometrical figure.

*an orthogonal line reflection and is left unaltered by any direct circular transformation.*

*Proof.* The statement with regard to direct circular transformations is an obvious consequence of the one with regard to inversions and orthogonal line reflections. What we have to prove is, therefore, the following:

Let  $\Pi$  be an inversion or an orthogonal line reflection, and let  $l_1$  and  $l_2$  be two lines meeting in a point  $P$  such that  $\Pi(P) = Q$  is an ordinary point. If  $l_1$  is carried by  $\Pi$  into a line, let this line be denoted by  $m_1$ ; and if  $l_1$  (together with  $l_\infty$ ) is carried to a circle  $C_1^2$ , let  $m_1$  denote the tangent to  $C_1^2$  at  $Q$ ; likewise, if  $l_2$  is carried by  $\Pi$  into a line, let this line be denoted by  $m_2$ ; and if  $l_2$  (together with  $l_\infty$ ) is carried to a circle  $C_2^2$ , let  $m_2$  denote the tangent to  $C_2^2$  at  $Q$ . The two ordered pairs of lines  $l_1 l_2$  and  $m_1 m_2$  are symmetric.

In case  $\Pi$  is an orthogonal line reflection,  $m_1 = \Pi(l_1)$  and  $m_2 = \Pi(l_2)$ , and the proposition is a direct consequence of the definition of the term "symmetric" (§ 57). Suppose, then, that  $\Pi$  is an inversion having a point  $O$  as center.

One of the lines  $l_1, l_2$ , say  $l_1$ , can be transformed into itself if and only if  $l_1$  is on  $O$ . By hypothesis  $O \neq P$ ; hence if  $\Pi(l_1) = l_1$ , the line  $l_2$  goes into the set of points different from  $O$  on a circle  $C_2^2$  through  $O$  and  $Q$ . Then  $m_2$  is the tangent to  $C_2^2$  at  $Q$ . Any line through  $O$  which meets  $l_2$  in an ordinary point  $X$  meets  $C_2^2$  in the point which corresponds to

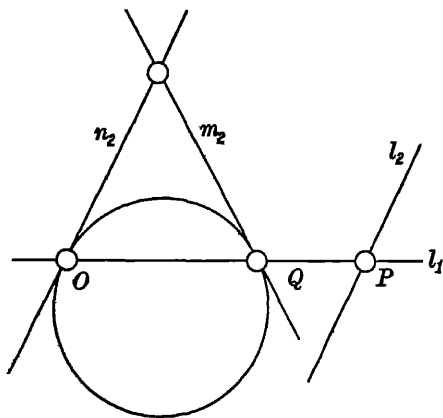


FIG. 71

$X$  under the inversion. Hence the line  $n_2$  through  $O$  and tangent to  $C_2^2$  cannot meet  $l_2$  in an ordinary point, and is therefore parallel to  $l_2$ . Hence the line pair  $l_1 l_2$  is congruent to the pair  $l_1 n_2$ . The line  $m_2$  is the tangent to  $C_2^2$  at  $Q$ . Since  $l_1 n_2$  is carried to  $l_1 m_2$  by the orthogonal line reflection whose axis is the perpendicular bisector of  $OQ$ , the pair  $l_1 n_2$  is symmetric with  $l_1 m_2$ . Hence  $l_1 l_2$  is symmetric with  $l_1 m_2$ .

If neither of the lines  $l_1, l_2$  is transformed into itself, neither passes through  $O$ . Let  $l$  denote the line  $OP$ . Then by the last paragraph  $ll_1$  is symmetric with  $lm_1$ , and  $ll_2$  with  $lm_2$ . But by Theorem 13, Chap. IV, the symmetry which carries  $ll_1$  to  $lm_1$  must be identical with that which carries  $ll_2$  to  $lm_2$ . Hence  $l_1l_2$  is symmetric with  $m_1m_2$ .

As an exercise in generalization by inversion let us prove the following :

**THEOREM 14.** *If three circles  $C_1^2, C_2^2, C_3^2$  meet in a point  $O$  in such a way that each pair of them makes an angle  $\frac{\pi}{3}$ , and also meet by pairs in three other points  $P, Q, R$ , the circle (or line) through  $P, Q$  and  $R$  makes with each of the other circles an angle  $\frac{\pi}{3}$ .*

*Proof.* The pair of circles which meet at  $O$  obviously make the angle  $\frac{\pi}{3}$  at each of the points  $P, Q, R$ . An inversion  $\Pi$  with respect to a circle having  $O$  as center must therefore change them into the sides of an equilateral triangle. The circle circumscribing this triangle makes the angle  $\frac{\pi}{3}$  with each of the sides. But since this circle is the transform of the circle  $PQR$  by  $\Pi$ , the conclusion of the theorem follows.

As a second application of the theory of inversion, in combination with projective methods, we may consider the theorem of Feuerbach on the nine point circle (cf. Ex. 2, § 73).

**THEOREM 15.** *The nine-point circle of a triangle touches the four inscribed circles.*

*Proof.* Let the given triangle be  $ABC$ , and let the mid-points of the pairs  $BC, CA, AB$  be  $A_1, B_1, C_1$  respectively. The nine-point circle is the circle containing  $A_1, B_1, C_1$ .

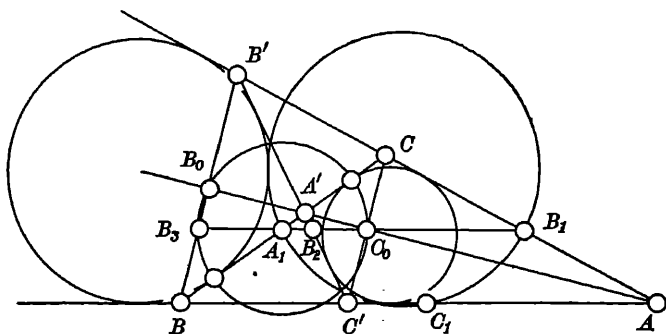


FIG. 72

Let  $K_1^2$  and  $K_2^2$  be the two inscribed circles whose centers are on one of the bisectors of  $\angle CAB$ . In case  $K_1^2$  and  $K_2^2$  touch the line  $BC$  at the same point, this is the mid-point  $A_1$  of the pair  $BC$ , the triangle  $ABC$  is isosceles, and the nine-point circle obviously touches  $K_1^2$  and  $K_2^2$  at  $A_1$ . In every other case there

is one line,  $l$ , besides  $AB$ ,  $BC$ ,  $CA$ , which touches both  $K_1^2$  and  $K_2^2$ . Let  $A'B'C'$  be the points in which  $l$  meets the sides  $BC$ ,  $CA$ ,  $AB$  respectively. Then  $AA'$ ,  $BB'$ ,  $CC'$  are the pairs of opposite vertices of a complete quadrilateral circumscribing both  $K_1^2$  and  $K_2^2$ , and the diagonal triangle of this quadrilateral is a self-polar triangle both for  $K_1^2$  and  $K_2^2$  (§ 44, Vol. I). Since the side  $AA'$  of this triangle is the line of centers of  $K_1^2$  and  $K_2^2$ , the other two sides,  $BB'$  and  $CC'$ , are parallel to each other and perpendicular to  $AA'$ . Let their points of intersection with  $AA'$  be  $B_0$  and  $C_0$  respectively. These two points are conjugate with respect to both circles, and hence must be the limiting points of the pencil of circles containing  $K_1^2$  and  $K_2^2$ . The radical axis of the pencil of circles is the perpendicular bisector of the pair  $B_0C_0$ , and hence (§ 40) passes through the mid-points of all the pairs  $BC$ ,  $B'C'$ ,  $BC'$ ,  $B'C$ ,  $B_0C_0$ . In particular the radical axis of  $K_1^2$  and  $K_2^2$  passes through  $A_1$ , the mid-point of  $BC$ . Hence there is a circle  $G^2$  with  $A_1$  as center and passing through  $B_0$  and  $C_0$ .

Let  $\Gamma$  be the inversion with respect to  $G^2$ . Since this circle passes through  $B_0$  and  $C_0$ , it is orthogonal both to  $K_1^2$  and  $K_2^2$  (Theorem 34, § 71), and hence  $\Gamma$  transforms each of these circles into itself. We shall now prove that  $\Gamma$  transforms  $l$  into the nine-point circle.

Let  $B_2$  be the point in which  $A_1B_1$  meets  $l$ . Since  $A_1B_1$  is parallel to  $AB$ , it is not parallel to  $l$ , and hence  $B_2$  is an ordinary point. Since  $A_1B_1$  contains the mid-point  $A_1$  of the pair  $CB$  and is parallel to  $BC'$ , it contains the mid-point  $C_0$  of the pair  $CC'$ . The involution which  $\Gamma$  effects on the line  $A_1B_1$  must have  $C_0$  as one of its double points and  $A_1$  as its center; hence the other double point must be the point  $B_2$  in which  $A_1B_1$  meets  $BB'$ , because  $A_1$  is the mid-point of the pair  $C_0B_2$ . Thus  $G^2$  passes through  $B_2$  as well as through  $C_0$ . But since

$$B_0A'C_0A \frac{B'}{\Lambda} B_2B_2C_0B_1,$$

$B_1$  and  $B_2$  are harmonically conjugate with respect to  $C_0$  and  $B_2$ . Hence  $\Gamma$  transforms  $B_2$  to  $B_1$ .

In like manner it can be shown that if  $C_2$  is the point in which  $A_1C_1$  meets  $l$ ,  $\Gamma$  transforms  $C_2$  to  $C_1$ . Since any line whatever is transformed by  $\Gamma$  to a circle through  $A_1$ , it follows that  $l$  is transformed to the circle through  $A_1$ ,  $B_1$ , and  $C_1$ , i.e. to the nine-point circle. By Theorem 11, Cor. 4, since  $l$  is tangent to  $K_1^2$  and  $K_2^2$ , the nine-point circle touches  $K_1^2$  and  $K_2^2$ . Since it has not been specified which of the bisectors of  $\angle CAB$  contains the centers of  $K_1^2$  and  $K_2^2$ , this argument shows that the nine-point circle touches all four inscribed circles.

### EXERCISES

1. Any three points can be carried by an inversion into three collinear points.
2. Two nonintersecting circles can be carried by an inversion into concentric circles.
3. Any direct circular transformation is a product of an inversion and an orthogonal line reflection.

4. A product of two inversions is an involution if and only if the circles are orthogonal.

5. Of four circles mutually perpendicular by pairs, three can be real.

6. The nine-point circle meets the circle through  $C_0$  having  $A_1$  as center in points of the line  $A'B'$ .

7. The nine-point circle of a triangle touches the sixteen circles inscribed to the triangle or to any of the triangles formed by pairs of its vertices with the orthocenter.

8. Let three circles  $C_1^2, C_2^2, C_3^2$  meet in a point  $O$ , and let  $P_1, P_2, P_3$  be the other points of intersection of the pairs  $C_2^2 C_3^2, C_3^2 C_1^2, C_1^2 C_2^2$  respectively. If  $Q_1$  be any point of  $C_1^2, Q_2$  the point of  $C_2^2$  collinear with and distinct from  $Q_1$  and  $P_3$ , and  $Q_3$  the point of  $C_3^2$  collinear with and distinct from  $Q_2$  and  $P_1$ , then  $Q_3, P_2$ , and  $Q_1$  are collinear.

9. *The problem of Apollonius.* Construct the circles touching three given circles. Cf. Pascal, Repertorium der Höheren Mathematik, II 1, Chap. II, on this and the following exercise.

10. *The problem of Malfatti.* Given a triangle, determine three circles each of which is tangent to the other two and also to two sides of the triangle.

**94. Inversions in the complex Euclidean plane.** Thus far we have dealt only with a real Euclidean plane. The definition of an inversion given in § 71, however, applies without change in the complex Euclidean plane; i.e. two points  $A_1, A_2$  are inverse with respect to a circle  $C^2$ , provided they are conjugate with respect to  $C^2$  and collinear with its center. The transformation thus defined is obviously one to one and reciprocal for all points of the complex projective plane except those on the sides of the triangle  $OI_1I_2$ , where  $O$  is the center of  $C^2$ , and  $I_1$  and  $I_2$  are the circular points at infinity. Any point of  $l_\infty$  is carried to  $O$  by the inversion, and  $O$  is carried to every point of  $l_\infty$ . The circular point  $I_1$  is transformed to every point of the line  $OI_2$ , and every point of the line  $OI_1$  is transformed to  $I_1$ . In like manner  $I_2$  is transformed to every point of the line  $OI_1$ , and every point of this line is carried to  $I_2$ .

**DEFINITION.** The sides of the triangle  $OI_1I_2$  are called the *singular lines* of the inversion with respect to  $C^2$ , and the points on these lines are called its *singular points*.

The principal properties of an inversion may be inferred from the following construction: If  $A_1$  is any point not on a side of the triangle  $OI_1I_2$ , let  $B_1$  and  $B_2$  be the points distinct from  $I_1$  and  $I_2$  (fig. 73) in which the lines  $A_1I_1$  and  $A_1I_2$ , respectively meet  $C^2$ . Let  $A_2$  be the point of intersection of  $I_1B_2$  and  $I_2B_1$ . The points  $A_1$  and  $A_2$

are mutually inverse because, by familiar theorems on conics, they are conjugate with regard to  $C^2$  and collinear with  $O$ .

From this construction it is evident in the first place that all points, except  $I_1$  of the line  $A_1I_1$ , are transformed into points of the line  $A_2I_2$ , and vice versa. Hence an inversion transforms the minimal lines through  $I_1$  into the minimal lines through  $I_2$ , and vice versa. Moreover, the correspondence between the two pencils of minimal lines is such that if  $B$  is a variable point of  $C^2$ , the line  $I_1B$  always corresponds to  $I_2B$ . In other words, the correspondence effected by an inversion between the two pencils of minimal lines is a projectivity generating the invariant circle  $C^2$ .

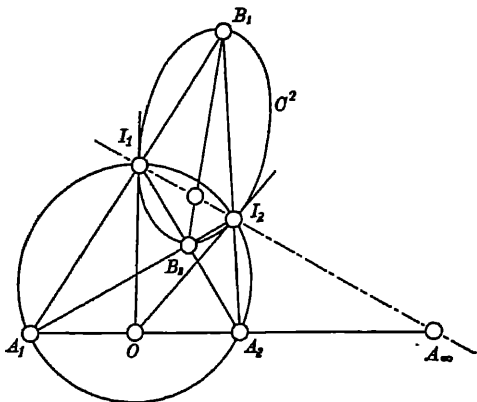


FIG. 78

The definitions of circular and of direct circular transformations, given in § 92, apply without change in the complex Euclidean plane. The result just obtained therefore implies that *any direct circular transformation transforms each pencil of minimal lines projectively into itself, and any nondirect circular transformation transforms each pencil of minimal lines projectively into the other.*

Now suppose that  $A_1$  is a variable point on any line  $l$  not containing  $I_1$  or  $I_2$ .

$$(4) \quad I_1[A_1] \stackrel{l}{\frown} I_2[A_1].$$

Since  $B_1$  and  $B_2$  are always on the conic  $C^2$ ,

$$(5) \quad I_1[A_1] \frown I_2[B_1],$$

and

$$(6) \quad I_2[A_1] \frown I_1[B_2].$$

Hence

$$(7) \quad I_2[B_1] \frown I_1[B_2].$$

But corresponding lines of these two pencils intersect in the variable point  $A_2$ , which is therefore always on a conic through  $I_1$  and  $I_2$ .

or on a line. In the projectivity (5) the line  $I_2O$  corresponds to  $l_\infty$ ; in (4)  $l_\infty$  corresponds to itself; and in (6)  $l_\infty$  corresponds to  $I_1O$ . Hence in (7) the line  $I_2O$  corresponds to  $I_1O$ , and so the circle or line generated by (7) passes through  $O$ .

This result may be stated in a form which takes account of the singular elements, as follows: *Any degenerate conic consisting of  $l_\infty$  and a nonminimal line is carried by an inversion with respect to  $C^2$  into a conic (degenerate or not) which passes through  $I_1$ ,  $I_2$ , and  $O$ .*

Next suppose  $A_1$  to be a variable point on any nondegenerate conic through  $I_1$  and  $I_2$ . In this case

$$(8) \quad I_1[A_1] \overline{\wedge} I_2[A_1],$$

and hence by the projectivities (5) and (6) we have

$$(9) \quad I_2[A_2] \overline{\wedge} I_1[A_2].$$

Hence  $A_2$  is again on a conic through  $I_1$  and  $I_2$ , which can degenerate only if  $l_\infty$  corresponds to itself under (9). The latter case implies, by (5) and (6), that  $I_1O$  and  $I_2O$  correspond under (8) or, in other words, that the locus of  $A_1$  passes through  $O$ . Hence *any nondegenerate conic  $K^2$  through  $I_1$  and  $I_2$  corresponds by the inversion with respect to  $C^2$  to a conic through  $I_1$  and  $I_2$ , which degenerates into a pair of lines, one of which is  $l_\infty$ , only in case  $K^2$  passes through  $O$ .*

This result, together with the other statement italicized above, amount to an extension of Cors. 1 and 3 of Theorem 11 to the complex Euclidean plane. From our present point of view we can also establish the following theorem, which did not come out of the reasoning in § 92.

**THEOREM 16.** *The correspondence between two circles which are homologous under an inversion is projective.*

*Proof.* If  $A_1$  is a variable point of one circle and  $A_2$  of the other, then, in the notation above,  $I_2B_1 = I_2A_2$ , and hence by (5)

$$I_1[A_1] \overline{\wedge} I_2[A_2],$$

which is a necessary and sufficient condition that the correspondence between the two circles be projective (cf. the corollary and definitions following Theorem 10, Chap. VIII, Vol. I).

The same reasoning also applies in case one or both of the conics which are the loci of  $A_1$  and  $A_2$  degenerate. We thus have

**COROLLARY.** *A projective correspondence is established by an inversion between any two homologous lines or between a line and its homologous set of points on a circle.*

The proof of Theorem 13 on the preservation of angles under a circular transformation applies without change in the complex Euclidean plane. This theorem can also be proved by the use of considerations with regard to the circular points. We shall give the argument for the case of orthogonal circles, leaving it as an exercise for the reader to derive the proof along these lines for the general case.

It has been proved in § 71 that the circles through two points  $A_1, A_2$  are orthogonal to the circles through two points  $B_1, B_2$  if and only if the pairs  $A_1A_2, B_1B_2$ , and  $I_1I_2$  are pairs of opposite vertices of a complete quadrilateral (cf. fig. 73). The sides  $I_1A_1, I_1A_2, I_2A_1, I_2A_2$  of such a quadrilateral are transformed by an inversion relative to any circle into four lines through  $I_1$  and  $I_2$ . Hence the points  $A_1, A_2, B_1, B_2$  are transformed into four points  $A'_1, A'_2, B'_1, B'_2$  such that  $I_1I_2, A'_1A'_2$ , and  $B'_1B'_2$  are pairs of opposite vertices of a complete quadrilateral. Hence the pencils of circles through  $A_1, A_2$  and  $B_1, B_2$  respectively are transformed into two pencils such that the circles of one pencil are orthogonal to those of the other.

With this result it is easy to prove that Theorems 8-11, 13, and their corollaries hold in the complex Euclidean plane, proper exceptions being made so as to exclude minimal lines and pairs of points on minimal lines. This is left as an exercise.

**95. Correspondence between the real Euclidean plane and a complex pencil of lines.** The correspondence between a complex one-dimensional form and the points of a real Euclidean plane, together with  $l_\infty$ , can be established in a particularly interesting way if the one-dimensional form be taken as the pencil of lines on one of the circular points of the line at infinity of the Euclidean plane.

Let  $l_\infty$  be the line at infinity, and  $I_1$  be one of the circular points. By Theorem 15, Chap. V, each line through  $I_1$  contains at least one real point. No line through  $I_1$ , except  $l_\infty$ , can contain more than one real point; for otherwise it would be a real line, and hence would meet  $l_\infty$  in a real point contrary to the fact that  $I_1$  is imaginary. Then each

line through  $I_1$ , except  $l_\infty$ , contains one and only one real point of the Euclidean plane. Let us denote by  $\Gamma'$  the correspondence by which  $l_\infty$  corresponds to itself and the other lines through  $I_1$  correspond each to the real point which it contains.

By § 94 a direct circular transformation transforms the pencil of lines on  $I_1$  projectively into itself. Hence every direct circular transformation corresponds under  $\Gamma'$  to a projectivity of the lines on  $I_1$ .

By Theorem 9 there is one and only one direct circular transformation carrying an ordered triad of distinct points to an ordered triad of distinct points; and by Assumption P there is one and only one projectivity carrying an ordered triad of lines of a pencil to any ordered triad of the pencil. Hence a given projectivity of the pencil of lines on  $I_1$  can correspond under  $\Gamma'$  to only one direct circular transformation. In other words,  $\Gamma'$  sets up a simple isomorphism between the projective group of a complex one-dimensional form and the group of direct circular transformations.

The correspondence between the points of a real line and the lines joining them to  $I_1$  is evidently projective. Since the cross ratio of four points of a real line is real, so is the cross ratio of the lines joining them to  $I_1$ . Hence any real line together with  $l_\infty$  corresponds under  $\Gamma'$  to a chain. Since any two chains of a one-dimensional form are projectively equivalent, and any circle of the Euclidean plane is equivalent under the inversion group to an ordinary line and  $l_\infty$ , it follows that under  $\Gamma'$  any chain corresponds to a circle and any circle to a chain.

The correspondence  $\Gamma'$  may be used to transfer the theory of involution from the complex pencil of lines to the Euclidean plane. Let  $AA', BB', CC'$  be pairs of opposite vertices of a complete quadrilateral of the Euclidean plane. The pairs of lines joining these point pairs to  $I_1$  are pairs of an involution. Hence

**THEOREM 17.** *The pairs of opposite vertices of a complete quadrilateral are pairs of an involution, i.e. they are pairs of homologous points in a direct circular transformation of period two.*

In other words, the pairs of opposite vertices of a complete quadrilateral constitute the image under  $\Gamma'$  (and hence under  $\Gamma$ ) of a quadrangular set. While the converse of this proposition is not true, the proposition can be generalized by inversion so as to give

a construction for the most general quadrangular set in which no four of the six points are on the same circle or line (cf. Ex. 1, below). We shall state the construction in terms of chains.\*

**THEOREM 18.** *Given two pairs of points  $AA'$  and  $BB'$  and a point  $C$  such that no four of the five points are on the same chain. The chains  $C(AB'C)$  and  $C(A'BC)$  either meet in a point  $D$  other than  $C$  or touch each other at  $C$ . In the latter case let  $D$  denote  $C$ . The chains  $C(DAB)$  and  $C(DA'B')$  meet in a point  $C'$  such that  $AA'$ ,  $BB'$ ,  $CC'$  are pairs of an involution.*

*Proof.* Consider the figure in the Euclidean plane (together with  $l_\infty$ ) corresponding under  $\Gamma'$  to the figure described in the theorem. If  $\Gamma'(D) \neq l_\infty$ ,  $\Gamma'(D)$  can be transformed to  $l_\infty$  by an inversion  $I$ . Under  $I\Gamma'$  the four chains  $C(AB'C)$ ,  $C(A'BC)$ ,  $C(DAB)$ , and  $C(DA'B')$  correspond to Euclidean lines (with  $l_\infty$ ), and hence  $AA'$ ,  $BB'$ ,  $CC'$  correspond to the vertices of a complete quadrilateral; so that the theorem reduces to Theorem 17. If  $\Gamma'(D) = l_\infty$ , the theorem reduces directly to Theorem 17.

**COROLLARY.** *Three pairs of points on a complex line  $AA'$ ,  $BB'$ ,  $CC'$ , such that the chains  $C(A'B'C')$ ,  $C(A'BC)$ ,  $C(AB'C)$ ,  $C(ABC')$  are distinct, are pairs of an involution if and only if the four chains have a point in common.*

### EXERCISES

1. Three pairs of points of the same chain  $AA'$ ,  $BB'$ ,  $CC'$  are in involution if for any point  $D$  not in the chain the chains  $C(DAA')$ ,  $C(DBB')$ ,  $C(DCC')$  are in the same pencil.
2. Derive Ex. 15, § 81, from the theory of involutions in a plane.
3. If  $AA'$ ,  $BB'$ ,  $CC'$  are pairs of opposite vertices of a complete quadrilateral, the three circles having  $AA'$ ,  $BB'$ ,  $CC'$  respectively as ends of their diameters belong to the same pencil, and the radical axis of this pencil passes through the center of the circle circumscribing the diagonal triangle of the quadrilateral.
4. Construct the double points of an involution in a Euclidean plane with ruler and compass.

\* This puts in evidence the fact that while the geometry of real one-dimensional forms depends essentially on constructions implying the existence of two-dimensional forms, the geometry of the complex projective line could be developed without supposing the existence of points outside the line.

**96. The real inversion plane.** In a real Euclidean plane an inversion has been seen to be a one-to-one and reciprocal transformation except in that it transforms  $l_{\infty}$  to the center of inversion, and the center to  $l_{\infty}$ . An inversion, therefore, is strictly one to one if we regard it as a transformation of the set of objects composed of the points of the real Euclidean plane together with  $l_{\infty}$  regarded as a single object.

**DEFINITION.** The set of points in a real Euclidean plane, together with the line at infinity regarded as a single object, is called a *real inversion plane*;  $l_{\infty}$  is called the *point at infinity* of the inversion plane. The set of points on a real circle, or on a real line  $l$  together with  $l_{\infty}$ , is called a *circle* of the inversion plane. An *inversion* is either an inversion in the sense of § 71 with respect to a real or imaginary circle or an orthogonal line reflection. Circular transformations, etc. are defined as in § 92. The set of theorems about the inversion plane, which remain valid when the figures to which they refer are subjected to every transformation of the inversion group, is called the *real inversion geometry*.

Although the point at infinity receives special mention in this definition, from the point of view of the inversion geometry it is not to be distinguished from any other point of the inversion plane. For any point of the inversion plane can be carried to any other point of it by an inversion. In a set of assumptions for the inversion geometry as a separate science, there would be no mention of a point at infinity; just as there is no mention of a line or a plane at infinity in our assumptions for projective geometry.

The inversion geometry has a relation to the Euclidean geometry which is entirely analogous to the relation of the projective geometry to the Euclidean; namely, the set of transformations of the inversion group which leaves one point of the inversion plane invariant is a parabolic metric group in the Euclidean plane obtained by omitting this point from the inversion plane.

A large class of theorems about circles can be stated with the utmost simplicity in terms of the geometry of inversion. For example, the propositions that three noncollinear ordinary points determine a circle and that two ordinary points determine a line combine into the single proposition:

**THEOREM 19.** *In the inversion plane any three distinct points are on one and but one circle.*

The theorem that there is one and only one circle touching a given circle  $C^2$  at a given point  $A$ , and passing through a given point  $B$  not on  $C^2$ , may be put in the following form, which also includes the proposition that through a given point not on a given line  $l$  there is one and but one line parallel to  $l$ .

**THEOREM 20.** *There is one and but one circle through a point  $A$  on a circle  $C^2$  and a point  $B$  not on  $C^2$ , and having no point except  $A$  in common with  $C^2$ .*

The theory of pencils of circles makes no special mention of the radical axis (§ 71), for the radical axis (with  $l_\infty$ ) is merely one circle of the pencil and is indistinguishable from the other circles. In like manner the center of a circle is not to be distinguished from any other point; for the center is merely the inverse of  $l_\infty$ , with respect to the circle, and the inversion group does not leave  $l_\infty$  invariant.

Thus the theory of pencils of circles in the inversion geometry involves no reference to the radical axis or to the line of centers. A pencil of circles may be defined as follows:

**DEFINITION.** A *pencil of circles* is either (a) the set of all circles through two distinct points, or (b) the set of all circles orthogonal to the circles of a pencil of Type (a), or (c) the set of all circles through a point of a given circle  $C^2$  and meeting  $C^2$  in no other point. A pencil of circles is said to be *hyperbolic*, *elliptic*, or *parabolic*, according as it is of Types (a), (b), or (c). Any point common to all circles of a pencil is called a *base point* of the pencil.

By comparison with the theorems in the preceding sections it is evident that the pencils of circles of these three types include all the pencils referred to in § 71 and also certain pencils of circles which are regarded as degenerate, from the Euclidean point of view. Thus, consider a pencil of lines through an ordinary point of a Euclidean plane. Each of these lines, with  $l_\infty$ , constitutes a degenerate circle, and the set of degenerate circles is a pencil according to the definition above. Again, a pencil of parallel lines in the Euclidean plane determines a set of circles  $[K^2]$  in the inversion plane which have in common only the one point  $l_\infty$ . By Theorem 11, Cor. 3, any inversion  $\Gamma$  with a center  $O$  transforms  $[K^2]$  into a set of circles  $[K_1^2]$  through  $O$  which have in common no other real points than  $O$ .

Since there is one and only one circle of the set  $[K_1^2]$  through every point of the Euclidean plane,  $[K_1^2]$  must be a pencil of circles of Type (c).

The fundamental theorems about circular transformations may be stated as follows:

**THEOREM 21.** *A circular transformation is a one-to-one transformation of the inversion plane which carries circles into circles. There is a unique direct circular transformation carrying three distinct points  $A, B, C$  to three distinct points  $A', B', C'$  respectively. A circular transformation leaving three points invariant is either an inversion relative to the circle through these three points or the identity.*

The theorems on orthogonal circles in § 71, together with the corresponding propositions on circles, lines, and orthogonal line reflections, become:

**THEOREM 22.** *Two circles are orthogonal if and only if one of them passes through two points which are inverse with respect to the other.*

**COROLLARY 1.** *Two circles are orthogonal if and only if they belong respectively to two pencils of circles such that the limiting points of one pencil are the common points of the circles of the other pencil.*

**COROLLARY 2.** *If  $A_1$  and  $A_2$  are inverse with respect to a circle  $C^2$ , all circles through  $A_1$  and orthogonal to  $C^2$  pass through  $A_2$ .*

The correspondence  $\Gamma$ , which was established in §§ 90, 91, between the Euclidean plane and the complex projective line, is one to one and reciprocal between the inversion plane and the complex line. Since circles and chains correspond under  $\Gamma$ , the inversion geometry is identical with the geometry of chains on a complex line. The direct circular transformations of the inversion plane correspond to the projectivities of the complex line.

It follows from § 90 that the inversion with respect to the chain  $C(Q_0, Q_1, Q_\infty)$  transforms every point  $z = x + iy$  into the conjugate imaginary point  $\bar{z} = x - iy$ . Hence an inversion with regard to any chain is a transformation projectively equivalent to that by which each point goes to its conjugate imaginary point (cf. § 78). For this reason we make the definition:

**DEFINITION.** Two points are said to be *conjugate* with respect to a chain if they are inverse with respect to it.

It is easily seen that any nondirect circular transformation is a product of a particular inversion and a direct circular transformation. Hence any nondirect transformation may be written in the form

$$z' = \frac{a\bar{z} + b}{c\bar{z} + d}.$$

We shall return to this subject in § 99.

### EXERCISES

1. Construct a set of assumptions for the inversion geometry as a separate science.\*

2. Work out the theorems analogous to those of §§ 71, 90-96 for the parabolic metric group in a modular space. Thus obtain a modular inversion geometry. The number of points in a finite inversion plane is  $p^2 + 1$  if the number of points on a circle is  $p + 1$ .

3. The double points of an involution leaving a chain invariant are inverse with respect to the chain.

**97. Order relations in the real inversion plane.** The more elementary theorems on order relations in the inversion plane follow readily from the corresponding theorems for the Euclidean and projective planes. Suppose we start with a projective plane  $\pi'$ . By leaving out a line  $l_\infty$  of  $\pi'$ , a Euclidean plane  $\pi$  is determined; and by regarding  $l_\infty$  as a point, an inversion plane  $\bar{\pi}$  is determined. Any line  $l$  of  $\pi'$  which is distinct from  $l_\infty$  determines a circle of the inversion plane  $\pi$ ; and we now define the order relations on this circle as identical with the projective order relations of  $l$ , the point  $l_\infty$  taking the place of the point in which  $l$  meets  $l_\infty$ . The order relations on any circle which does not contain  $l_\infty$  are determined by § 20.

Since the correspondence effected between any two circles by an inversion is projective (Theorem 16), it follows that the order relations among the points on any circle are unaltered by inversion. Hence order relations on circles are unaltered by circular transformations.

On a complex line the order relations in a chain are identical with the order relations on a real line as developed in §§ 18, 19, 21-24. The correspondence  $\Gamma$  (§§ 90, 91) is such that the order relations of corresponding sets of points on a chain  $C(Q_0 Q_1 Q_\infty)$  and the circle  $P_0 P_1 l_\infty$  are identical. Since order relations on circles are unaltered by

\* This question has been treated for the three-dimensional case by M. Pieri, *Giornale di Matematiche*, Vol. XLIX (1911), p. 49, and Vol. L, p. 106.

circular transformations, and order relations on chains are unaltered by projectivities, it follows that  $\Gamma$  is such that the order relations of corresponding sets of points on any chain and the corresponding circle are identical. Therefore the theory of order in the inversion plane applies also to the complex line.

Returning to the Euclidean plane  $\pi'$ , we know by § 28 that the points not on an ordinary line  $l$  fall into two classes such that any two points of the same class are joined by a segment not meeting  $l$ , whereas a line joining two points of different classes always meets  $l$ . By § 64 any circle containing two points of different classes meets  $l$  in two points. We thus have

**THEOREM 23. DEFINITION.** *The points of an inversion plane not on a circle  $C^2$  fall into two classes, called the two sides of  $C^2$ , such that two points on the same side of  $C^2$  are joined by a segment of a circle which does not contain any point of  $C^2$ , and such that any circle containing two points on different sides of  $C^2$  contains two points of  $C^2$ .*

Since order relations on circles are not altered by inversion, there follows:

**COROLLARY 1.** *If two points are on opposite sides of a circle  $C^2$ , the points to which they are transformed by an inversion  $\Pi$  are on opposite sides of  $\Pi(C)$ .*

On a complex line the points on one side of the chain  $C(Q_0, Q_1, Q_\infty)$  are evidently those whose coördinates relative to the scale  $Q_0, Q_1, Q_\infty$  are  $x + iy$ , where  $x$  is real and  $y$  real and positive, and those on the other side are those whose coördinates are  $x - iy$ . Hence, in general,

**COROLLARY 2.** *The points  $D$  and  $D'$  are on opposite sides of a circle through  $A, B, C$  if and only if  $y$  and  $y'$  are of opposite sign in the following two equations:*

$$\Re(AB, CD) = x + iy, \quad \Re(AB, CD') = x' + iy',$$

where  $x, y, x', y'$  are all real.

**DEFINITION.** A throw  $T(AB, CD)$  is said to be *neutral* if  $\Re(AB, CD)$  is real. Two throws  $T(AB, CD)$  and  $T(A'B', C'D')$  are *similarly* or *oppositely sensed* according as  $y$  and  $y'$  are of the same or of opposite signs in the equations

$$\Re(AB, CD) = x + iy \quad \text{and} \quad \Re(A'B', C'D') = x' + iy',$$

$x, y, x', y'$  being real.

From this definition it is obvious that a direct circular transformation transforms any non-neutral throw into a similarly sensed throw. It is also obvious that an inversion which reduces in the Euclidean plane  $\pi$  to an orthogonal line reflection changes non-neutral throws into oppositely sensed throws. Hence we have

**THEOREM 24.** *A direct circular transformation carries non-neutral throws into similarly sensed throws, and a nondirect circular transformation carries them into oppositely sensed throws.*

### EXERCISES

1. Two circles  $C^2, K^2$  intersecting in two distinct points separate the inversion plane into four classes of points such that two points of the same class are joined by a segment of a circle containing no points of  $C^2$  and  $K^2$ , whereas any circle containing points of different classes contains points of  $C^2$  and  $K^2$ .

2. Two points which are inverse with respect to a circle are on opposite sides of it.

3. What is the relation between the sense of throws as defined above and the sense of noncollinear point triads in a Euclidean plane as defined in § 30?

4. In a Euclidean plane if a triangle  $ABC$  is carried to a triangle  $A'B'C'$  by an inversion, the sense  $S(ABC)$  is the same as or different from  $S(A'B'C')$  according as the center of the inversion is or is not interior to the circle  $ABC$ .

5. In the notation of Ex. 7, § 92, if  $O$  is interior to a circle  $C^2$ , then  $O'$  is interior to  $\Pi(C^2)$ , and every point interior to  $C^2$  is transformed by  $\Pi$  to a point exterior to  $O'$ .

**98. Types of circular transformations.** By § 5 every projectivity on a complex line has one or two double points. On account of the correspondence  $\Gamma$  the same result holds for the direct circular transformations of the real inversion plane.

Let us consider first a transformation  $\Pi$  having but one double point. In the theory of projectivities such a transformation has been called parabolic; and it has been proved that there is one and but one parabolic projectivity leaving a point  $M$  invariant and carrying a point  $A_0$  to a point  $A_1$ . We have also seen that if  $A_{-1}$  is the point which goes to  $A_0$ ,  $\Re(MA_0, A_1A_{-1}) = -1$ . Hence  $A_{-1}, A_0, A_1$  are on the same chain through  $M$ . Since  $A_{-1}, A_0, M$  are transformed into  $A_0, A_1, M$  respectively, this chain is left invariant by  $\Pi$ .

In like manner any other point  $B_0$  not on the chain  $C(A_0A_1M)$  determines a chain which is left invariant by  $\Pi$ . These two chains cannot have another point than  $M$  in common, because this point

would have to be left invariant by  $\Pi$ . Thus  $\Pi$  leaves invariant a set of chains through  $M$  no two of which have a point in common, and such that there is one and only one chain of the set through any point except  $M$ .

If  $\Pi$  be regarded as a transformation of the inversion plane, this means that  $\Pi$  leaves invariant each circle of a pencil of circles of the parabolic type. In the Euclidean plane  $\epsilon$ , obtained by leaving  $M$  out of the inversion plane, this pencil of circles is a system of parallel lines and  $\Pi$  is a direct similarity transformation. Now let us regard  $\epsilon$  from the projective point of view. The transformation  $\Pi$  leaves all points of the line at infinity of  $\epsilon$  invariant, because it leaves each of the circular points invariant as well as the point at infinity of the system of parallel lines. Hence  $\Pi$  is a translation in the Euclidean plane  $\epsilon$ .

This result may be expressed in terms of the inversion plane as follows:

**THEOREM 25.** *Any direct circular transformation with only one invariant point transforms into itself every pencil of circles of the parabolic type having this point as base point. One and only one of these pencils is such that each circle of the pencil is invariant.*

Returning to the Euclidean plane we have

**THEOREM 26.** *Any direct similarity transformation which is not a translation or the identity leaves invariant one and only one ordinary point.*

*Proof.* Regard the Euclidean plane as obtained by omitting one point from an inversion plane. A direct similarity transformation effects a transformation of the direct inversion group and leaves this point invariant. In case it leaves only this point invariant, it has just been seen to be a translation in the Euclidean plane. If not, by the first paragraph of this section it has one and only one other invariant point unless it reduces to the identity.

A similarity transformation leaving an ordinary point  $O$  invariant must transform into itself the pencil of lines through this point and the pencil of circles having this point as center.

Two important special cases arise, namely, a rotation about  $O$  and a dilation with  $O$  as center. Moreover, since there is one and only one direct similarity transformation leaving  $O$  invariant and carrying

a point  $P$ , distinct from  $O$ , to a point  $P'$ , distinct from  $O$ , any non-parabolic direct similarity transformation is expressible as a product of a rotation and a dilation.

A rotation which is not a point reflection leaves all circles with  $O$  as center invariant, and changes every line through  $O$  into another line through  $O$ . A dilation which is not a point reflection leaves every line through  $O$  invariant, and changes every circle with  $O$  as center into another such circle. Hence a product of a dilation and a rotation, neither of which is of period two, leaves invariant no line through  $O$  and no circle with  $O$  as center. Since either a rotation or a dilation of period two is a point reflection, any direct circular transformation falls under one of the three cases just mentioned or else is a point reflection. Stated in terms of the inversion plane these results become (cf. fig. 56, p. 158):

**THEOREM 27.** *A direct circular transformation having two fixed points transforms into itself the pencil of circles through the fixed points and also the pencils of circles about these points. The transformation either leaves invariant every circle of one pencil and no circle of the other pencil, or it leaves invariant no circle of either pencil, or it leaves invariant every circle of both pencils and is of period two.*

**DEFINITION.** A direct circular transformation is said to be *parabolic* if it leaves invariant only one point; to be *hyperbolic* if it leaves invariant two points and all circles through these points; to be *elliptic* if it leaves invariant two points and all circles about these points; to be *loxodromic* if it leaves invariant two points and no circle through the invariant points or about them.

The theorems above are all valid for the complex line if circles be replaced by chains and direct circular transformations by projectivities. The definition is to be understood to apply in the same fashion. Since every nonidentical projectivity on the complex line has one or two double points, the discussion above gives the theorem:

**THEOREM 28.** *A direct circular transformation (or a projectivity on a complex line) is either parabolic, hyperbolic, elliptic, or loxodromic.*

**COROLLARY.** *An involution on a complex line is both hyperbolic and elliptic; and any projectivity which is both hyperbolic and elliptic is an involution.*

## EXERCISES

1. A projectivity whose double points  $x_1$  and  $x_2$  are distinct from each other and from the point  $P_\infty$  of a scale  $P_0, P_1, P_\infty$ , and whose characteristic cross ratio (§ 73, Vol. I) is  $k$ , may be written

$$(10) \quad \frac{x' - x_1}{x' - x_2} = k \frac{x - x_1}{x - x_2}.$$

If one of the double points is  $P_\infty$  and the other is  $x_1$ , the projectivity may be written

$$(11) \quad x' - x_1 = k(x - x_1).$$

The projectivity is hyperbolic if  $k$  is real, elliptic if  $k = e^{i\theta}$ , where  $\theta$  is real, and loxodromic if neither of these conditions is satisfied.

2. The parabolic projectivities with  $x_1$  as double point may be written in the form

$$(12) \quad \frac{1}{x' - x_1} = \frac{1}{x - x_1} + at,$$

or, in case the double point is  $P_\infty$ , in the form

$$x' = x + at.$$

In either case a subgroup is obtained by requiring  $t$  to be real. The locus of the points to which an arbitrary point is transformed by the transformation of this subgroup is a chain, and the set of such chains constitutes a parabolic pencil of chains.

3. The projectivities (10) and (11) for which

$$k = a^t,$$

where  $a$  is constant and  $t$  a real variable, form a group (a continuous group of one real parameter, in fact). The locus of the points to which a given point is carried by the transformations of this group or the group considered in Ex. 2 is called a *path curve*. In the nonparabolic cases, if  $a$  is real the path curves are chains through the double points. If  $a$  is complex and  $|a| = 1$ , they are chains about the double points. If  $a$  satisfies neither of these conditions, and the double points are  $P_0$  and  $P_\infty$ , the path curves are the loci of  $x = re^{i\theta}$  satisfying the condition

$$(13) \quad r = ae^{\beta\theta},$$

where  $\alpha$  and  $\beta$  are real constants; if the double points are not specialized, the path curves are projectively equivalent to the system (13). Diagrams illustrating the three types of path curves will be found in Klein and Fricke's *Elliptische Modulfunktionen*, Vol. I, Abschnitt II.

4. From the Euclidean point of view the  $r$  and  $\theta$  in Ex. 3 are *polar coordinates*, and the loci (13) are *logarithmic spirals* meeting the lines through the origin at the angle  $\tan^{-1}(1/\beta)$ . (A generalization of the notion of angle analogous to that in § 93 is here taken for granted.) The path curves of a

one-parameter group of Euclidean transformations may be a pencil of parallel lines or a pencil of concentric circles or a set of logarithmic spirals congruent to (13).

5. A projectivity having a finite period must be elliptic. A direct similarity transformation having a finite period must be a rotation.

6. A loxodromic projectivity is a product of an elliptic and a hyperbolic projectivity.

7. A projectivity leaving a chain invariant is either hyperbolic or elliptic.

**99. Chains and antiprojectivities.** The theory of chains on a complex line has been developed in the sections above by combining the general theory of one-dimensional projectivities with the Euclidean theory of circles. It is of course possible, and from some points of view desirable, to develop the theory of chains entirely independently of the Euclidean geometry. The reader is referred for the outlines of such a theory to an article by J. W. Young in the *Annals of Mathematics*, 2d Series, Vol. XI (1909), p. 33. Many of the properties of chains may be generalized to  $n$  dimensions, an  $n$ -dimensional chain or an  $n$ -chain being defined as a real  $n$ -dimensional space contained in an  $n$ -dimensional complex space in such a way that any three points on a line of the real space are on a line of the complex space. (This is the relation between  $S$  and  $S'$  in §§ 6 and 70.) A discussion of the theory of these generalized chains will be found in the articles by C. Segre and C. Juel referred to below, and also in those by J. W. Young, *Transactions of the American Mathematical Society*, Vol. XI (1910), p. 280, and H. H. MacGregor, *Annals of Mathematics*, 2d Series, Vol. XIV (1912), p. 1.

The transformations,

$$(14) \quad z' = \frac{az + b}{\bar{c}z + d}, \quad \begin{vmatrix} a & b \\ c & d \end{vmatrix} = 0,$$

of the complex line which were mentioned at the end of § 96 are analogous to the following class of transformations of the complex projective plane:

$$(15) \quad \begin{aligned} x'_0 &= a_{00}\bar{x}_0 + a_{01}\bar{x}_1 + a_{02}\bar{x}_2, \\ x'_1 &= a_{10}\bar{x}_0 + a_{11}\bar{x}_1 + a_{12}\bar{x}_2, \\ x'_2 &= a_{20}\bar{x}_0 + a_{21}\bar{x}_1 + a_{22}\bar{x}_2, \end{aligned} \quad \begin{vmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{vmatrix} \neq 0,$$

where  $\bar{x}_i$  denotes the complex number conjugate to  $x_i$ . These transformations are collineations, because they transform collinear points

to collinear points,\* but they are not projective collineations. If  $x'_0$ ,  $x'_1$ ,  $x'_2$  be replaced by  $u'_0$ ,  $u'_1$ ,  $u'_2$ , (15) gives the equation of a non-projective correlation. The analogous formulas in four homogeneous variables will define nonprojective collineations and correlations in space.

DEFINITION. A nonprojective collineation or correlation or a one-dimensional transformation of the type (14) is called an *anti-projectivity*.

The theory of antiprojectivities has been studied by C. Juel, *Acta Mathematica*, Vol. XIV (1890), p. 1, and more fully by C. Segre, *Torino Atti*, Vol. XXV (1890), pp. 276, 430 and Vol. XXVI, pp. 35, 592. Their rôle in projective geometry may be regarded as defined by the following theorem due to G. Darboux, *Mathematische Annalen*, Vol. XVII (1880), p. 55. In this paper Darboux also points out the connection of the geometrical result with the functional equation,

$$f(x+y) = f(x) + f(y).$$

THEOREM 29. *Any one-to-one reciprocal transformation of a real projective line which carries harmonic sets into harmonic sets is projective.†*

*Proof.* Let  $\Pi$  be any transformation satisfying the hypotheses of the theorem,  $A, B, C$  any three points of the line,  $\Pi(ABC) = A'B'C'$ , and  $\Pi'$  the projectivity such that  $\Pi'(A'B'C') = ABC$ . Then  $\Pi'\Pi(ABC) = ABC$ . If we can prove that  $\Pi'\Pi$  is the identity, it will follow that  $\Pi = \Pi'^{-1}$ , and hence that  $\Pi$  is a projectivity.

If  $\Pi'\Pi$  were not the identity, it would transform a point  $P$  to a point  $Q$  distinct from  $P$ , while it left invariant all points of the net of rationality  $R(ABC)$ . Let  $L_1, L_2, L_3$  be points of this net in the order

$$\{PL_1L_2QL_3\}.$$

By Theorem 8, Chap. V, there would exist two real points  $S, T$  which harmonically separate the pairs  $PL_1$  and  $L_2L_3$ . The transformation  $\Pi'\Pi$  must carry  $S$  and  $T$  into two points harmonically separating the pairs  $QL_1$  and  $L_2L_3$ . But since the latter two pairs separate each

\* Cf. § 28, Vol. I.

† Von Staudt, *Geometrie der Lage* (Nürnberg, 1847), § 9, defined a projectivity of a real line as a transformation having this property. We are using Cremona's definition of a projectivity as a resultant of perspectivities (cf. Vol. I, § 22).

other, by Theorem 8, Chap. V, there is no pair separating them both harmonically. Hence the assumption that  $\Pi'\Pi$  is not the identity leads to a contradiction.

**COROLLARY 1.** *Any collineation or correlation in a real projective space is projective.*

*Proof.* Since a collineation transforms collinear points into collinear points, it transforms nets of rationality into nets of rationality in such a way that the correspondence between any two homologous nets is projective (cf. §§ 33-35, Vol. I). Hence, according to the theorem above, the correspondence effected by the collineation between any two lines is projective. Hence the collineation is projective.

A like argument proves that a correlation is projective. The reasoning holds without change in a real projective space of  $n$  dimensions.

**COROLLARY 2.** *Any one-to-one reciprocal transformation of the real inversion plane which carries points into points and circles into circles is a transformation of the inversion group.*

*Proof.* Regard the inversion plane  $\pi$ , minus a point  $P_\infty$ , as a Euclidean plane  $\pi'$ ; let  $\Pi$  be any transformation satisfying the hypotheses of the corollary, let  $\Pi(P_\infty) = P'$ , and let  $\Pi'$  be an inversion carrying  $P'$  to  $P_\infty$ . Then  $\Pi'\Pi$  is a transformation satisfying the hypotheses of the corollary and leaving  $P_\infty$  invariant.

Since  $\Pi'\Pi$  carries circles through  $P_\infty$  into circles, it effects a collineation in  $\pi$ . By the first corollary this collineation is projective. Since it carries circles into circles, it is a similarity transformation. Hence  $\Pi'\Pi$  is a transformation, say  $\Pi''$ , of the inversion group in  $\pi'$ . Since  $\Pi = \Pi'^{-1}\Pi''$ ,  $\Pi$  is also in the inversion group.

Translated into the geometry of the complex projective line the last corollary states:

**COROLLARY 3.** *Any transformation which carries chains into chains is either a projectivity or an antiprojectivity.*

In the light of Corollary 2 it is clear that the whole theory of the inversion group can be developed from the definition of a circular transformation as one which carries points into points and circles into circles. This is the point of view adopted by Möbius in his *Theorie der Kreisverwandtschaft*, where, however, he used also the unnecessary assumption that the transformation is continuous.

## EXERCISES

1. Derive the formulas for antiprojectivities in a modular geometry. Cf. O. Veblen, Transactions of the American Mathematical Society, Vol. VIII (1907), p. 866.

2. Which if any of the following propositions are true? Any one-to-one and reciprocal transformation of a complex projective line which carries harmonic sets of points into harmonic sets of points is either projective or antiprojective. Any one-to-one and reciprocal transformation of a complex projective line which carries quadrangular sets of points into quadrangular sets is either projective or antiprojective. Any collineation or correlation of a complex projective space is either projective or antiprojective.

3. An antiprojectivity carries four collinear points having an imaginary cross ratio into four points whose cross ratio is the conjugate imaginary.

100. Tetracyclic coördinates. The general equation of a circle in a Euclidean plane  $\pi$  with respect to the coördinate system employed in Chap. IV is

$$(16) \quad \alpha_0(x^2 + y^2) + 2\alpha_1x + 2\alpha_2y + \alpha_3 = 0.$$

DEFINITION. A *degenerate circle* is either a pair of lines joining an ordinary point to the circular points at infinity or a pair of lines  $l_\infty$ , where  $l_\infty$  is the line at infinity.

Thus (16) represents a nondegenerate circle, provided that the following condition is not satisfied:

$$(17) \quad 0 = \begin{vmatrix} \alpha_3 & \alpha_1 & \alpha_2 \\ \alpha_1 & \alpha_0 & 0 \\ \alpha_2 & 0 & \alpha_0 \end{vmatrix} \equiv \alpha_0(\alpha_0\alpha_3 - \alpha_1^2 - \alpha_2^2).$$

The condition  $\alpha_0 = 0$  clearly means that (16) represents a degenerate circle consisting of  $l_\infty$  and an ordinary line, unless  $\alpha_1 = \alpha_2 = 0$  also, in which case (16) reduces to  $\alpha_3 = 0$ . The condition

$$(18) \quad \alpha_0\alpha_3 - \alpha_1^2 - \alpha_2^2 = 0$$

means in case  $\alpha_0 \neq 0$  that (16) represents a pair of ordinary lines through the circular points. In case  $\alpha_0, \alpha_1, \alpha_2, \alpha_3$  are real, these two lines must be conjugate imaginaries. In the rest of this section the  $\alpha$ 's are supposed real.

Let us now interpret the ordered set of numbers  $(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$  as homogeneous coördinates of a point in a projective space of three dimensions,  $S_3$ . For every point of  $S_3$ , except those satisfying (18), there is a unique circle or line pair  $l_\infty$ , where  $l$  is ordinary, and vice versa. Hence there is a one-to-one and reciprocal correspondence

between the points of  $S_s$  not on the locus (18) and the circles of the inversion plane  $\bar{\pi}$  obtained by adjoining  $l_\infty$  (regarded as a point) to  $\pi$ .

The points of  $S_s$  which are on the locus (18) and not on  $\alpha_0 = 0$  represent pairs of conjugate imaginary lines joining ordinary points of  $\pi$  to  $I_1$  and  $I_2$  respectively. There is one such pair of conjugate imaginary lines of  $\pi$  through each ordinary point of  $\pi$ . The points of  $S_s$  on the locus (18) and not on  $\alpha_0 = 0$  may therefore be regarded as corresponding to the points of  $\bar{\pi}$ , with the exception of  $l_\infty$ . The only point of  $S_s$  common to  $\alpha_0 = 0$  and (18) is  $(0, 0, 0, 1)$ , and this point may be taken to correspond to  $l_\infty$ . Thus the points of  $S_s$  not on (18) represent circles of the inversion plane  $\bar{\pi}$ , and the points of  $S_s$  on (18) represent the points of  $\bar{\pi}$ .

Stated without the intervention of  $S_s$ , this means that the ordered set of numbers  $(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$  taken homogeneously and subject to the relation (18) may be regarded as coördinates of the points of  $\bar{\pi}$ . When not subject to the relation (18) they may be regarded as coördinates of the circles and points in  $\bar{\pi}$ .

DEFINITION. The ordered sets of four numbers  $(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$  subject to (18) are called *tetracyclic coördinates* of the points in  $\bar{\pi}$ . The same term is applied to any set of coördinates  $(\beta_0, \beta_1, \beta_2, \beta_3)$  such that

$$\beta_i = \sum_{j=0}^3 a_{ij} \alpha_j, \quad |a_{ij}| \neq 0. \quad (i = 0, 1, 2, 3)$$

The circles (real or imaginary or degenerate) represented by  $(1, 0, 0, 0)$ ,  $(0, 1, 0, 0)$ ,  $(0, 0, 1, 0)$ ,  $(0, 0, 0, 1)$  are called the *base* or *fundamental* circles of the coördinate system.

A second particular choice of tetracyclic coördinates is given below.

The points of  $S_s$  on (18) evidently constitute the set of all real points on the lines of intersection of corresponding planes of the two projective pencils

$$(19) \quad \alpha_0 = \sigma(\alpha_1 + \sqrt{-1}\alpha_2) \quad \text{and} \quad \alpha_1 - \sqrt{-1}\alpha_2 = \sigma\alpha_3,$$

where the planes determined by the same value of  $\sigma$  are homologous. For (18) is obtained by eliminating  $\sigma$  between these two equations. The lines of intersection of homologous planes are all imaginary, but each contains one real point. This system of lines is, by § 103, Vol. I, a regulus, and the set of points on the lines, by § 104, Vol. I, a quadric surface. The locus (18) is therefore a real quadric surface all of whose rulers are imaginary (cf. also § 105, Vol. I).

The correspondence between the points of  $S_3$  and the circles and points of the inversion plane  $\bar{\pi}$  is such that a range of points corresponds to a pencil of circles. For the points of the line joining  $(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$  and  $(\beta_0, \beta_1, \beta_2, \beta_3)$  correspond to the circles given by the equation

$$(\lambda\alpha_0 + \mu\beta_0)(x^2 + y^2) + (\lambda\alpha_1 + \mu\beta_1)x + (\lambda\alpha_2 + \mu\beta_2)y + (\lambda\alpha_3 + \mu\beta_3) = 0,$$

which represents a pencil of circles, together with its limiting points in case the latter are real.

Any collineation  $\Gamma$  of  $S_3$  which carries the quadric (18) into itself must correspond to a transformation  $\bar{\Gamma}$  of  $\bar{\pi}$  which carries points into points, circles into circles, and pencils of circles into pencils of circles.  $\bar{\Gamma}$  therefore has the property that if a point  $P$  of  $\bar{\pi}$  is on a circle  $C^2$  of  $\bar{\pi}$ , then  $\bar{\Gamma}(P)$  is on  $\bar{\Gamma}(C^2)$ . By Theorem 29, Cor. 2,  $\bar{\Gamma}$  is a circular transformation. Conversely, any circular transformation of  $\bar{\pi}$  carries points to points, circles to circles, and pencils of circles to pencils of circles, and therefore corresponds to a collineation of  $S_3$  which carries the quadric into itself. By Theorem 29, Cor. 1, this collineation is projective. In other words,

**THEOREM 30.** *The real inversion geometry is equivalent to the projective geometry of the quadric (18).*

**COROLLARY.** *The projective geometry of the real quadric (18) is equivalent to the complex projective geometry of a one-dimensional form.*

A one-to-one correspondence between a complex line and the real quadric (18) may also be set up as follows: Let  $l$  be any complex line in the regulus conjugate to that composed of the lines (19). Each of these lines contains one real point,  $P$ , of the quadric (18) and one point,  $Q$ , of  $l$ . The correspondence required is that in which  $Q$  corresponds to  $P$ .

By properly choosing the constants which enter in the equation of a circle, we may set up the correspondence between the circles of the inversion plane and the points of an  $S_3$  in such a way that the equation of the quadric surface corresponding to the points of the inversion plane has a particularly simple form. The equation of a circle in  $\pi$  may be written

$$(20) \quad \xi_0(x^2 + y^2 + 1) + \xi_1(x^2 + y^2 - 1) + 2\xi_2x + 2\xi_3y = 0.$$

The points  $(\xi_0, \xi_1, \xi_2, \xi_3)$  which correspond to points of the inversion plane now satisfy the equation

$$(21) \quad \xi_0^2 = \xi_1^2 + \xi_2^2 + \xi_3^2,$$

and the circles corresponding to the four points  $(1, 0, 0, 0)$ ,  $(0, 1, 0, 0)$ ,  $(0, 0, 1, 0)$ , and  $(0, 0, 0, 1)$  are mutually orthogonal, one of them being imaginary. The coördinates  $(\xi_0, \xi_1, \xi_2, \xi_3)$  are connected with  $(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$  by the equations

$$\alpha_0 = \xi_0 + \xi_1, \quad \alpha_1 = \xi_2, \quad \alpha_2 = \xi_3, \quad \alpha_3 = \xi_0 - \xi_1,$$

which represent a collineation carrying the quadric (18) into the quadric (21).

If  $\xi_1/\xi_0$ ,  $\xi_2/\xi_0$ ,  $\xi_3/\xi_0$  are regarded as nonhomogeneous coördinates with respect to a properly chosen frame of reference in a Euclidean space of three dimensions (cf. Chap. VII), (21) is the equation of a sphere. Hence the real inversion geometry is equivalent to the projective geometry of a sphere.

The latter equivalence may be established very neatly, with the aid of theorems of Euclidean three-dimensional geometry, by the method of stereographic projection. This discussion would naturally come as an exercise in the next chapter. It is to be found in books on function theory. On the whole subject of inversion geometry from this point of view, compare Bôcher, *Reihenentwicklungen der Potentialtheorie* (Leipzig, 1894), Chap. II.

**DEFINITION.** A circle  $C_3^2$  is *linearly dependent* on two circles  $C_1^2$  and  $C_2^2$  if and only if it is in the pencil determined by  $C_1^2$  and  $C_2^2$ . A circle  $C^2$  is *linearly dependent* on  $n$  circles  $C_1^2, \dots, C_n^2$  if and only if it is a member of some finite set of circles  $C_{n+1}^2, \dots, C_{n+k}^2$  such that  $C_{n+i}^2$  is linearly dependent on two of  $C_1^2, \dots, C_{n+i-1}^2$  ( $i = 1, 2, \dots, k$ ). A set of  $n$  circles is *linearly independent* if no one of them is linearly dependent on the rest. The set of all circles linearly dependent on three linearly independent circles is called a *bundle*.

### EXERCISES

1. The tetracyclic coördinates of a point are proportional to the powers of the point with respect to four fixed circles. If the four circles are mutually orthogonal, the identity which they satisfy reduces to (21).

2. A homogeneous equation of the first degree in tetracyclic coördinates represents a circle.

3. What kind of coördinates are obtained by taking as the base (a) two orthogonal circles and the two points in which they meet? (b) four points?

4. Two points of  $S_8$  correspond to orthogonal circles if and only if they are conjugate with regard to the quadric (21).

5. What set of circles corresponds to the conics in which the quadric (21) is met by the planes of a self-polar tetrahedron?

6. The direct circular transformations of  $\pi$  correspond to collineations of  $S_3$  which leave each imaginary regulus of (21) invariant, while the others correspond to collineations interchanging the two reguli. The direct circular transformations of  $\pi$  correspond to direct collineations of  $S_3$  in the sense of § 31, Chap. II.

7. The circles of a bundle correspond to the points of a plane of  $S_3$ .

8. The circles common to two bundles constitute a pencil and hence correspond to a line of  $S_3$ . Determine the projectively distinct types of pencils of circles on this basis.

9. All circles are linearly dependent on four linearly independent circles.

10. For any bundle of circles there is a point  $O$  which has the same power,  $c^2$ , with respect to every circle of the bundle. The radical axes of all pairs of circles in the bundle pass through  $O$ . In case there is more than one point  $O$ , the radical axes of all pairs of circles of the bundle coincide.

11. A bundle of circles may consist of all circles through a point (the set of all lines in a Euclidean plane is a special case of this). In every other case there is a nondegenerate circle orthogonal to all circles of the bundle. This circle has the point  $O$  (Ex. 10) as center and consists of the points  $C$  such that  $\text{Dist}(OC) = c$ . It is real if and only if  $c$  is real. In case  $c$  is imaginary let  $C^2$  be the real circle consisting of points  $C'$  such that  $\text{Dist}(OC') = c$ ; any circle of the bundle meets  $C^2$  in the ends of a diameter.

**101. Involutoric collineations.** In view of the isomorphism between the real inversion group and the projective group of the real quadric (21), a further consideration of the group of a general quadric will be found apropos. In this connection we need to define certain particular types of involutoric collineations in any projective space. The theorems are all based on Assumptions A, E, P,  $H_0$ .

It is proved in § 29, Vol. I, that if  $\omega$  is any plane and  $O$  any point not on  $\omega$ , there exists a homology carrying any point  $P$  to a point  $P'$ , provided that  $O, P, P'$  are distinct and collinear and  $P$  and  $P'$  are not on  $\omega$ . It follows by the constructions given in that place that if one point  $P$  is transformed into its harmonic conjugate with regard to  $O$  and the point in which the line  $OP$  meets  $\omega$ , every point is transformed in this way. It is also obvious that a homology is of period two if and only if it is of this type. Hence we make the following definition:

**DEFINITION.** A homology of a three-space is said to be *harmonic* if and only if it is of period two. A harmonic homology is also called a *point-plane reflection* and is denoted by  $\{O\omega\}$  or  $\{\omega O\}$ , where  $O$  is the center and  $\omega$  the plane of fixed points.

DEFINITION. If  $l$  and  $l'$  are two nonintersecting lines of a projective space  $S_3$ , the transformation of  $S_3$  leaving each point of  $l$  and  $l'$  invariant, and carrying any other point  $P$  to the point  $P'$  such that the line  $PP'$  meets  $l$  and  $l'$  in two points harmonically conjugate with regard to  $P$  and  $P'$ , is called a *skew involution* or a *line reflection in  $l$  and  $l'$* . It is denoted by  $\{ll'\}$ , and  $l$  and  $l'$  are called its *axes* or *directrices*.

THEOREM 31. A line reflection  $\{ll'\}$  is a product of two point-plane reflections  $\{O\omega\} \cdot \{P\pi\}$ , where  $O$  and  $P$  are any two distinct points of  $l$ ,  $\omega$  is the plane on  $P$  and  $l'$ , and  $\pi$  is the plane on  $O$  and  $l'$ .

*Proof.* Consider any plane through  $l$ , and let  $L$  be the point in which it meets  $l'$ . In this plane  $\{O\omega\}$  and  $\{P\pi\}$  effect harmonic homologies whose centers are  $O$  and  $P$  respectively and whose axes are  $PL$  and  $OL$  respectively. The product is therefore the harmonic homology whose center is  $L$  and axis  $l$ . Hence the product  $\{O\omega\} \cdot \{P\pi\}$  satisfies the definition of a line reflection whose axes are  $l$  and  $l'$ .

COROLLARY. A line reflection is a projective collineation of period two, and any projective collineation of period two leaving invariant the points of two skew lines is a line reflection.

### EXERCISES

1. A projective collineation of period two in a plane is a harmonic homology.
2. A projective collineation of period two in a three-space is a point-plane reflection or a line reflection.

3. Let  $A, B, C, D$  be the vertices of a tetrahedron and  $\alpha, \beta, \gamma, \delta$  the respectively opposite faces. The transformations obtainable as products of the three harmonic homologies  $\{A\alpha\}, \{B\beta\}, \{C\gamma\}$  constitute a commutative group of order 8 consisting of four point-plane reflections, three line reflections, and the identity. If the transformations other than the identity be denoted by 0, 1, 2, 3, 4, 5, 6, the multiplication table may be indicated by the modular plane given by the table (1) on p. 3, Vol. I, the rule being that the product of any two transformations corresponding to points  $i, j$  of the modular plane is the one which corresponds to the third point on the line joining  $i$  and  $j$ .

4. Generalize the last exercise to  $n$  dimensions. The group of involutonic transformations carrying  $n + 1$  independent points into themselves is commutative, and such that its multiplication table may be represented by means of a finite projective space of  $n - 1$  dimensions in which there are three points on each line.

5. A projectivity  $\Gamma$  of a complex line such that for one point  $P$  which is not invariant,  $\Gamma^n(P) = P$  is such that  $\Gamma^n$  is the identity. If  $n$  is the least positive integer for which  $\Gamma^n = 1$ ,  $\Gamma$  is said to be *cyclic of degree  $n$* ; the

characteristic cross ratio of  $\Gamma$  is an  $n$ th root of unity; in case  $n = 3$ , this cross ratio is said to be *equianharmonic*, and a set of four points having this cross ratio is said to be *equianharmonic*. As a transformation of the involution group,  $\Gamma$  is equivalent to a rotation of period  $n$ .

6. A planar projective collineation of period  $n$  ( $n > 2$ ) is of Type I and the set of transforms of any point is on a conic, or else the collineation is a homology. In the first case, it is projectively equivalent to a rotation; in the second case, to a dilation (in general, imaginary). Consider the analogous problem in three dimensions. (For references on this and the last exercise cf. *Encyclopédie des Sc. Math.* III 8, § 14. The statements in the *Encyclopédie* on the planar case are not strictly correct, since they do not sufficiently take the existence of homologies of finite period into account.)

**102. The projective group of a quadric.** According to the definition in § 104, Vol. I, a quadric may be regarded as the set of points of intersection of the lines of two conjugate reguli. These two reguli may be improper in the sense of Chap. IX, Vol. I, and in the following theorems improper elements are supposed adjoined when needed for the constructions employed.

**DEFINITION.** If there are proper lines on a quadric, the quadric is said to be *ruled*, otherwise it is said to be *unruled*.

**THEOREM 32.** *A harmonic homology whose center is the pole of its plane of fixed points with regard to a quadric surface  $Q^2$  transforms  $Q^2$  into itself in such a way that the two lines of  $Q^2$  through any fixed point are interchanged.*

*Proof.* Let  $O$  be a point not on  $Q^2$ , and  $\omega$  its polar plane. Any line  $l$  of  $Q^2$  meets  $\omega$  in a unique point  $K$ . The plane  $Ol$  contains one other line  $l'$  of  $Q^2$ , and (cf. § 104, Vol. I)  $l'$  passes through  $K$ . Any line joining  $O$  to a point  $L$  of  $l$  other than  $K$  must meet  $l'$  in a point  $L'$  such that  $L$  and  $L'$  are harmonically conjugate (§ 104, Vol. I) with regard to  $O$  and the point in which  $OL$  meets  $\omega$ . Hence  $\{O\omega\}$  interchanges  $l$  and  $l'$ . From this result the theorem follows at once.

Comparing Theorems 31 and 32, we have

**COROLLARY.** *A line reflection  $\{ab\}$  such that  $a$  and  $b$  are polar with respect to a quadric  $Q^2$  transforms  $Q^2$  into itself in such a way that each regulus on  $Q^2$  is transformed into itself.*

**THEOREM 33.** *A projective collineation of a quadric which leaves three points of the quadric invariant, no two of the three points being on the same ruler, is either the identity or a harmonic homology whose center and plane of fixed points are polar with respect to the quadric.*

*Proof.* Denote the three points by  $A, B, C$ , the plane containing them by  $\omega$ , and the pole of  $\omega$  by  $O$ . Since no two of  $A, B, C$  are on a line of  $Q^2$ ,  $\omega$  contains no line of  $Q^2$  and hence is not on  $O$ . Since three points of the conic in which  $\omega$  meets the quadric are invariant, all such points are invariant, as is also  $O$ . Hence the given collineation is either the identity or a homology. In the latter case it must be a harmonic homology, since any two points of the quadric collinear with  $O$  are harmonically conjugate with respect to  $O$  and the point in which the line joining them meets  $\omega$ .

**THEOREM 34.** *There exists one and only one projective collineation transforming each line of a regulus into itself and effecting a given projectivity on one of these lines. Such a collineation is a product of two line reflections whose axes are lines of the conjugate regulus.*

*Proof.* Let  $R_1^2$  be a regulus and  $R_2^2$  the conjugate regulus. A projectivity on a line,  $l$ , of  $R_1^2$  is by § 78, Vol. I, a product of two involutions, say  $I$  and  $I'$ . Let  $\{m_1 m_2\}$  be a line reflection such that  $m_1$  and  $m_2$  are lines of  $R_2^2$  through the double points of  $I$ , and let  $\{m'_1 m'_2\}$  be a line reflection such that  $m'_1$  and  $m'_2$  are lines of  $R_2^2$  through the double points of  $I'$ . The product of  $\{m'_1 m'_2\}$  and  $\{m_1 m_2\}$  effects the given projectivity on  $l$  and transforms each line of  $R_1^2$  into itself.

Conversely, any projectivity  $\Gamma$  leaving all lines of  $R_1^2$  invariant effects a projectivity on  $l$  which is a product of two involutions  $I$  and  $I'$ . The line reflections  $\{m_1 m_2\}$  and  $\{m'_1 m'_2\}$  being defined as before,

$$\{m'_1 m'_2\} \cdot \{m_1 m_2\} \cdot \Gamma^{-1}$$

leaves all points of  $l$  invariant and hence leaves all lines of  $R_1^2$  as well as all lines of  $R_2^2$  invariant. Hence

$$\begin{aligned} \{m'_1 m'_2\} \cdot \{m_1 m_2\} \cdot \Gamma^{-1} &= 1, \\ \{m'_1 m'_2\} \cdot \{m_1 m_2\} &= \Gamma. \end{aligned}$$

**COROLLARY.** *The group of permutations of the lines of a regulus effected by the projective collineations transforming the regulus into itself is simply isomorphic with the projective group of a line.*

**DEFINITION.** A collineation of a quadric which carries each regulus on the quadric into itself is said to be *direct*.

**THEOREM 35.** *There is one and but one direct collineation of a quadric surface  $Q^2$  carrying an ordered triad of points of  $Q^2$ , no two of which are on a line of  $Q^2$ , to an ordered triad of points of  $Q^2$  no two of which are on a line of  $Q^2$ .*

*Proof.* Let  $ABC$  and  $PQR$  be the given ordered triads of points, let  $a, b, c, p, q, r$  be the lines of one regulus through the points  $A, B, C, P, Q, R$  respectively, and let  $a', b', c', p', q', r'$  respectively be the lines of the conjugate regulus through the same points. By the last theorem there is a projective collineation  $\Gamma$  carrying  $a, b, c$  to  $p, q, r$  respectively while leaving all lines of the conjugate regulus invariant, and also a projective collineation  $\Gamma'$  carrying  $a'b'c'$  to  $p'q'r'$  respectively while leaving all of the lines  $a, b, c, p, q, r$  invariant. The product of  $\Gamma$  and  $\Gamma'$  carries  $A, B, C$  to  $P, Q, R$  respectively. That there is only one direct collineation having this effect is a corollary of Theorem 33.

Let  $R_1^2$  be the regulus containing the lines  $a, b, c$ , and  $R_2^2$  the regulus containing  $a', b', c'$ . The two collineations  $\Gamma$  and  $\Gamma'$  which have been used in the proof above are commutative as transformations of  $R_1^2$  because  $\Gamma'$  leaves all lines of  $R_1^2$  invariant, and are commutative as transformations of  $R_2^2$  because  $\Gamma$  leaves all lines of  $R_2^2$  invariant. Hence

$$\Gamma\Gamma' = \Gamma'\Gamma.$$

$$\text{By Theorem 34, } \Gamma\Gamma' = \{lm\} \cdot \{rs\} \cdot \{l'm'\} \cdot \{r's'\},$$

where  $l, m, r, s$  are lines of  $R_1^2$ , and  $l', m', r', s'$  are lines of  $R_2^2$ . The collineations  $\{rs\}$  and  $\{l'm'\}$  are commutative for the same reason that  $\Gamma$  and  $\Gamma'$  are commutative. Hence

$$\Gamma\Gamma' = \{lm\} \cdot \{l'm'\} \cdot \{rs\} \cdot \{r's'\}.$$

The pairs  $lm$  and  $l'm'$  are two pairs of opposite edges of a tetrahedron the other two edges of which may be denoted by  $a$  and  $b$ . The product  $\{lm\} \cdot \{l'm'\}$  leaves each point of  $a$  and  $b$  invariant and is involutonic on each of the lines  $l, l', m, m'$ . Hence

$$\{lm\} \cdot \{l'm'\} = \{ab\}.$$

The lines  $a$  and  $b$  are polar with respect to  $R_1^2$  because one of them is the line joining the point  $ll'$  to the point  $mm'$ , and the other the line of intersection of the plane  $ll'$  with the plane  $mm'$  (cf. § 104, Vol. I).

$$\text{In like manner } \{pq\} \cdot \{p'q'\} = \{cd\},$$

where  $c$  and  $d$  are polar with respect to  $R_2^2$ . Hence we have

**THEOREM 36.** *Any direct projective collineation  $T$  of a quadric surface is expressible in the form*

$$T = \{ab\} \cdot \{cd\},$$

*where the line  $a$  is polar to the line  $b$ , and the line  $c$  is polar to the line  $d$ .*

Since any line reflection whose axes are polar with respect to a quadric is a product of two harmonic homologies whose centers are polar to their planes of fixed points (cf. Theorem 31), the last theorem implies

**COROLLARY 1.** *Any direct projective collineation of a quadric is a product of four harmonic homologies whose centers are polar to their respective planes of fixed points.*

**COROLLARY 2.** *Any nondirect projective collineation of a quadric is a product of an odd number of harmonic homologies whose centers are polar to their respective planes of fixed points.*

*Proof.* If a projective collineation  $\Gamma$  interchanges the two reguli, and  $\Lambda$  is a harmonic homology of the sort described in the statement of the corollary, then  $\Gamma\Lambda = \Delta$  is a projective collineation leaving each regulus invariant. By Cor. 1,  $\Delta$  is a product of an even number of harmonic homologies of the required sort, and hence  $\Gamma = \Delta\Lambda$  is a product of an odd number.

**103. Real quadrics.** The isomorphism between the real inversion group and the projective collineation group of the real quadric (or sphere) (21) may now be studied more in detail. Since a circular transformation leaving three given points of the inversion plane  $\bar{\pi}$  invariant is the identity or an inversion (Theorem 21), and since a collineation of  $S_3$  leaving three points of the quadric (21) invariant is the identity or a harmonic homology whose center is polar to its plane of fixed points, it follows that inversions in  $\bar{\pi}$  correspond to homologies of  $S_3$ . Hence the direct circular transformations of  $\bar{\pi}$  correspond to the direct collineations of  $S_3$  transforming (21) into itself.

An involution in  $\bar{\pi}$  is a product of two inversions whose invariant circles intersect and are perpendicular. To say that the invariant circles intersect and are perpendicular is to say that they intersect in such a way that one of the circles is transformed into itself by the inversion with respect to the other. Now suppose that  $\{O\omega\}$  and  $\{P\pi\}$  are the harmonic homologies corresponding to the two inversions. If the points of the quadric on the plane  $\omega$  are to be transformed among themselves by  $\{P\pi\}$ ,  $\omega$  must pass through  $P$ . In like manner  $\pi$  must pass through  $O$ . Hence

$$\{O\omega\} \cdot \{P\pi\} = \{\mathcal{U}\},$$

where  $l$  is the line  $OP$ ,  $l'$  the line  $\omega\pi$ , and the lines  $l$  and  $l'$  are polar with respect to the quadric. Hence the involutions in the group of direct circular transformations correspond to the line reflections whose axes are polar with respect to (21).

Thus the theorem that any direct circular transformation of  $\bar{\pi}$  is a product of two involutions is equivalent to Theorem 36 applied to the quadric (21). Since an involution in  $\bar{\pi}$  always has two double points, we have the additional information, not contained in § 102, that every line reflection transforming the quadric (21) into itself has two and only two fixed points on the quadric. The line joining these two points is obviously one of the axes of the line reflection. Hence the line reflection has two real axes one of which meets the quadric (21) and the other of which does not.

These remarks are enough to show how the real inversion geometry can be made effective in obtaining the theory of the real quadric (21). We shall now show that any real nonruled quadric is projectively equivalent to the quadric (21), from which it follows that the real inversion geometry is equivalent to the projective geometry of any real nonruled quadric.

A nonruled quadric is obviously nondegenerate. In the complex space any two nondegenerate quadrics are projectively equivalent, because any two reguli are projectively equivalent. Since (18) represents a quadric, it therefore follows that every nondegenerate quadric may be represented by an equation of the second degree.

Now let  $Q^2$  be any quadric whose polar system transforms real points into real planes, and let the frame of reference be chosen so that  $(1, 0, 0, 0)$ ,  $(0, 1, 0, 0)$ ,  $(0, 0, 1, 0)$ , and  $(0, 0, 0, 1)$  are vertices of a real self-polar tetrahedron. The plane section by the plane  $x_0 = 0$  must be a conic whose equation is of the form

$$a_1x_1^2 + a_2x_2^2 + a_3x_3^2 = 0,$$

and similar remarks can be made about the sections by the planes  $x_1 = 0$ ,  $x_2 = 0$ , and  $x_3 = 0$ . From this it follows that  $Q^2$  has the equation

$$(22) \quad a_0x_0^2 + a_1x_1^2 + a_2x_2^2 + a_3x_3^2 = 0,$$

where  $a_0, a_1, a_2, a_3$  are real. The projective collineation

$$(23) \quad x'_0 = \sqrt{|a_0|}x_0, \quad x'_1 = \sqrt{|a_1|}x_1, \quad x'_2 = \sqrt{|a_2|}x_2, \quad x'_3 = \sqrt{|a_3|}x_3$$

transforms  $Q^2$  into a quadric having one of the following equations

$$x_0'^2 \pm x_1'^2 \pm x_2'^2 \pm x_3'^2 = 0.$$

Any one of the eight quadrics thus represented is obviously equivalent projectively to one of the following three:

$$(24) \quad x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0,$$

$$(25) \quad -x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0,$$

$$(26) \quad -x_0^2 - x_1^2 + x_2^2 + x_3^2 = 0.$$

It is also obvious that (24) is imaginary, that (26) has real rulers, and that (25) is equivalent to (21).

### EXERCISES

1. Determine the types of collineations transforming into itself (1) a real unruled quadric, (2) a real ruled quadric, (3) an imaginary quadric having a real polar system.

2. Discuss the projective groups of the three types of quadrics enumerated in the last exercise.

**104. The complex inversion plane.** A projective plane may be obtained from a Euclidean plane (cf. Introduction, Vol. I) by adjoining ideal points and an ideal line in such a way as to make it possible to regard every collineation as a one-to-one reciprocal transformation of all points in the plane. In like manner the real inversion plane has been obtained from the real Euclidean plane by adjoining a single ideal point which serves as the correspondent of the center of each inversion. Similar considerations will now be adduced showing that an inversion in the complex plane may be rendered one to one and reciprocal by introducing *two intersecting ideal lines*.

In the complex projective plane an inversion has been seen (§ 94) to be a one-to-one reciprocal transformation of all points not on the sides of the singular triangle  $OI_1I_2$ , and to effect a projective transformation interchanging the pencil of lines on  $I_1$  with the pencil of lines on  $I_2$ . In this projectivity the line  $I_1I_2$  is homologous both with  $OI_1$  and with  $OI_2$ .

In the Euclidean plane obtained by omitting the line  $I_1I_2$  from the projective plane, it follows that the inversion is one to one and reciprocal except for points on the two minimal lines,  $p_0$  and  $m_0$ , through  $O$ . Moreover, it effects a projective correspondence between the set of minimal lines  $[p]$  parallel with and distinct from  $p_0$  and the set of minimal lines  $[m]$  parallel with and distinct from  $m_0$ .

The correspondence between any line  $p$  and the homologous line  $m$  is incomplete because there is no point on  $p$  corresponding to the intersection of  $m$  with  $p_0$  and no point on  $m$  corresponding to the intersection of  $p$  with  $m_0$ . This correspondence, however, may be made completely one to one and reciprocal by introducing an ideal point  $M_\infty$  on  $m$  as the correspondent of the point  $pm_0$  and an ideal point  $P_\infty$  on  $p$  as the correspondent of the point  $mp_0$ . In order to treat all the minimal lines symmetrically, ideal points  $P'_\infty$  and  $M'_\infty$  must be introduced on  $p_0$  and  $m_0$ , respectively, as mutually corresponding points. Also one other ideal point  $O_\infty$  is introduced as the correspondent of  $O$ .

According to these conventions the line  $p_0$  together with its ideal point  $P'_\infty$  is transformed into a set of points consisting of  $O_\infty$ ,  $M'_\infty$ , and all the points  $M_\infty$ . This set of points is therefore called an ideal line  $\bar{m}_\infty$ . In like manner the line  $m_0$  together with its ideal point  $M'_\infty$  is transformed into a set of points consisting of  $O_\infty$ ,  $P'_\infty$ , and all the points  $P_\infty$ ; and this set of points is called an ideal line  $\bar{p}_\infty$ . The Euclidean plane with the lines  $\bar{p}_\infty$  and  $\bar{m}_\infty$  adjoined is called an inversion plane. Or to state the definition formally and without reference to a particular inversion:

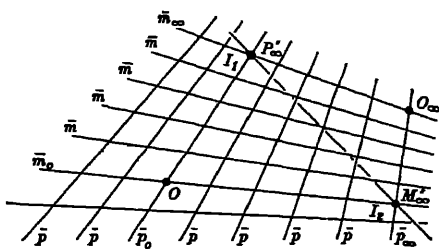


FIG. 74

DEFINITION. Given a complex Euclidean plane  $\pi$  and in it two pencils of minimal lines  $[p]$  and  $[m]$ . By a *complex inversion plane*  $\bar{\pi}$  is meant the set of all points of  $\pi$  (referred to as *ordinary points*) together with a set of elements called *ideal points* of which there is one, denoted by  $P_\infty$ , for each  $p$ , and one, denoted by  $M_\infty$ , for each  $m$ , distinct  $p$ 's and  $m$ 's determining distinct ideal points, and also one other ideal point which shall be denoted by  $O_\infty$ . By a *minimal line* of  $\bar{\pi}$  is meant (1) the set of points on a  $p$  together with the corresponding  $P_\infty$ , or (2) the set of points on an  $m$  together with the corresponding  $M_\infty$ , or (3) the set of all  $P_\infty$ 's together with  $O_\infty$ , or (4) the set of all  $M_\infty$ 's together with  $O_\infty$ . The minimal lines of Types (1) and (2) are called *ordinary*, and the lines (3) and (4) are called *ideal*.

A minimal line of Type (1) or (4) will be denoted by  $\bar{p}$ , of Type (2) or (3) by  $\bar{m}$ ; the minimal lines of Types (3) and (4) are denoted by  $\bar{m}_x$  and  $\bar{p}_x$  respectively.

This definition is evidently such that each point of  $\bar{\pi}$  is on a unique  $\bar{p}$  and on a unique  $\bar{m}$ .

DEFINITION. By an *inversion*  $\bar{I}$  of  $\bar{\pi}$  is meant a transformation defined as follows by an inversion  $I$  of  $\pi$ : If  $p_0$  and  $m_0$  are the singular lines of  $I$ ,  $\bar{I}$  interchanges  $\bar{p}_0$  with  $\bar{m}_x$ ,  $\bar{m}_0$  with  $\bar{p}_x$ , and each  $\bar{p}$  containing a  $p$  with the  $\bar{m}$  containing the  $m$  to which  $p$  is transformed by  $I$ . A point of  $\bar{\pi}$  which is the intersection of a  $\bar{p}$  and an  $\bar{m}$  is transformed to the point which is the intersection of  $\bar{I}(\bar{p})$  and  $\bar{I}(\bar{m})$ . The set of points of  $\bar{\pi}$  left invariant by an inversion is called a *nondegenerate circle* of  $\bar{\pi}$ . A pair of minimal lines, one a  $\bar{p}$  and the other an  $\bar{m}$ , is called a *degenerate circle* of  $\bar{\pi}$ .

By reference to § 94 it is evident that every circle of  $\pi$  is a subset of the points on a circle of  $\bar{\pi}$ .

The complex inversion plane is perhaps best understood by setting it in correspondence with a quadric surface, the lines of one regulus on the quadric being homologous with  $[\bar{p}]$  and those of the other with  $[\bar{m}]$ . This correspondence may be studied by means of tetracyclic coördinates as in § 100, but it can also be set up by means of a geometric construction as follows:

Regard the complex Euclidean plane  $\pi$  with which we started as immersed in a complex Euclidean space. Let  $Q^2$  be a quadric surface such that  $OI_1$  is a line of one ruling and  $OI_2$  of the other (fig. 74). Through  $I_1$  and  $I_2$  there are two other lines of the two rulings which intersect in a point  $O_\infty$ . Any point  $P$  of the Euclidean plane is joined to  $O_\infty$  by a line which meets the quadric  $Q^2$  in a unique point  $Q$  other than  $O_\infty$  and, conversely, any point of  $Q^2$  which is not on either of the lines  $O_\infty I_1$  or  $O_\infty I_2$  is joined to  $O_\infty$  by a line which meets the Euclidean plane in a point  $P$ . Thus there is a correspondence  $T$  between the Euclidean plane and the points of  $Q^2$  not on  $O_\infty I_1$  or  $O_\infty I_2$ . This correspondence is such that every minimal line in  $\pi$  of the pencil on  $I_1$  corresponds to a line of the quadric which is in the same ruling with  $OI_1$ , and every line of  $\pi$  of the pencil on  $I_2$  corresponds to a line of the quadric which is in the same ruling with  $OI_2$ . From this it is evident that if ideal elements are adjoined to  $\pi$  as explained above, the ideal points can be regarded as corresponding to

the points of the lines  $O_\pi I_1$  and  $O_\pi I_2$  so that there is a one-to-one reciprocal correspondence between  $\overline{\pi}$  and  $Q^2$ .

Now any nondegenerate circle of  $\pi$  is a conic through  $I_1$  and  $I_2$ . This is projected from  $O_\pi$  by a cone of lines having in common with  $Q^2$  the two lines  $O_\pi I_1$  and  $O_\pi I_2$ . It follows that the cone and  $Q^2$  have also a conic section in common. For let  $Q_1, Q_2, Q_3$  be three of the common points which are not on the lines  $O_\pi I_1$  and  $O_\pi I_2$ ; the plane  $Q_1 Q_2 Q_3$  meets the cone in a conic  $K_1^2$  and  $Q^2$  in a conic  $K_2^2$ . These two conics have also in common the points in which they meet the lines  $O_\pi I_1$  and  $O_\pi I_2$  (if these points coincide,  $K_1^2$  and  $K_2^2$  have a common tangent at this point), and hence  $K_1^2 = K_2^2$ . The conic  $K_1^2$  is nondegenerate, because a nondegenerate cone through  $O_\pi$  can have no other line than  $O_\pi I_1$  and  $O_\pi I_2$  in common with  $Q^2$ . Hence every nondegenerate circle of  $\pi$  corresponds under T to a section of  $Q^2$  by a nontangent plane.

Conversely, if  $K^2$  is any nondegenerate conic section which is a plane section of  $Q^2$ , it is projected from  $O_\pi$  by a cone two of whose lines are  $O_\pi I_1$  and  $O_\pi I_2$ . Hence  $K^2$  corresponds under T to a nondegenerate circle of  $\pi$ .

An inversion in  $\pi$  with respect to a circle  $C^2$  transforms every minimal line of the pencil  $[p]$  into that one of  $[m]$  which meets it on  $C^2$ . Let  $K^2$  be the conic section on  $Q^2$  corresponding under T to  $C^2$ . The inversion corresponds under T to a transformation of  $Q^2$  by which every line of one regulus is transformed into the line of the other regulus which meets it in a point of  $K^2$ . This is the transformation (Theorem 32) effected by a harmonic homology whose plane of fixed points contains  $K^2$  and whose center is the polar to this plane with respect to  $Q^2$ . Hence every inversion in  $\pi$  corresponds under T to a collineation of  $Q^2$  effected by a harmonic homology whose center and plane of fixed points are polar with regard to  $Q^2$ . Conversely, every such collineation of  $Q^2$  evidently corresponds under T to an inversion in  $\pi$ . Hence (Theorem 36, Cors. 1 and 2) the inversion group in  $\pi$  is isomorphic under T with the group of projective collineations of  $Q^2$ , and the direct circular transformations of  $\pi$  correspond to the projective collineations of  $Q^2$  which carry each regulus into itself.

### EXERCISE

Develop the theory of the modular inversion plane, using improper elements in the sense of Chap. IX, Vol. I.

105. **Function plane, inversion plane, and projective plane.** In the theory of functions of two complex variables

$$F(xy)$$

the two variables  $x$  and  $y$  are thought of as completely independent of each other. The domain of each is the set of all complex numbers, including  $\infty$ . This domain is therefore equivalent to the complex line or to the real inversion plane. Thus the domain of  $x$  may be taken to be a real unruled quadric (in particular, a sphere) and the domain of  $y$  another real unruled quadric. Or the pair of values  $(x, y)$  may be regarded as an *ordered pair of points* on the same real unruled quadric.

Now consider a regulus in the complex projective space and, adopting the notation of the last section (fig. 74), let a scale be established on the lines  $\bar{p}_0$  and  $\bar{m}_0$  so that  $O$  is the zero in each scale. Let  $x$  be the coördinate of any point on  $\bar{p}_0$  and  $y$  of any point on  $\bar{m}_0$ . Then a pair of values  $(x, y)$  determines a unique point on the quadric, i.e. the point of intersection of the line  $\bar{m}$  through the point with  $x$  as its coördinate, and the line  $\bar{p}$  through the point with  $y$  as its coördinate. Conversely, the same construction determines a pair of numbers  $(x, y)$  for each point of the quadric.

**DEFINITION.** The set of all ordered pairs  $(x, y)$  where  $x$  and  $y$  are complex numbers, including  $\infty$ , is called a *complex function plane*, or the *plane of the theory of functions* of complex variables, or the *complex plane of analysis*. The ordered pairs  $(x, y)$  are called *points*. Any point for which  $x = \infty$  or  $y = \infty$  is said to be *ideal* or *at infinity*, and all other points are called *ordinary*.

The points at infinity of the function plane can be represented conveniently by replacing  $x$  by a pair of homogeneous coördinates  $x_0, x_1$  such that  $x_1/x_0 = x$ , and  $y$  by a pair  $(y_0, y_1)$  such that  $y_1/y_0 = y$ . Thus the points of the function plane are represented by

$$(x_0, x_1; y_0, y_1),$$

and the ideal points are those satisfying the condition

$$x_0 y_0 = 0.$$

The set of ordinary points of the function plane obviously forms a Euclidean plane in which a line is the locus of an equation of the form

$$ax + by + c = 0.$$

This is equivalent in homogeneous coördinates to

$$(27) \quad ax_1y_0 + by_1x_0 + cx_0y_0 = 0,$$

an equation which is linear both in the pair of variables  $x_0, x_1$  and in the pair  $y_0, y_1$ . The most general equation which is linear in both pairs is

$$(28) \quad \alpha x_0y_0 + \beta x_0y_1 + \gamma x_1y_0 + \delta x_1y_1 = 0.$$

This reduces to (27) if the condition be imposed that the locus shall contain the point  $(\infty, \infty)$  which in homogeneous coördinates is  $(0, 1; 0, 1)$ .

DEFINITION. The set of points of the function plane satisfying (28) is called a *circle* (or a *bilinear curve*), and any circle of the form (27) is called a *line*.

The group of transformations which is indicated as most important by problems of elementary function theory has the equations

$$(29) \quad \begin{aligned} x' &= \frac{p_1x + q_1}{r_1x + s_1}, & \begin{vmatrix} p_1 & q_1 \\ r_1 & s_1 \end{vmatrix} &\neq 0, \\ y' &= \frac{p_2y + q_2}{r_2y + s_2}, & \begin{vmatrix} p_2 & q_2 \\ r_2 & s_2 \end{vmatrix} &\neq 0, \end{aligned}$$

or, in homogeneous coördinates,

$$(30) \quad \begin{aligned} x'_1 &= p_1x_1 + q_1x_0, & y'_1 &= p_2y_1 + q_2y_0, \\ x'_0 &= r_1x_1 + s_1x_0, & y'_0 &= r_2y_1 + s_2y_0. \end{aligned}$$

This group of transformations clearly transforms circles into circles. The subgroup obtained by imposing the conditions,

$$r_1 = 0, \quad r_2 = 0,$$

transforms lines into lines because it leaves  $(\infty, \infty)$  invariant.

Returning to the interpretation of the coördinates  $x$  and  $y$  on a quadric, it is clear (cf. § 102) that every transformation (29) represents a direct collineation of the quadric, the formula in  $x$  determining the transformation of one regulus and the formula in  $y$  the transformation of the conjugate regulus. Hence the fundamental group of the function plane is isomorphic with the group of direct projective collineations of a quadric surface.

The parameters  $x$  and  $y$  which determine the points of a regulus may be connected with the three-dimensional coördinates  $(\xi_0, \xi_1, \xi_2, \xi_3)$  by means of the following equations:

$$(31) \quad \begin{aligned} \xi_0 &= x_1 y_1 + x_0 y_0, \\ \xi_1 &= x_1 y_1 - x_0 y_0, \\ \xi_2 &= x_1 y_0 + x_0 y_1, \\ \xi_3 &= i(x_1 y_0 - x_0 y_1), \end{aligned}$$

where  $i^2 = -1$ . For the set of all points  $(\xi_0, \xi_1, \xi_2, \xi_3)$  given by these equations are the points on the quadric,

$$(21) \quad \xi_0^2 = \xi_1^2 + \xi_2^2 + \xi_3^2.$$

Any plane section of this quadric is given by a linear equation in  $\xi_0, \xi_1, \xi_2, \xi_3$ , which by (31) reduces to a relation of the form (28) among the parameters  $x_0, x_1; y_0, y_1$ . Hence the circles of the function plane correspond to the plane sections of the quadric (21). In view of the relation already established between the groups it follows that the geometry of a quadric in a complex projective space is identical with that of a complex function plane. In view of § 104 both these geometries are identical with the complex inversion geometry.\*

The complex projective plane may be contrasted with the complex inversion plane or function plane in an interesting manner as follows: The homogeneous coördinates  $(\alpha_0, \alpha_1, \alpha_2)$  may be regarded as the coefficients of a quadratic equation

$$(32) \quad \alpha_0 z_0^2 + \alpha_1 z_0 z_1 + \alpha_2 z_1^2 = 0.$$

Every such equation determines two and only two values of  $z_1/z_0$ , which may coincide or become infinite (if  $\alpha_2 = 0$ ); and, moreover, two distinct points of the projective plane determine distinct quadratic equations and hence distinct pairs of values of  $z_1/z_0$ .

\*If one were to confine attention to real values, the definition of the plane of analysis given above would determine a set of elements abstractly equivalent to a real ruled quadric. This is distinct from the real inversion plane, because the latter is equivalent to a real nonruled quadric. For the purposes of the theory of functions of a real variable, however, it is usually desirable to distinguish between  $+\infty$  and  $-\infty$ . If this be done, the function plane is easily seen to be a figure analogous to a rectangle in a Euclidean plane. The group of transformations of such a function plane does not seem to be of great interest from the projective point of view.

The numbers  $(z_0, z_1)$  may be taken as homogeneous coördinates on a projective line. Thus there is a one-to-one and reciprocal correspondence between the points of a complex projective plane and the pairs of points on a complex projective line. It is important to notice that the pairs of points on the line *are not ordered pairs*, because a pair of values of  $z_1/z_0$  taken in either order would be the pair of roots of the same quadratic.

Now representing the points of a complex line on a real unruled quadric (e.g. a sphere), we have that the projective plane is in one-to-one reciprocal correspondence with the *unordered* pairs of points of the quadric. On the other hand, we have already seen that the complex projective plane is in one-to-one reciprocal correspondence with the *ordered* pairs of points of the quadric. In either case the points of a pair may coincide.

For further discussion of the subject of this section see "The Infinite Regions of Various Geometries" by M. Bôcher, Bulletin of the American Mathematical Society, Vol. XX (1914), p. 185.

**106. Projectivities of one-dimensional forms in general.** The theorems of the last four sections have established and made use of the fact that the permutations effected among the lines of a regulus by projective collineations form a group isomorphic with the projective group of a line. Now a regulus is a one-dimensional form of the second degree,\* and the notion of one-dimensional projective transformation has been extended to all the other one-dimensional forms (Chap. VIII, Vol. I, particularly § 76). It is therefore to be expected that an analogous extension can be made to the regulus. This we shall now make, but instead of dealing with the regulus in particular, we shall restate the old definition in a form which includes the cases where the regulus is in question.

**DEFINITION.** A correspondence between any two one-dimensional forms whose elements are of different kinds and not such that all elements of one form are on every element of the other form is said to be *perspective* if it is one-to-one and reciprocal and such that each element of either form is on the corresponding element of the other form.

\*The one-dimensional forms of the first and second degrees in three-space are the pencil of points, the flat pencil of lines, the pencil of planes, the point conic, the line conic, the cone of lines, the cone of planes, and the regulus.

This covers the notion of perspectivity as defined in Vol. I between a pencil of points and a pencil of lines or between a pencil of lines and a point conic, etc. It also defines perspectivities between (1) the lines of a regulus and the points on a line of the conjugate regulus, (2) the lines of a regulus and the planes on a line of the conjugate regulus, (3) the lines of a regulus and the points of a conic which is a plane section of the regulus, (4) the lines of a regulus and the planes of a cone tangent to the regulus.

DEFINITION. A correspondence between two one-dimensional forms or among the elements of a single one-dimensional form is *projective* if and only if it is the resultant of a sequence of perspectivities.

This definition comprehends that made in § 22, Vol. I, for forms of the first degree, and extended in § 76, Vol. I, so as to include those of the second degree equivalent under duality to a point conic. In order to justify the new definition, it is necessary to prove that it does not lead to any modification of the relation of perspectivity between one-dimensional forms of the first degree. In other words, we must prove that *any correspondence between two one-dimensional forms of the first degree is projective according to the new definition only if it is projective according to the definition of § 22, Vol. I.*

To prove this theorem it is sufficient to show that a sequence of perspectivities beginning and ending with forms of the first degree and involving forms of the second degree can be replaced by one involving only forms of the first degree. This follows directly from the fact that each one-dimensional form of the second degree is generated by projective one-dimensional forms of the first degree. For example, if a pencil of points  $[P]$  is perspective with a regulus  $[L]$  and the regulus with a point conic and the point conic with something else, it follows by the theorems of § 103, Vol. I, that  $[P]$  is perspective with the pencil of planes  $[ml]$ , where  $m$  is a line of the conjugate regulus and  $[ml]$  is perspective with the point conic. Thus the regulus  $[L]$  in this sequence of perspectivities is replaced by the pencil of planes  $[ml]$ . In similar fashion it can be shown by a consideration of the finite number of possible cases that however a form of the second degree may intervene in a sequence of perspectivities, it can be replaced by a form or forms of the first degree. The enumeration of the possible cases is left to the reader, the argument required in each case being obvious.

From this theorem it follows that *the group of projective correspondences of any one-dimensional form with itself is isomorphic with the projective group of a line*. For let  $\Gamma$  be any projectivity of a one-dimensional form  $F^2$  of the second degree (e.g. a regulus), and let  $\Pi$  represent a perspectivity between  $F^2$  and a one-dimensional form  $F^1$  of the first degree (e.g. a line of the conjugate regulus). Then  $\Pi\Gamma\Pi^{-1}$  is a projectivity of  $F^1$ . In like manner, if  $\Gamma'$  is a projectivity of  $F^1$ ,  $\Pi^{-1}\Gamma'\Pi$  is a projectivity of  $F^2$ . Hence  $\Pi$  establishes an isomorphism between the two groups.

**\*107. Projectivities of a quadric.** An involution on a regulus is the transformation of the lines of the regulus effected by a line reflection whose axes are the double lines of the involution. Since any projectivity of a regulus is a product of two involutions, it may be regarded as effected by a three-dimensional projective collineation which transforms the regulus into itself. Conversely, any direct projective collineation transforming a quadric into itself is a product of two line reflections (Theorem 36) each of which effects an involution on each of the reguli on the quadric.

This relation between the theory of one-dimensional projectivities and the projective group of a quadric may be used to obtain properties of the quadric analogous to the properties of conic sections studied in Chap. VIII, Vol. I. The discussion is based on Assumptions A, E, P,  $H_0$ , improper points being adjoined to the space whenever this is required for quadratic constructions.

In Chap. VIII, Vol. I, we have seen that any projectivity on a conic determines a unique point, the center of the projectivity, and that the axes of any two involutions into which the projectivity may be resolved pass through its center. If, now, a projectivity  $\Gamma$  be given on a regulus, any plane  $\pi$  meets the regulus in a conic  $C^2$  on which is determined a projectivity  $\Gamma'$  having a point  $P$  as center. This determines a correspondence between the planes  $\pi$  and points  $P$  of space which is a null system (§ 108, Vol. I), and hence the axes of the involutions into which the projectivity  $\Gamma'$  can be resolved form a linear complex. The formal proof of this statement follows.

**THEOREM 37.** *For any nonidentical projectivity of a regulus there exists a linear complex of lines  $[l]$  having the property that if  $l_1$  is any line of the complex not tangent to the regulus, there are three lines  $l_2$ ,*

$l_3, l_4$  such that  $l_2$  is polar to  $l_1$  and  $l_3$  to  $l_4$  with respect to the regulus, and such that the collineation

$$\{l_1 l_2\} \cdot \{l_3 l_4\}$$

effects the given projectivity on the regulus. Moreover, every line  $l_i$  having this property belongs to the complex, and so do  $l_2, l_3, l_4$ .

*Proof.* Let  $R^2$  be a regulus and  $\Gamma$  a projectivity of  $R^2$ . If  $l_1 l_2$  and  $l_3 l_4$  are pairs of polar lines such that  $\{l_1 l_2\} \cdot \{l_3 l_4\}$  effects the given projectivity on  $R^2$ , let  $\pi$  be any plane containing  $l_1$  and not tangent to  $R^2$ . The projectivity  $\Gamma$  on  $R^2$  is perspective with a projectivity  $\Gamma'$  on the conic  $C^2$  in which  $\pi$  meets  $R^2$ . Moreover,  $\{l_1 l_2\}$  and  $\{l_3 l_4\}$  effect involutions on  $R^2$  which are perspective with involutions  $I'$  and  $I''$  on  $C^2$ . Thus on  $C^2$

$$\Gamma' = I' I''.$$

But (cf. § 77, Vol. I)  $l_1$  is the axis of  $I'$  and hence passes through the center of  $\Gamma'$ . A similar argument shows that  $l_i$  ( $i = 2, 3, 4$ ) passes through the center of the projectivity perspective with  $\Gamma$  on the conic in which  $R^2$  is met by any plane containing  $l_i$  and not tangent to  $R^2$ .

Hence all lines  $l_1, l_2, l_3, l_4$  defined as above are contained in the set  $[l]$  of all lines  $l$  such that if  $\pi$  is any plane on  $l$  and not tangent to  $R^2$ ,  $l$  is also on a point  $P$  defined as follows: Let  $C^2$  be the conic in which  $\pi$  meets  $R^2$  and  $\Gamma'$  the projectivity on  $C^2$  perspective with the projectivity  $\Gamma$  on  $R^2$ ; then  $P$  is the center of  $\Gamma'$ .

The set  $[l]$  obviously contains all lines tangent to  $R^2$  at points of the double lines (if existent) of  $\Gamma$ . If  $l_1$  is any other line of  $[l]$  let  $\pi$  be a plane on  $l_1$  and not tangent to  $R^2$ , let  $C^2$  be the conic in which  $\pi$  meets  $R^2$ , and let  $\Gamma'$  be the projectivity on  $C^2$  perspective with  $\Gamma$ . By § 79, Vol. I, and the definition of  $[l]$ ,  $\Gamma'$  is a product of two involutions having  $l_1$  and another line,  $l_3$ , as axes. Let  $l_2$  and  $l_4$  be the polars of  $l_1$  and  $l_3$  respectively. Then  $\{l_1 l_2\} \cdot \{l_3 l_4\}$  effects the perspectivity  $\Gamma'$  on  $C$  and hence effects  $\Gamma$  on  $R^2$ . By the first paragraph of the proof  $l_2, l_3, l_4$  are all lines of  $[l]$ . Hence all lines of  $[l]$  have the property enunciated in the theorem. It remains to prove that  $[l]$  is a linear complex.

By definition, if  $\pi$  is a plane not tangent to  $R^2$  the lines of  $[l]$  in  $\pi$  form a flat pencil. If  $\pi$  is tangent to  $R^2$  let  $p$  be the line of  $R^2$  on  $\pi$  and  $q$  the line of the conjugate regulus on  $\pi$ . In case  $p$  is a fixed line of  $\Gamma$ , the lines  $l$  on  $\pi$  are the tangents to  $R^2$ , i.e. the pencil of lines on  $\pi$  and the point  $pq$ . In case  $p$  is not a fixed line of  $\Gamma$ ,  $q$  is a tangent to  $R^2$  which meets a fixed line of  $\Gamma$  and hence is

a line of  $[l]$ . Any other line  $l_1$  of  $[l]$  in  $\pi$  must have a polar line  $l_2$  passing through the point  $pq$ . Let  $\Gamma''$  be the projectivity on  $q$  perspective with  $\Gamma$ . If  $\Gamma$  is effected by  $\{l_1 l_2\} \cdot \{l_3 l_4\}$ , then  $\Gamma''$  is the product of two involutions,  $I'$  and  $I''$ , which are perspective with the involutions effected on  $R^2$  by  $\{l_1 l_2\}$  and  $\{l_3 l_4\}$  respectively. Since  $l_2$  must pass through the point  $pq$ , the latter is a double point of  $I'$ . But when  $\Gamma''$  is expressed as a product of two involutions, one of these involutions is fully determined by one of its double points in case the latter is not a double point of  $\Gamma''$  (cf. § 78, Vol. I). Hence the other double point,  $P$ , is fixed; and since  $l_1$  must pass through it, it follows that all lines of  $[l]$  on  $\pi$  pass through  $P$ . Moreover, it is evident that if  $l_1$  is any line (except  $q$ ) on  $\pi$  and  $P, l_2$  its polar line, and  $\{l_3 l_4\}$  any line reflection effecting an involution on  $R^2$  which is perspective with  $I''$ , the projectivity  $\Gamma$  is effected by  $\{l_1 l_2\} \cdot \{l_3 l_4\}$ . Hence  $[l]$  contains all lines on  $\pi$  and  $P$ . Hence  $[l]$  is a linear complex by Theorem 24, Chap. XI, Vol. I.

**THEOREM 38.** *A direct projectivity  $\Gamma$  of a quadric surface  $Q^2$  which does not leave all lines of either regulus invariant determines a linear congruence of lines having the property that if  $a_1$  is any line of the congruence not tangent to  $Q^2$  there exist lines  $a_2, b_1, b_2$  of the congruence such that*

$$(33) \quad \Gamma = \{a_1 a_2\} \cdot \{b_1 b_2\}.$$

*Moreover, each line  $a_1$  having this property belongs to the congruence, and so do  $a_2, b_1, b_2$ .*

*Proof.*  $\Gamma$  effects a projectivity on each regulus of  $Q^2$ , and each of these reguli by the last theorems determines a linear complex of lines. The two complexes are obviously not identical and hence have a linear congruence in common. Any line  $a_1$  of this congruence is either tangent to  $Q^2$ , or such that there exist lines  $a_2, b_1, b_2$  which are in both complexes and such that  $\{a_1 a_2\} \cdot \{b_1 b_2\}$  effects the same projectivity as  $\Gamma$  on both reguli. Hence  $\{a_1 a_2\} \cdot \{b_1 b_2\} = \Gamma$ . Moreover, any  $a_1$  for which  $a_2, b_1, b_2$  exist satisfying this condition must, by the last theorem, belong to both complexes and hence belong to this congruence.

**COROLLARY 1.** *The congruence referred to in the theorem may be degenerate and consist of all lines on a point of  $Q^2$  and on a plane tangent to  $Q^2$  at this point; or it may be parabolic and have a line of the quadric as directrix; or it may be hyperbolic and have a pair*

of polar lines as directrices; or it may be elliptic and have a pair of improper polar lines as directrices.

*Proof.* Let  $C$  denote the congruence referred to in the theorem and let  $\Pi$  be the polarity by which every point is transformed into its polar plane with respect to  $Q^2$ . This polarity transforms any line  $a_1$  of  $C$  into its polar line, and the latter, by the theorem, is in  $C$ . Hence  $\Pi$  transforms  $C$  into itself.

According to § 107, Vol. I, any congruence is either degenerate, parabolic, hyperbolic, or elliptic. If degenerate, it consists of all lines on a point  $R$  or a plane  $\rho$ ,  $R$  being on  $\rho$ . If  $\Pi$  transforms such a congruence into itself, it must interchange  $R$  and  $\rho$ , and hence  $R$  must be on  $Q^2$  and  $\rho$  tangent to  $Q^2$  at  $R$ . The congruence  $C$  will be of this type if  $b_1$  meets  $a_1$  in a point of  $Q^2$  and does not meet  $a_2$ .

If  $C$  is parabolic, its one directrix must be transformed into itself by  $\Pi$ , and hence must be a line of  $Q^2$ . This case arises if  $a_1, a_2, b_1, b_2$  all meet the same line of  $Q^2$  and do not meet any other line of  $Q^2$ .

If  $C$  is hyperbolic,  $\Pi$  must either leave the two directrices fixed individually or interchange them. In the first case each directrix must be a line of  $Q^2$ , which implies that  $a_1, a_2, b_1, b_2$  all meet two lines of  $Q^2$  and hence that all lines of one regulus are left invariant by  $\Gamma$ , contrary to hypothesis. Hence the second case is the only possible one. It occurs when  $a_1, a_2, b_1, b_2$  do not all meet any line of  $Q^2$ , but are met by a pair of real lines.

If  $C$  is elliptic, it has two improper directrices\* and the reasoning is the same as for the hyperbolic case.

**DEFINITION.** A line  $l$  is said to *meet* or *to be met by* a pair of lines  $pq$  if and only if it meets both of them. A pair of lines  $lm$  is said to *meet* or *cross* a pair  $pq$  if both  $l$  and  $m$  meet  $pq$ .

### EXERCISES

1. The lines which cross the distinct pairs of an involution on a regulus together with the lines tangent to the regulus at points of the double lines (if existent) of the involution form a nondegenerate linear complex.

2. If two pairs of polar lines,  $a_1a_2$  and  $b_1b_2$ , of a regulus meet each other, the involutions effected by  $\{a_1a_2\}$  and  $\{b_1b_2\}$  are harmonic (commutative) and their double lines form a harmonic set.

\* This may be proved as follows: Let  $l_1, l_2, l_3, l_4$  be lines of  $C$  not on the same regulus. Any plane on  $l_1$  meets the regulus  $R^2$  containing  $l_1, l_2, l_3$  in a conic, and  $l_4$  meets this conic in two improper points  $P_1, P_2$ . The two lines of the regulus conjugate to  $R^2$  which pass through  $P_1, P_2$  meet  $l_1, l_2, l_3, l_4$  and hence meet all lines of  $C$ .

3. Let  $\Gamma$  be a projectivity on a regulus  $R^2$ . A variable plane meets  $R^2$  in a conic  $C^2$  on which there is a projectivity  $\Gamma'$  perspective with  $\Gamma$ . The axes of the projectivities  $\Gamma'$  are lines of a linear congruence.

4. Enumerate the types of collineations leaving invariant a quadric (1) in the complex space, (2) in a real space, (3) in various modular spaces.

**\*108. Products of pairs of involutoric projectivities.**

**THEOREM 39.** *A direct projective collineation of a quadric surface is a line reflection whose axes are polar, if it interchanges two points of the quadric which are not joined by a line of the quadric.*

*Proof.* Denote the collineation by  $\Gamma$ , the quadric by  $Q^2$ , the two reguli on it by  $R_1^2$  and  $R_2^2$ , and the two points which  $\Gamma$  interchanges by  $A$  and  $B$ . Let  $a$  and  $b$  be the lines of  $R_1^2$  on  $A$  and  $B$  respectively, and  $a'$  and  $b'$  those of  $R_2^2$  on  $A$  and  $B$  respectively. Since  $\Gamma$  interchanges  $a$  and  $b$  it effects an involution on  $R_1^2$ , and since it interchanges  $a'$  and  $b'$  it effects an involution on  $R_2^2$ . Let  $l, m$  be the double lines of the involution on  $R_1^2$ , and  $p, q$  those of the involution on  $R_2^2$ .  $\Gamma$  is evidently the product of  $\{lm\}$  by  $\{pq\}$  and hence is a line reflection whose axes are the line joining the points  $lp$  and  $mq$  and the line joining the planes  $lp$  and  $mq$ . These two lines are polar with regard to  $Q^2$ .

**THEOREM 40.** *Two lines which are not on a quadric  $Q^2$  and do not meet the same line of  $Q^2$  are met by one and but one polar pair of lines.*

*Proof.* Let one of the given lines meet the quadric in  $A$  and  $A'$  and the other meet it in  $B$  and  $B'$ . By Theorem 35 there is a unique direct projective collineation of the quadric which carries  $A$  to  $A'$ ,  $A'$  to  $A$ , and  $B$  to  $B'$ . By Theorem 39 this is a line reflection  $\{lm\}$  and  $l$  and  $m$  are polar with respect to  $Q^2$ . Since  $\{lm\}$  transforms  $A$  to  $A'$ ,  $l$  and  $m$  both meet the line  $AA'$ , and since  $\{lm\}$  transforms  $B$  to  $B'$ ,  $l$  and  $m$  both meet the line  $BB'$ .

If there were another pair of polar lines  $l', m'$  meeting  $AA'$  and  $BB'$ ,  $\{l'm'\}$  would interchange  $A$  and  $A'$  and  $B$  and  $B'$ . By Theorem 35  $\{lm\} = \{l'm'\}$ .

**COROLLARY.** *Two lines which are not on a quadric  $Q^2$  and do not meet the same line of  $Q^2$  are met by two and only two lines which are conjugate to them both with regard to  $Q^2$ .*

*Proof.* This follows directly from the theorem, because two mutually polar lines  $a, b$  meeting two lines  $l$  and  $m$  are both conjugate to

$l$  and  $m$  and, moreover, if a line  $a$  meets and is conjugate to both  $l$  and  $m$  its polar line also meets and is conjugate to both  $l$  and  $m$ .

**THEOREM 41.** *If a simple hexagon is inscribed in a quadric surface in such a way that no two of its vertices are on a line of the quadric, the three pairs of opposite edges are met each by a polar pair of lines, and these three polar pairs of lines are in the same linear congruence.*

*Proof.* Let  $A_1B_2C_1A_2B_1C_2$  be the simple hexagon. By the last theorem the pair of opposite edges  $A_1B_2$ ,  $A_2B_1$  is met by a pair of lines  $c_1$ ,  $c_2$  which are polar with respect to the quadric. In like manner  $B_2C_1$ ,  $B_1C_2$  are met by a polar pair  $a_1$ ,  $a_2$ , and  $C_1A_2$ ,  $C_2A_1$  by a polar pair  $b_1$ ,  $b_2$ . Consider the product of line reflections,

$$\Gamma = \{c_1c_2\} \cdot \{b_1b_2\} \cdot \{a_1a_2\}.$$

The line reflection  $\{a_1a_2\}$  carries  $B_1$  to  $C_2$ ,  $\{b_1b_2\}$  carries  $C_2$  to  $A_1$ , and  $\{c_1c_2\}$  carries  $A_1$  to  $B_2$ . Likewise  $\{a_1a_2\}$  carries  $B_2$  to  $C_1$ ,  $\{b_1b_2\}$  carries  $C_1$  to  $A_2$ , and  $\{c_1c_2\}$  carries  $A_2$  to  $B_1$ . Hence  $\Gamma$  interchanges  $B_1$  and  $B_2$ , and by Theorem 39 it is a line reflection. Denoting  $\Gamma$  by  $\{d_1d_2\}$  we have

$$\{c_1c_2\} \cdot \{d_1d_2\} = \{b_1b_2\} \cdot \{a_1a_2\}.$$

By Theorem 38 the axes of the four line reflections in this equation are all lines of the same congruence.

In view of the corollaries of Theorems 38 and 40 this theorem may be restated in the following forms:

**COROLLARY 1.** *If a simple hexagon is inscribed in a quadric in such a way that no two of its vertices are on a line of the quadric, the three polar pairs of lines which meet the pairs of opposite edges are met by a polar pair of lines (which may coincide).*

**COROLLARY 2.** *If a simple hexagon is inscribed in a quadric surface in such a way that no two of its vertices are on a line of the quadric, each pair of opposite edges is met by a unique pair of lines conjugate to both edges, and the latter three pairs of lines are met by a pair of lines conjugate to each of them. The lines of the last pair may coincide.\**

\* Bulletin of the American Mathematical Society, Vol. XVI (1909), pp. 55 and 62. A theorem of non-Euclidean geometry from which this may be obtained by generalization has been given by F. Klein, *Mathematische Annalen*, Vol. XXII (1883), p. 248.

This theorem is closely analogous to Pascal's theorem on conic sections (Chap. V, Vol. I). In the Pascal hexagon the pairs of opposite sides determine three points  $A, B, C$  which are collinear. In the hexagon inscribed in a quadric they determine three pairs of lines  $a_1a_2, b_1b_2, c_1c_2$  which are in a linear congruence. In case the vertices of the hexagon are coplanar, the theorem on the quadric reduces directly to Pascal's.

The Pascal theorem may be proved by precisely the method used above. For let  $A_1B_2C_1A_2B_1C_2$  be a hexagon inscribed in a conic and let  $A$  be the point  $(B_1C_2, C_1B_2)$ ,  $B$  be  $(C_1A_2, A_1C_2)$ , and  $C$  be  $(A_1B_2, B_1A_2)$ . Let  $\{Aa\}$ ,  $\{Bb\}$ , and  $\{Cc\}$  be the harmonic homologies effecting the involutions having  $A, B, C$  as centers. By construction the projectivity effected by  $\{Cc\} \cdot \{Bb\} \cdot \{Aa\}$  on the conic carries  $B_1$  to  $B_2$ , and  $B_2$  to  $B_1$ , and hence is an involution. Denoting its center and axis by  $D$  and  $d$ , we have

$$\{Cc\} \cdot \{Bb\} \cdot \{Aa\} = \{Dd\}.$$

This implies

$$\{Bb\} \cdot \{Aa\} = \{Cc\} \cdot \{Dd\}.$$

By the theorems of Chap. VIII, Vol. I, the line  $AB$  is the axis of the projectivity effected by  $\{Bb\} \cdot \{Aa\}$  and must contain  $C$  and  $D$ . Hence  $A, B, C$  are collinear.

Pascal's theorem is thus based on the proposition that the product of three involutions on a conic is itself an involution if and only if the centers of the three involutions are collinear, i.e. if and only if their axes are concurrent. Let us denote an involution whose double points are  $L$  and  $M$  by  $\{LM\}$ , as in Ex. 11, § 52. If the involution is represented on a conic, the double points are joined by the axis of the involution. The proposition above then takes the form: The product  $\{L_3M_3\} \cdot \{L_2M_2\} \cdot \{L_1M_1\}$  is an involution if and only if the lines  $L_1M_1, L_2M_2, L_3M_3$  concur. The concurrence of the three lines means either that the three point pairs have a point in common or that they are themselves pairs of an involution. Thus the theorem on involutions may be stated as follows:

**THEOREM 42.** *In any one-dimensional form a product of three involutions  $\{L_1M_1\}, \{L_2M_2\}, \{L_3M_3\}$  is an involution in case the pairs of points  $L_1M_1, L_2M_2, L_3M_3$  have a point in common or are pairs of an involution; and the product is not an involution in any other case.*

The double points of the involutions may be either proper or improper (real or imaginary). In order to state the result entirely in terms of proper elements, the involutions may be represented on a conic and the condition stated in terms of the concurrence of their axes, as above; or it may be expressed by saying that they all belong to the same pencil of involutions, or by saying that they are all harmonic to the same projectivity.

This theorem on involutions in a one-dimensional form is fundamental in the theory of those groups of projectivities, in a space of any number of dimensions, which are products of involutonic projectivities. For example,

it is essentially the same as Theorem 8, Chap. IV, which was fundamental in the theory of the parabolic metric group in the plane. Corresponding theorems in the Euclidean geometry of three dimensions will be found in §§ 114 and 121, Chap. VII. The same principle appears as Theorem 27, Cor. 1, Chap. III, in connection with the equiaffine group.

These groups are all projective and on that account related to the projective group of a one-dimensional form. But the essential feature which they have in common is that *every transformation of each group is a product of two involutories of the same group*. On this account, even without their common projective basis, the geometries corresponding to these groups must have many features in common. In particular, whenever there is some class of figures such that if two of the figures are interchanged by a transformation, the transformation is of period two, there must exist a theorem analogous to Pascal's theorem. As examples of this may be cited Theorem 41 above; Ex. 6, § 80, Vol. I; Ex. 1, § 122, below; and in the list of exercises below, Ex. 4, referring to the group of point reflections and translations, Exs. 5, 6 referring to the Euclidean group in a plane, Ex. 7 referring to the equiaffine group. On this subject in particular and also on the general theory of groups generated by transformations of period two, the reader should consult a series of articles by H. Wiener in the *Berichte der Gesellschaft der Wissenschaften zu Leipzig*, Vol. XLII (1890), pp. 13, 71, 245; Vol. XLIII (1891), pp. 424, 644; and also the article by Wiener referred to in § 45, above. Cf. also § 80, Vol. I.

### EXERCISES

1. (Converse of Theorem 41.) If the three pairs of opposite edges of a simple hexagon are met by three pairs of lines  $a_1a_2$ ,  $b_1b_2$ ,  $c_1c_2$  in pairs of points which are harmonically conjugate to the pairs of vertices with which they are collinear, and if the lines  $a_1$ ,  $a_2$ ,  $b_1$ ,  $b_2$ ,  $c_1$ ,  $c_2$  are in the same linear congruence, then the vertices of the hexagon are on a quadric surface with regard to which  $a_1a_2$ ,  $b_1b_2$ ,  $c_1c_2$  are polar pairs of lines.

2. Two pairs of lines which are polar with regard to the same regulus cannot consist of lines of a common regulus.

3. If two lines  $l$  and  $m$  are met by two pairs of lines which are polar with respect to a quadric,  $l$  and  $m$  are polar.

4. In a Euclidean plane let  $A$ ,  $B$ ,  $C$  be the three points of intersection of pairs of opposite sides of a simple hexagon. If  $A$  and  $B$  are mid-points of the sides containing them, and  $C$  is the mid-point of one side containing it, then  $C$  is also a mid-point of the other side containing it.

5. Let  $A_1B_2C_1A_2B_1C_2$  be a simple hexagon in a Euclidean plane. If the perpendicular bisector of the point pair  $A_1B_2$  coincides with that of  $A_2B_1$ , and the perpendicular bisector of  $B_2C_1$  with that of  $B_1C_2$ , and the perpendicular bisector of  $C_2A_1$  with that of  $C_1A_2$ , then the three perpendicular bisectors meet in a point.

6. Let  $a, b, c, a', b', c'$  be six concurrent lines of a Euclidean plane. If there is a pair of lines bisecting each of the pairs  $ab'$  and  $a'b$ , and a pair bisecting  $bc'$  and  $b'c$ , there is a pair bisecting  $ca'$  and  $c'a$ .

7. If the pairs of opposite sides of a simple hexagon are parallel, the lines joining their mid-points are concurrent.

109. **Conjugate imaginary lines of the second kind.** The theory of antiprojectivities (§ 99) and the extended theory of projectivities of one-dimensional forms (§ 106) will now enable us to complete the theory of conjugate imaginary elements in certain essential details which we were not ready to discuss in § 78. Let  $S'$  be a complex projective space and let  $S$  be a three-chain of  $S'$ , i.e. a space related to  $S'$  in the manner described in §§ 6 and 70, and let us use the definitions and notations of § 70. The simplest type of antiprojective collineation of  $S'$  is given by the equations

$$(34) \quad x'_0 = \bar{x}_0, \quad x'_1 = \bar{x}_1, \quad x'_2 = \bar{x}_2, \quad x'_3 = \bar{x}_3.$$

The frame of reference is such that the points of  $S$  have real coördinates. The transformation changes each point

$$(\alpha_0 + i\beta_0, \alpha_1 + i\beta_1, \alpha_2 + i\beta_2, \alpha_3 + i\beta_3),$$

where the  $\alpha$ 's and  $\beta$ 's are real, into the point

$$(\alpha_0 - i\beta_0, \alpha_1 - i\beta_1, \alpha_2 - i\beta_2, \alpha_3 - i\beta_3).$$

These two points if distinct are joined by the real line

$$(\alpha_0 + \lambda\beta_0, \alpha_1 + \lambda\beta_1, \alpha_2 + \lambda\beta_2, \alpha_3 + \lambda\beta_3)$$

and are the double points of the involution determined by the transformation of the parameter  $\lambda$ ,

$$\lambda' = -\frac{1}{\lambda}.$$

Comparing with the definition of conjugate imaginary points in § 78, it is clear that (34) is the transformation by which every point of  $S'$  goes to its conjugate imaginary point, the points of  $S$  being regarded as real.

From the fact that the transformation (34) leaves no imaginary point invariant, it follows that it cannot leave any imaginary line or plane invariant. For the real line through an imaginary point  $P$  of the given line or plane is left invariant by (34), and hence  $P$  would be left invariant by (34). On the other hand, (34) leaves every real

element invariant and hence leaves every elliptic involution in a real one-dimensional form invariant. Since (34) cannot leave the double elements of such an involution invariant, it must interchange them. Hence (34) interchanges any element of  $S'$  with the element which is its conjugate imaginary according to the definition of § 78.

The definition of § 78 defines the notion of conjugate imaginary elements for all one-dimensional forms of the first or second degrees, and the theorems of that section cover all cases except that of a pair of conjugate imaginary lines which are the double lines of an elliptic involution in the lines of a regulus.

DEFINITION. An imaginary line which is a double line of an elliptic involution in a flat pencil is said to be of *the first kind*, and one which is a double line of an elliptic involution in a regulus is said to be of *the second kind*.

THEOREM 43. *Any imaginary line is either of the first or of the second kind.*

*Proof.* Let  $l$  be an imaginary line. It cannot contain two real points, else it would be a real line (§ 70). Hence it contains one or no real point. In the first case let  $O$  be the real point on  $l$ ,  $P$  one of the imaginary points on  $l$ , and  $\bar{P}$  the imaginary point conjugate to  $P$ . The line  $P\bar{P}$  is real, and hence the plane  $OP\bar{P}$  is real. Hence by § 78 the lines  $OP$  and  $O\bar{P}$  are the double lines of an elliptic involution in the pencil of real lines on the point  $O$  and the plane  $OP\bar{P}$ .

In the second case let  $P, Q$  and  $R$  be three points of  $l$  and let  $\bar{P}, \bar{Q}$  and  $\bar{R}$  be their respective conjugate imaginary points. The lines  $P\bar{P}, Q\bar{Q}, R\bar{R}$  are real and no two of them can intersect, for if they did  $l$  would be on a real plane, and we should have the case considered in the last paragraph. Hence these lines determine a regulus  $R_1^2$  in  $S$ . On the real line  $P\bar{P}$  there is by § 78 an elliptic involution having  $P$  and  $\bar{P}$  as its imaginary double points. Hence there is an elliptic involution in the regulus  $R_1^2$ , conjugate to  $R_1^2$ , having  $l$  as one double line and a line  $\bar{l}$  through  $\bar{P}$  as the other. The lines  $l$  and  $\bar{l}$  are conjugate imaginary lines by definition, and satisfy the definition of imaginary lines of the second kind. Since (34) transforms each element into its conjugate element, it is clear that  $\bar{l}$  contains  $\bar{Q}$  and  $\bar{R}$  as well as  $\bar{P}$ .

The system of real lines obtained by joining each point of  $l$  to its conjugate imaginary point on  $\bar{l}$  is, by the reasoning above, a set of

lines of the real space  $S$ , no two of which intersect. Any four of them determine a linear congruence (§ 107, Vol. I)  $C$  in  $S$  and also a linear congruence  $\bar{C}$  of  $S'$ . The congruence  $C$  has the property that each of its lines is contained in a line of  $\bar{C}$ , and  $\bar{C}$  evidently is the set of all lines joining points of  $l$  to points of  $\bar{l}$ . Hence  $C$  is an elliptic congruence according to the definition of § 107, Vol. I, and consists of all real lines meeting  $l$  and  $\bar{l}$ . Hence the system of real lines joining points of  $l$  to their conjugate imaginary points is an elliptic congruence in  $S$ , or in other words:

**THEOREM 44.** *An imaginary line of the second kind is a directrix of an elliptic congruence.*

The observation, made in the argument above, that there is one line of a certain elliptic congruence through each point of an imaginary line of the second kind, shows that an elliptic congruence may be taken as a real image of a complex one-dimensional form. This of course implies that the whole of the real inversion geometry can be carried over into the theory of the elliptic congruence and *vice versa*. Cf. the exercises below.

The relations between the imaginary lines of the second kind and the regulus and elliptic congruence are fundamental in the von Staudt theory of imaginaries which has been referred to in § 6. In addition to the references given in that place, the reader may consult the *Encyclopédie des Sciences Mathématiques*, III 8, § 19, and III 3, §§ 14, 15.

### EXERCISES

1. An elliptic congruence in a real space has a pair of conjugate imaginary lines of the second kind as directrices.

2. The correspondence by which each point of an imaginary line  $l$  corresponds to its conjugate imaginary point is an antiprojectivity between  $l$  and its conjugate imaginary line.

3. Under the projective group of a real space any imaginary point is transformable into any other imaginary point, any imaginary line of the first kind into any imaginary line of the first kind, and any imaginary line of the second kind into any imaginary line of the second kind; an imaginary line of the first kind is not transformable into one of the second kind.

4. There is a one-to-one reciprocal correspondence between the points of a complex line and the lines of an elliptic congruence in a real space in which the points of a chain correspond to the lines of a regulus. By means of this correspondence, make a study of the elliptic congruence and its group.

5. Let  $S'_3$  be a three-dimensional complex space. Any five noncoplanar points of  $S'_3$  determine a unique three-chain, which is a real  $S_3$ . This  $S_3$  is related to  $S'_3$  in the manner described in §§ 6 and 70. Through any point  $P$  of  $S'_3$  not on  $S_3$ , there is (§ 78) a unique line which contains a line of  $S_3$  (i.e. a chain  $C_1$ ) as a subset. On this chain  $C_1$  there is a unique elliptic involution having  $P$  as a double point. Let  $\bar{P}$  be the other double point of this involution.  $P$  and  $\bar{P}$  are the conjugate imaginary points with regard to the real space  $S_3$ , and the transformation of  $S'_3$  by which each point  $P$  not on  $S_3$  goes to  $\bar{P}$ , and each point on  $S_3$  is left invariant, may be called a *reflection in the three-chain  $S_3$* . Any transformation which is a product of an odd number of reflections in three-chains is an antiprojective collineation, and any transformation which is a product of an even number of reflections in three-chains is a projective collineation. Every collineation is expressible in this form.

110. *The principle of transference.* We have seen how the geometry of the inversion group in the plane, arising initially as an extension of the Euclidean group, is equivalent to the projective geometry of the complex line and also to that of a real quadric which may be specialized as a sphere. We have also seen the equivalence of the projective groups of all one-dimensional forms in any properly projective space. Since the regulus is a one-dimensional form, this gave a hold on the group of the general quadric. The latter group in a complex space has been seen to be isomorphic with the complex inversion group and also with the fundamental group of the function plane.

At each step we have helped ourselves forward by transferring the results of one geometry to another, combining these with easily obtained theorems of the second geometry, and thus extending our knowledge of both. This is one of the characteristic methods of modern geometry. It was perhaps first used with clear understanding by O. Hesse,\* and was formulated as a definite geometrical principle (Uebertragungsprinzip) by F. Klein in the article referred to in § 34.

This principle of transference or of carrying over the results of one geometry to another may be stated as follows: *Given a set of elements  $[e]$  and a group  $G$  of permutations of these elements, and a set of theorems  $[T]$  which state relations left invariant by  $G$ . Let  $[e']$  be another set of elements, and  $G'$  a group of permutations of  $[e']$ . If there is a one-to-one reciprocal correspondence between  $[e]$  and  $[e']$*

\* Gesammelte Werke, p. 531.

in which  $G$  is simply isomorphic with  $G'$ , the set of theorems  $[T]$  determines by a mere change of terminology a set of theorems  $[T']$  which state relations among elements  $e'$  which are left invariant by  $G'$ .

This principle becomes effective when the method by which  $[e]$  and  $G$  are defined is such as to make it easy to derive theorems which are not so easily seen for  $[e']$  and  $G'$ . This has been abundantly illustrated in the present chapter, but the series of geometries equivalent to the projective geometry on a line could be much extended. Some of the possible extensions are mentioned in the exercises below.

From the example of the conic and the quadric surface (§ 107) it is clear that in order to carry results over from the theory of a set  $[e]$  and a group  $G$  to a set  $[e']$  and a group  $G'$  it is not necessary that the correspondence be one-to-one. The transference of theorems is, however, no longer a mere translation from one language, as it were, to another, but involves a study of the nature of the correspondence.

DEFINITION. Given a set of elements  $[e]$  and a group  $G$  of permutations of  $[e]$ , the set of theorems  $[T]$  which state relations among the elements of  $[e]$  which are left invariant by  $G$  and are not left invariant by any group of permutations containing  $G$  is called a *generalized geometry* or a *branch of mathematics*.\*

This is, of course, a generalization of the definition of a geometry employed in §§ 34 and 39. At the time when the rôle of groups in geometry was outlined by Klein, the only sets  $[e]$  under consideration were continuous manifolds, i.e. complex spaces of  $n$  dimensions or loci defined by one or more analytic relations among the coördinates of points in such spaces. The older writers restrict the term "geometry" by means of this restriction on the set  $[e]$ . But in view of the existence of modular spaces and other sets of elements determining sets of theorems more nearly identical with ordinary geometry than some of those admitted by Klein's original definition, it seems desirable to state the definition in the form adopted above.

In case the set of theorems  $[T]$  is arranged deductively, as explained in the introduction to Vol. I, it becomes a mathematical science. The problem of the foundation of such a science is that of determining, if possible, a finite set of assumptions from which  $[T]$  may be deduced.

\* The generalized conception of a geometry is discussed very clearly in the article by G. Fano in the *Encyclopädie der Math. Wiss.* III AB 4b. A number of special cases are outlined in the latter half of the article.

## EXERCISES

1. If a projective collineation interchanges the two reguli on a quadric, homologous lines of the two reguli meet in points of a plane.

\*2. Let  $R^2$  be a regulus,  $\omega$  a plane not tangent to  $R^2$ , and  $O$  the pole of  $\omega$  ( $\omega$  may conveniently be regarded as the plane at infinity of a Euclidean space). A projectivity  $\Gamma$  of  $R^2$  may be effected by a collineation  $\Gamma'$  leaving all lines of the conjugate regulus invariant. This collineation multiplied by the harmonic homology  $\{O\omega\}$  gives a collineation  $\Gamma''$  interchanging the two reguli. By Ex. 1,  $\Gamma''$  determines a unique plane. Let  $P$  be the point polar to  $\Gamma''$  with regard to  $R^2$ . The correspondence thus determined between the projectivities  $\Gamma$  of  $R^2$  and the points of space not on  $R^2$  is one to one and reciprocal. It is such that projectivities which are harmonic (§ 80, Vol. I) correspond to conjugate points with respect to  $R^2$ , and all the involutions correspond to points of  $\omega$ .

\*3. The construction of Ex. 2 sets up a correspondence between the projectivities of a one-dimensional form and the points of a three-dimensional space which are not on a certain quadric. The same correspondence may be obtained by letting a projectivity

$$x' = \frac{a_0x + a_1}{a_2x + a_3}$$

correspond to the point  $(a_0, a_1, a_2, a_3)$ . The relations between the one-dimensional and three-dimensional projective geometries thus obtained have been studied by C. Stéphanos, *Mathematische Annalen*, Vol. XXII (1883), p. 299.

\*4. Develop the theory of the twisted cubic curve in space along the following lines: (1) Define it algebraically. (2) Give a geometric definition. (3) Prove that Definitions (1) and (2) are equivalent. (4) Derive the further theorems on the cubic as far as possible from the geometric definition. It will be found that the properties of this cubic can be obtained largely from those of conic sections and one-dimensional projectivities in view of an isomorphism of the groups in question. The theorems should be classified according to the principle laid down in § 83.

\*5. A *rational curve* in a space of  $k$  dimensions is a locus given parametrically as follows:

$$x_0 = R_0(t), \quad x_1 = R_1(t), \quad \dots, \quad x_n = R_n(t),$$

where  $R_0(t), \dots, R_n(t)$  are rational functions of  $t$ . In case  $k = n$  and the locus is not contained in any space of less than  $n$  dimensions, the curve is a *normal curve*. Develop the theories of various rational curves along the lines outlined in Ex. 4. For reference cf. § 23 of the encyclopedia article by Fano referred to above and articles by several authors in recent volumes of the *American Journal of Mathematics*.

\*6. The linear dependence of conic sections may be defined by substituting "point conic" or "line conic," as the case may be, for "circle" in the definition given at the end of § 100. Develop the theory of linear families of conics of one, two, three, and four dimensions, using the principle of correspondence whenever possible and classifying theorems according to the principle laid down in § 83. Cf. *Encyclopédie des Sc. Math.* III 18.

## CHAPTER VII

### AFFINE AND EUCLIDEAN GEOMETRY OF THREE DIMENSIONS

111. **Affine geometry.** DEFINITION. Let  $\pi_{\infty}$  be an arbitrary but fixed plane of a projective space  $S$ . The set of points of  $S$  not on  $\pi_{\infty}$  is called a *Euclidean space* and  $\pi_{\infty}$  is called the *plane at infinity* of this space. The plane  $\pi_{\infty}$  and the points and lines on  $\pi_{\infty}$  are said to be *ideal* or *at infinity*; all other points, lines, and planes of  $S$  are said to be *ordinary*. When no other indication is given, a point, line, or plane is understood to be ordinary. Any projective collineation transforming a Euclidean space into itself is said to be *affine*; the group of all such collineations is called the *affine group of three dimensions*, and the corresponding geometry the *affine geometry of three dimensions*.

DEFINITION. Two ordinary lines which have an ideal point in common are said to be *parallel* to each other. Two ordinary planes which have an ideal line in common, or an ordinary line and an ordinary plane which have an ideal point in common, are said to be *parallel* to each other.

In particular, a line or plane is said to be parallel to itself or to any plane or line which it is on. For ordinary points, lines, and planes we have as an obvious consequence of the assumptions and definitions of Chap. I, Vol. I, the following theorem:

THEOREM 1. *Through a given point there is one and only one line parallel to a given line. Through a given point there is one and only one plane parallel to a given plane. If two lines,  $l$  and  $l'$ , are not in the same plane there is one and only one plane through a given point parallel to  $l$  and  $l'$ . If  $l$  and  $l'$  are parallel, any plane through  $l$  is parallel to  $l'$ .*

Another obvious though important theorem is the following:

THEOREM 2. *The transformations effected in an ordinary plane  $\pi$  by the affine group in space constitute the affine group of the Euclidean plane consisting of the ordinary points of  $\pi$ .*

In consequence of this theorem we have the whole affine plane geometry as a part of the affine geometry of three dimensions, and we shall take all the definitions and theorems of Chap. III for granted without further comment.

This discussion is valid for any space satisfying Assumptions A, E. The affine geometry of an ordered space (A, E, S) has already been considered in § 31, and certain additional theorems are given in Exs. 5-7 below.

### EXERCISES

1. The lines joining the mid-points of the pairs of vertices of a tetrahedron meet in a point.

2. Classify the quadric surfaces from the point of view of real affine geometry. Develop the theory of diametral lines and planes. The real projective classification of the nondegenerate quadrics has been given in § 103. The affine classification is given in the *Encyclopédie des Sc. Math.* III 22, § 19.

\*3. Classify the linear congruences from the point of view of the real affine geometry. Cf. § 107, Vol. I.

\*4. Classify the linear complexes from the point of view of real affine geometry. Cf. § 108, Vol. I.

5. With respect to the coordinate system used in § 31 the points of the line joining  $A = (a_1, a_2, a_3)$  and  $B = (b_1, b_2, b_3)$  are

$$\left( \frac{a_1 + \lambda b_1}{1 + \lambda}, \frac{a_2 + \lambda b_2}{1 + \lambda}, \frac{a_3 + \lambda b_3}{1 + \lambda} \right),$$

$B$  corresponding to  $\lambda = \infty$  and the point at infinity to  $\lambda = -1$ . The segment  $AB$  consists of the points for which  $\lambda > 0$  and its two prolongations of those for which  $\lambda < -1$  and  $-1 < \lambda < 0$  respectively.

6. Two points  $D$  and  $D'$  are on the same side of the plane  $ABC$  if and only if

$$S(ABCD) = S(ABCD').$$

7. Using the notation of § 101 and dealing with an ordered Euclidean space,  $\{O\omega\}$  is an affine collineation which alters sense if  $O$  or  $\omega$  is at infinity and  $\{l\}$  is an affine collineation which does not alter sense if  $l$  or  $l'$  is at infinity. In an ordered projective space  $\{l\}$  is, and  $\{O\omega\}$  is not, a direct collineation.

112. Vectors, equivalence of point triads, etc. DEFINITION. An elation having  $\pi_\infty$  as its plane of fixed points is called a *translation*. If  $l$  is an ordinary line on the center of the translation, the translation is said to be *parallel* to  $l$ .

The properties of the group of translations follow in large part from the following evident theorem.

THEOREM 3. *The transformations effected in an ordinary plane  $\pi$  by the translations leaving  $\pi$  invariant constitute the group of translations of the Euclidean plane composed of the ordinary points of  $\pi$ .*

As corollaries of this we have statements about translations in space which are verbally identical with Theorems 3-7, Chap. III. Theorem 8, Chap. III, generalizes as follows:

COROLLARY. *If  $OX, OY$ , and  $OZ$  are three noncoplanar lines and  $T$  any translation, there exists a unique triad of translations  $T_x, T_y, T_z$  parallel to  $OX, OY, OZ$  respectively and such that*

$$T = T_x T_y T_z.$$

The theory of congruence under translations generalizes to space without change, and the contents of §§ 39 and 40 may be taken as applying to the affine geometry in three-space. In like manner the definition of a field of vectors and of addition of vectors is carried over to space if the words "Euclidean plane" be replaced by "Euclidean space." The theorems of § 42 then apply without change.

We arrive at this point on the basis of Assumptions A, E,  $H_0$ . Adding Assumption P we take over the theory of the ratio of collinear vectors from §§ 43, 44. Some of the theorems to which it may be applied without essential modifications of the methods used in the planar case are given in the exercises below.

The definition of equivalence of ordered point triads in § 48 is such that if a plane  $\pi$  be carried by an affine collineation to a plane  $\pi'$ , any two equivalent point triads of  $\pi$  are carried to two equivalent point triads of  $\pi'$ . Moreover, the definition of measure of ordered point triads in § 49 is such that if two coplanar ordered point triads  $ABC, DEF$  are carried by an affine collineation to  $A'B'C', D'E'F'$  respectively,

$$(1) \quad \frac{m(ABC)}{m(DEF)} = \frac{m(A'B'C')}{m(D'E'F')}.$$

This result in view of Theorem 39, Chap. III, depends on the corresponding theorem about the ratios of collinear vectors. In (1) the unit of measure in any plane is regarded as entirely independent of the unit of measure in every other plane, but nevertheless the ratio of the measures is an invariant of the affine group. Certain ratios of ratios of measures are invariants of the projective group (cf. Ex. 17 below).

The notion of equivalence of ordered point triads may be extended as follows:

DEFINITION. Two ordered point triads  $ABC$  and  $A'B'C'$  are *equivalent* if and only if  $ABC$  may be carried by a translation to an ordered triad  $A''B''C''$  which is equivalent in the sense of § 48, Chap. III, to  $A'B'C'$ .

The fundamental propositions with regard to equivalence, as developed in § 48, remain valid under the extended definition. Thus if  $ABC \asymp A_1B_1C_1$  and  $A_1B_1C_1 \asymp A_2B_2C_2$ ,  $ABC \asymp A_2B_2C_2$ ; if  $ABC \asymp A_1B_1C_1$ ,  $A_1B_1C_1 \asymp ABC$ , etc.

This extension of the notion of equivalence carries with it a corresponding restriction of the idea of measure, i.e. measure is now defined as in § 49, with the added proviso that the unit triad in any plane shall be equivalent to the unit triad in any parallel plane.

The method by which the theory of equivalence of ordered point triads was developed in Chap. III does not generalize directly to the case of ordered tetrads in three-dimensional space.\* We shall therefore give an algebraic definition of the measures of an ordered set of four points, leaving it to the reader to develop the corresponding synthetic theory (cf. Ex. 13 below).

DEFINITION. By the *measure* of an ordered tetrad of points  $A_1, A_2, A_3, A_4$  relative to an ordered tetrad  $OPQR$  as unit is meant the number

$$(2) \quad \begin{vmatrix} 1 & a_{11} & a_{12} & a_{13} \\ 1 & a_{21} & a_{22} & a_{23} \\ 1 & a_{31} & a_{32} & a_{33} \\ 1 & a_{41} & a_{42} & a_{43} \end{vmatrix} = m(A_1A_2A_3A_4),$$

where  $(a_{i1}, a_{i2}, a_{i3})$  are nonhomogeneous coördinates of  $A_i$  ( $i=1, 2, 3, 4$ ) in a coördinate system in which  $O, P, Q, R$  are  $(0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1)$  respectively. Two ordered tetrads are said to be *equivalent* if and only if they have the same measure. In real affine geometry the number  $\frac{1}{6}|m(A_1A_2A_3A_4)|$  is called the *volume* of the tetrahedron  $A_1A_2A_3A_4$  relative to the unit tetrahedron  $OPQR$  and is denoted by  $v(A_1A_2A_3A_4)$ .

The theory of the equivalence of point pairs, triads, tetrads, etc. is the most elementary part of vector analysis and the Grassmann *Ausdehnungslehre*. This subject in particular, and the affine geometry

\* Cf. M. Dehn, *Mathematische Annalen*, Vol. LV (1902), p. 465.

of three dimensions in general, is worthy of a much more extensive treatment than it is receiving here. We have referred only to that part of the subject which is essential to the study of the Euclidean geometry of three dimensions.

In the following exercises the coördinate system is understood to be that which is described in the definition of measure of ordered tetrads above. The vectors  $OP$ ,  $OQ$ ,  $OR$  are taken as units of measure for the respectively parallel systems of vectors. The ordered point triads  $OPQ$ ,  $OQR$ ,  $ORP$  are taken as units of measure for the respectively parallel systems of ordered point triads.

DEFINITION. By the *projection* of a set of points  $[X]$  on the  $x$ -axis is meant the set of points in which this axis is met by the planes through the points  $X$  and parallel to the plane  $x=0$ ; and the projection on the  $y$ - and  $z$ -axes have analogous meanings.

By the *projection* of a set of points  $[X]$  on the plane  $x=0$  is meant the set of points in which this plane is met by the lines on points  $X$  and parallel to the  $x$ -axis; and the projections on the planes  $y=0$  and  $z=0$  have analogous meanings.

### EXERCISES

1. The measures of ordered tetrads of points are unaltered by transformations

$$(3) \quad \begin{aligned} x' &= b_{11}x + b_{12}y + b_{13}z + b_{10}, \\ y' &= b_{21}x + b_{22}y + b_{23}z + b_{20}, \\ z' &= b_{31}x + b_{32}y + b_{33}z + b_{30}, \end{aligned}$$

subject to the condition  $\Delta = 1$ , where

$$(4) \quad \Delta = \begin{vmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{vmatrix}.$$

This group is called the *equiaffine* group and also the *special linear* group. The group for which  $\Delta^2 = 1$  leaves volumes invariant.

2. Ratios of measures of ordered tetrads of points are left invariant by the affine group.

3. In an ordered space two ordered sets of points  $ABCD$  and  $A'B'C'D'$  are in the same sense or not according as  $m(ABCD)$  and  $m(A'B'C'D')$  have the same sign or not.

4. The product of two line reflections  $\{l\}$  and  $\{mm'\}$  (cf. § 101) is a translation if  $l'$  and  $m'$  are at infinity and  $l$  and  $m$  are parallel.

5. Determine the subgroups of the group of translations in space.

6. The projections of a point pair  $P_1P_2$  on the  $x$ -,  $y$ -, and  $z$ -axes respectively have the measures

$$\alpha = x_2 - x_1, \quad \beta = y_2 - y_1, \quad \gamma = z_2 - z_1,$$

and those of the ordered point triad  $OP_1P_2$  on the planes  $x = 0$ ,  $y = 0$ ,  $z = 0$  respectively have the measures

$$\lambda = \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix}, \quad \mu = \begin{vmatrix} z_1 & x_1 \\ z_2 & x_2 \end{vmatrix}, \quad \nu = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}.$$

These numbers satisfy the relation

$$\alpha\lambda + \beta\mu + \gamma\nu = 0.$$

Any two points  $P'_1P'_2$  of the line  $P_1P_2$  such that  $\text{Vect } P_1P_2 = \text{Vect } P'_1P'_2$  determine the same six numbers  $\alpha, \beta, \gamma, \lambda, \mu, \nu$ . These numbers are proportional to the Plücker coördinates (cf. § 109, Vol. I) of the line  $P_1P_2$ .

7. Using the notations of Ex. 6,  $\lambda = m(OPP_1P_2)$ ,  $\mu = m(OQP_1P_2)$ ,  $\nu = m(ORP_1P_2)$ . If  $\alpha', \beta', \gamma', \lambda', \mu', \nu'$  are the numbers analogous to  $\alpha, \beta, \gamma, \lambda, \mu, \nu$  determined by an ordered pair  $P_3P_4$ ,

$$m(P_1P_2P_3P_4) = \alpha\lambda' + \beta\mu' + \gamma\nu' + \lambda\alpha' + \mu\beta' + \nu\gamma'.$$

8. The measures of the projections of an ordered point triad  $P_1P_2P_3$  on the planes  $x = 0$ ,  $y = 0$ ,  $z = 0$  respectively are

$$u_1 = \begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix}, \quad u_2 = - \begin{vmatrix} x_1 & z_1 & 1 \\ x_2 & z_2 & 1 \\ x_3 & z_3 & 1 \end{vmatrix}, \quad u_3 = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

The homogeneous coördinates of the plane  $P_1P_2P_3$  are  $(u_0, u_1, u_2, u_3)$ , where

$$u_0 = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = m(OP_1P_2P_3).$$

9. If  $P_1, P_2, P_3, P_4$  are four noncoplanar points and  $P'_3, P'_4$  are two points collinear with  $P_3$  and  $P_4$ , then  $\text{Vect}(P'_3P'_4) = \text{Vect}(P_3P_4)$  if and only if  $m(P_1P_2P_3P_4) = m(P_1P_2P'_3P'_4)$ .

10. If  $P_1, P_2, P_3, P_4$  are four noncoplanar points and the lines  $P_1P_2, P'_1P'_2, P''_1P''_2$  have a point in common and

$$\text{Vect}(P_1P_2) = \text{Vect}(P'_1P'_2) + \text{Vect}(P''_1P''_2),$$

then

$$m(P_1P_2P_3P_4) = m(P'_1P'_2P_3P_4) + m(P''_1P''_2P_3P_4).$$

\*11. Study barycentric coördinates and the barycentric calculus for three-dimensional space. Cf. § 51, § 27, and references to Möbius in § 49.

\*12. Study the measure of  $n$ -points in space, generalizing the exercises in § 49.

\*13. Define two ordered tetrads  $ABCD$  and  $A'B'C'D'$  as equivalent provided that (1)  $A = A', B = B', C = C'$ , and the line  $DD'$  is parallel to the plane  $ABC$ , or (2) if there are a finite number of ordered tetrads  $t_1, \dots, t_n$  such that  $ABCD$  is in relation (1) to  $t_1, t_1$  in a like relation to  $t_2, t_3$  to  $t_4, \dots$

and  $t_n$  to  $A'B'C'D'$ . Develop a theory of equivalence as nearly as possible analogous to that of § 48. Show that two tetrads are equivalent in this sense if and only if they are equivalent according to the definition in the text.

\*14. An elation whose center is at infinity and whose plane of fixed points is ordinary is called a *simple shear*. The set of all products of simple shears is the equiaffine group. Develop the theory of the equiaffine group on this basis. Is it possible to generalize § 52 to space?

15. If a plane meets the sides  $A_0A_1, A_1A_2, \dots, A_nA_0$  of a simple polygon  $A_0A_1A_2 \dots A_n$  in points  $B_0, B_1, \dots, B_n$ , respectively,

$$\frac{A_0B_0}{A_1B_0} \cdot \frac{A_1B_1}{A_2B_1} \dots \frac{A_nB_n}{A_0B_n} = 1.$$

16. If a quadric surface (§ 104, Vol. I) meets the lines  $A_0A_1, A_1A_2, \dots, A_nA_0$  respectively in the pairs of points  $B_0C_0, B_1C_1, \dots, B_nC_n$ , respectively,

$$\frac{A_0B_0}{A_1B_0} \cdot \frac{A_0C_0}{A_1C_1} \cdot \frac{A_1B_1}{A_2B_1} \cdot \frac{A_1C_1}{A_2C_1} \dots \frac{A_nB_n}{A_0B_n} \cdot \frac{A_nC_n}{A_0C_n} = 1.$$

\*17. Six points of a plane no three of which are collinear satisfy the following identity:

$$m(123)m(456) - m(124)m(563) + m(125)m(634) - m(126)m(345) \equiv 0.$$

The ratio of any two terms in this sum is a projective invariant. These propositions are given by W. K. Clifford in the Proceedings of the London Mathematical Society, Vol. II (1866), p. 3, as the foundation of the theory of two-dimensional projectivities. Develop the details of the theory outlined by Clifford. Cf. also Möbius, Der barycentrische Calcul, § 221.

### 113. The parabolic metric group. Orthogonal lines and planes.

DEFINITION. Let  $\Sigma_\infty$  be an arbitrary but fixed polar system in the plane at infinity  $\pi_\infty$ . This polar system shall be called the *absolute* or *orthogonal polar system*. The conic whose points lie on their polar lines with respect to  $\Sigma_\infty$  is, if existent, called the *circle at infinity*. The group of all collineations leaving  $\Sigma_\infty$  invariant is called the *parabolic metric group* and its transformations are called *similarity transformations*. Two figures conjugate under this group are said to be *similar*.

DEFINITION. Two ordinary planes or two ordinary lines are *orthogonal* or *perpendicular* if and only if they meet  $\pi_\infty$  in conjugate lines or points of the absolute polar system  $\Sigma_\infty$ . An ordinary line and plane are *orthogonal* or *perpendicular* if and only if they meet  $\pi_\infty$  in a point and line which are polar with regard to  $\Sigma_\infty$ . A line perpendicular to itself, i.e. a line through a point of the circle at infinity, is

called a *minimal* or *isotropic line*. A plane perpendicular to itself, i.e. a plane meeting  $\pi_\infty$  in a tangent to the circle at infinity, is called a *minimal* or *isotropic plane*.

As the analogue of Theorems 2 and 3 we have

**THEOREM 4.** *The similarity transformations which leave an ordinary nonminimal plane  $\pi$  invariant, effect in  $\pi$  the transformations of a parabolic metric group in the Euclidean plane consisting of the ordinary points of  $\pi$ .*

Generalizing Theorem 1, Chap. IV, we have

**THEOREM 5.** *At every point  $O$  of a Euclidean space the correspondence between the lines and their perpendicular planes is a polar system, the projection of  $\Sigma_\infty$ . All the lines through  $O$  perpendicular to a given line are on the plane perpendicular to the given line at  $O$ ; and all the planes through  $O$  perpendicular to a given plane are on the line through  $O$  perpendicular to this plane. If existent, the isotropic lines through a point  $O$  constitute a cone of lines, and the isotropic planes through  $O$  the cone of planes tangent to this cone of lines.*

**COROLLARY 1.** *Two perpendicular nonminimal planes meet in a nonminimal line, and two perpendicular nonminimal lines are parallel to a nonminimal plane.*

**COROLLARY 2.** *If a plane 1 is perpendicular to a plane 2, and 2 is parallel to a plane 3, then 1 is perpendicular to 3. If a plane 1 is perpendicular to a line 2, and 2 is parallel to a line or plane 3, then 1 is perpendicular to 3. If a line 1 is perpendicular to a plane 2, and 2 is parallel to a line or plane 3, then 1 is perpendicular to 3. If a line 1 is perpendicular to a line 2, and 2 is parallel to a line 3, then 1 is perpendicular to 3.*

**THEOREM 6.** *Two nonparallel lines not both parallel to the same minimal plane are met by one and only one line perpendicular to them both; this line is not minimal.*

*Proof.* Let  $A_\infty$  and  $B_\infty$  be the points in which the given lines meet  $\pi_\infty$ . By hypothesis  $A_\infty \neq B_\infty$ , and the line  $A_\infty B_\infty$  is not tangent to the circle at infinity. Let  $C_\infty$  be the pole of the line  $A_\infty B_\infty$  with respect to  $\Sigma_\infty$ . The required common intersecting perpendicular is the line through  $C_\infty$  meeting the two given lines; this line is obviously unique and not minimal.

## EXERCISE

The planes perpendicular to the edges of a tetrahedron at the mid-points of the pairs of vertices meet in a point  $O$ . The line perpendicular to any face of the tetrahedron at the center of the circle through the three vertices in this face passes through  $O$ .

**114. Orthogonal plane reflections.** DEFINITION. A homology of period two whose center,  $P$ , is a point at infinity polar in the absolute polar system to the line at infinity of its plane of fixed points,  $\pi$ , is called an *orthogonal reflection in a plane* or an *orthogonal plane reflection* or a *symmetry with respect to a plane*, and may be denoted by  $\{\pi P\}$ .\* The plane of fixed points is called the *plane of symmetry* of any two figures which correspond in the homology.

Since the center and the line at infinity of the plane of fixed points of an orthogonal reflection in a plane are pole and polar with respect to  $\Sigma_\infty$ , we have

**THEOREM 7.** *An orthogonal reflection in a plane is a transformation of the parabolic metric group.*

By a direct generalization of Theorems 3 and 4, Chap. IV, we obtain the following:

**THEOREM 8.** (1) *If  $\pi$  and  $\rho$  are two parallel nonminimal planes, the product  $\{\rho R\} \cdot \{\pi P\}$  is a translation parallel to any line perpendicular to  $\pi$  and  $\rho$ .* (2) *If  $T$  is a translation parallel to a non-minimal line  $l$ ,  $\pi$  any plane perpendicular to  $l$ , and  $\rho$  the plane perpendicular to  $l$  passing through the mid-point of the point pair in which  $\pi$  and  $T(\pi)$  meet  $l$ , then*

$$T = \{\rho R\} \cdot \{\pi P\};$$

*and if  $\sigma$  is the plane perpendicular to  $l$  passing through the mid-point of the pair in which  $\pi$  and  $T^{-1}(\pi)$  meet  $l$ ,*

$$T = \{\pi P\} \cdot \{\sigma S\}.$$

(3) *A translation parallel to a minimal line  $l$  is a product of four orthogonal plane reflections.*

**THEOREM 9.** *A product  $\Lambda_n \Lambda_{n-1} \dots \Lambda_1$  of orthogonal plane reflections is expressible in the form  $\Lambda'_n \Lambda'_{n-1} \dots \Lambda'_1 T$  or  $T' \Lambda'_n \Lambda'_{n-1} \dots \Lambda'_1$ , where  $\Lambda'_1, \Lambda'_2, \dots, \Lambda'_n$  are orthogonal plane reflections whose planes of*

\*In the rest of this chapter this notation will be used in the sense here defined and not in the more general sense of § 101.

fixed points all contain an arbitrary point  $O$ , and  $T$  and  $T'$  are translations. In case  $O$  is left invariant by  $\Lambda_n \Lambda_{n-1} \dots \Lambda_1$ ,  $T$  and  $T'$  reduce to the identity.

*Proof.* Let  $\Lambda'_i$  ( $i = 1, 2, \dots, n$ ) denote the orthogonal plane reflection whose plane of fixed points is the plane through  $O$  parallel to the plane of fixed points of  $\Lambda_i$ . Then by Theorem 8,  $\Lambda_i \Lambda'_i = T_i$ ,  $T_i$  being a translation. Hence  $\Lambda_i = T_i \Lambda'_i$  and

$$(5) \quad \Lambda_n \Lambda_{n-1} \dots \Lambda_1 = T_n \Lambda'_n T_{n-1} \Lambda'_n \dots T_1 \Lambda'_1.$$

By the generalization to space of Theorem 11, Cor. 2, Chap. III, if  $\Sigma$  is any affine collineation and  $T$  a translation,  $T\Sigma = \Sigma T'$ , where  $T'$  is a translation. By repeated application of this proposition, (5) reduces to

$$\Lambda_n \Lambda_{n-1} \dots \Lambda_1 = \Lambda'_n \Lambda'_{n-1} \dots \Lambda'_1 T = T' \Lambda'_n \Lambda'_{n-1} \dots \Lambda'_1,$$

where  $T$  and  $T'$  are translations.

In case  $O$  is a fixed point for the product  $\Lambda_n \Lambda_{n-1} \dots \Lambda_1$ , since it is also left invariant by each of the reflections  $\Lambda'_i$ , it is left invariant by  $T$  and  $T'$ . Hence in this case  $T$  and  $T'$  reduce to the identity.

**THEOREM 10.** *If  $\Lambda_1, \Lambda_2, \Lambda_3$  are three orthogonal plane reflections whose planes of fixed points meet in a line  $l$ , ordinary or ideal, the product  $\Lambda_3 \Lambda_2 \Lambda_1$  is an orthogonal plane reflection whose plane of fixed points contains  $l$ .*

*Proof.* One of the chief results obtained in Chap. VIII, Vol. I, can be put in the following form: \* If  $T_1, T_2, T_3$  are harmonic homologies leaving a conic invariant and such that their centers are collinear,  $T_3 T_2 T_1$  is a harmonic homology leaving the conic invariant. For by Theorem 19 of that chapter, and its corollary, the product  $T_2 T_1$  is expressible in the form  $T_2 T$ , where  $T$  is a harmonic homology whose center and axis are polar with respect to the conic, the axis being concurrent with those of  $T_1, T_2$ , and  $T_3$ ; and from  $T_2 T_1 = T_2 T$  follows  $T_3 T_2 T_1 = T_3 T_2 T = T$ .

Now if  $\Lambda_1, \Lambda_2, \Lambda_3$  are orthogonal plane reflections whose planes of fixed points meet in an ordinary line  $l$  their centers are collinear. Hence they effect in  $\pi_\infty$  three harmonic homologies whose centers are the poles of their axes with respect to the absolute polar system and whose centers are collinear. Hence  $\Lambda_3 \Lambda_2 \Lambda_1$  effects a harmonic homology in the plane at infinity and its axis,  $m_\infty$ , passes

\* Cf. the fine print in § 108.

through the point at infinity of  $l$ . Since  $l$  and  $m_\infty$  are both lines of fixed points of  $\Lambda_3\Lambda_2\Lambda_1$ , all points of the plane  $\pi$  containing  $l$  and  $m_\infty$  are invariant. Hence  $\Lambda_3\Lambda_2\Lambda_1$  effects a homology having the pole of  $m_\infty$  with respect to  $\Sigma_\infty$  as center. Since this homology is of period two in  $\pi_\infty$  it must be an orthogonal plane reflection.

In case the planes of fixed points of  $\Lambda_1, \Lambda_2, \Lambda_3$  are parallel we have by Theorem 8 (1) that  $\Lambda_3\Lambda_2\Lambda_1$  is a translation parallel to a line perpendicular to these planes, i.e. parallel to a nonminimal line. Hence by Theorem 8 (2) there exists an orthogonal plane reflection,  $\Lambda_4$ , such that

$$\Lambda_2\Lambda_1 = \Lambda_3\Lambda_4$$

or

$$\Lambda_3\Lambda_2\Lambda_1 = \Lambda_4.$$

COROLLARY. If  $\{\lambda_1 L_1\}$  and  $\{\lambda_2 L_2\}$  are two orthogonal plane reflections, and  $\lambda'_1$  is any ordinary nonminimal plane in the same pencil with  $\lambda_1$  and  $\lambda_2$ , there exists a plane  $\lambda'_2$  and points  $L'_1$  and  $L'_2$  such that

$$\{\lambda_2 L_2\} \cdot \{\lambda_1 L_1\} = \{\lambda'_2 L'_2\} \cdot \{\lambda'_1 L'_1\}.$$

*Proof.* By the theorem, if  $L'_1$  is the point at infinity of a line perpendicular to  $\lambda'_1$ , there exists an orthogonal plane reflection  $\{\lambda'_2 L'_2\}$  such that

$$\{\lambda_2 L_2\} \cdot \{\lambda_1 L_1\} \cdot \{\lambda'_1 L'_1\} = \{\lambda'_2 L'_2\},$$

and hence

$$\{\lambda_2 L_2\} \cdot \{\lambda_1 L_1\} = \{\lambda'_2 L'_2\} \cdot \{\lambda'_1 L'_1\}.$$

**115. Displacements and symmetries. Congruence.** We may now generalize directly from § 57, Chap. IV:

DEFINITION. A product of an even number of orthogonal plane reflections is called a *displacement* or *rigid motion*. A product of an odd number of orthogonal plane reflections is called a *symmetry*.

THEOREM 11. *The set of all displacements and symmetries is a self-conjugate subgroup of the parabolic metric group and contains the set of all displacements as a self-conjugate subgroup.*

DEFINITION. Two figures such that one can be transformed into the other by a displacement are said to be *congruent*. Two figures such that one can be transformed into the other by a symmetry are said to be *symmetric*.

THEOREM 12. *If a figure  $F_1$  is congruent to a figure  $F_2$ , and  $F_2$  to a figure  $F_3$ , then  $F_1$  is congruent to  $F_3$ . If  $F_1$  is symmetric with  $F_2$ , and  $F_2$  with  $F_3$ , then  $F_1$  is congruent to  $F_3$ . If  $F_1$  is congruent to  $F_2$ , and  $F_2$  symmetric with  $F_3$ , then  $F_1$  is symmetric with  $F_3$ .*

THEOREM 13. *Any displacement leaving an ordinary point  $O$  invariant is a product of two orthogonal plane reflections whose planes of fixed points contain  $O$ .*

*Proof.* Consider a product of four orthogonal plane reflections, whose planes of fixed points pass through  $O$ .

$$\Gamma = \{\lambda_4 L_4\} \cdot \{\lambda_3 L_3\} \cdot \{\lambda_2 L_2\} \cdot \{\lambda_1 L_1\}.$$

Let  $l$  be the line of intersection of  $\lambda_1$  and  $\lambda_3$ ,  $m$  that of  $\lambda_3$  and  $\lambda_4$ , and let  $\lambda$  be a plane containing  $l$  and  $m$ , where in case  $l = m$ ,  $\lambda$  is chosen so as not to be minimal. If  $\lambda$  is nonminimal, by the corollary of Theorem 10 there exist orthogonal plane reflections  $\{\mu M\}$ ,  $\{\nu N\}$  such that

$$\{\lambda_3 L_3\} \cdot \{\lambda_1 L_1\} = \{\lambda L\} \cdot \{\mu M\},$$

$$\text{and} \quad \{\lambda_4 L_4\} \cdot \{\lambda_3 L_3\} = \{\nu N\} \cdot \{\lambda L\}.$$

$$\text{Hence} \quad \Gamma = \{\nu N\} \cdot \{\lambda L\} \cdot \{\lambda L\} \cdot \{\mu M\} = \{\nu N\} \cdot \{\mu M\}.$$

In case  $\lambda$  is minimal\*  $\{\lambda_1 L_1\}$  transforms  $\lambda$  to the other minimal plane through  $l$  (i.e. the other plane containing  $l$  and a tangent to the circle at infinity), and  $\{\lambda_3 L_3\}$  transforms this plane back to  $\lambda$ . In like manner the product  $\{\lambda_4 L_4\} \cdot \{\lambda_3 L_3\}$  leaves  $\lambda$  invariant. Hence  $\lambda$  is left invariant by  $\Gamma$ . On the other hand the line  $l$  is obviously not left invariant by  $\Gamma$ , and therefore  $\Gamma$  does not leave all points at infinity invariant. Hence  $\Gamma$  leaves at most two tangents to the circle at infinity invariant, and thus leaves at most two minimal planes through  $O$  invariant. Let  $\lambda'_2$  be any plane of the bundle containing  $\lambda_2$  and  $\lambda_3$  which does not meet  $\lambda_1$  in a line of an invariant minimal plane of  $\Gamma$ . By the corollary of Theorem 10 there exists a plane  $\lambda'_3$  and points  $L'_2$  and  $L'_3$  such that

$$\{\lambda_3 L_3\} \cdot \{\lambda_2 L_2\} = \{\lambda'_3 L'_3\} \cdot \{\lambda'_2 L'_2\},$$

and hence such that

$$\Gamma = \{\lambda_4 L_4\} \cdot \{\lambda'_3 L'_3\} \cdot \{\lambda'_2 L'_2\} \cdot \{\lambda_1 L_1\}.$$

Now let  $l$  be the line of intersection of  $\lambda_1$  and  $\lambda'_2$ ,  $m$  that of  $\lambda'_3$  and  $\lambda_4$ , and  $\lambda'$  the plane containing  $l$  and  $m$ . If  $\lambda'$  were minimal it would, as argued above for  $\lambda$ , be invariant under  $\Gamma$ , whereas  $\lambda'_2$  was so chosen that  $l$  cannot be in such a plane. Hence the argument in the previous paragraph can be applied to the last expression obtained for  $\Gamma$ .

\* This case obviously does not arise in the real Euclidean geometry (§ 116), so that this paragraph may be omitted if one is interested only in that case. It is needed, however, in complex geometry.

Thus, in any case, a product of four orthogonal plane reflections whose planes of fixed points pass through  $O$  reduces to a product of two such reflections. By Theorem 9 any displacement leaving  $O$  invariant is a product of an even number, say  $2n$ , of orthogonal reflections in planes through  $O$ . This may be reduced to a product of two orthogonal reflections in planes through  $O$  by  $n-1$  applications of the result proved above.

COROLLARY. *An orthogonal plane reflection is not a displacement.*

*Proof.* Let  $O$  be a point of the plane of fixed points of an orthogonal plane reflection  $\Lambda$ . If  $\Lambda$  were a displacement it would, by the theorem, be a product of two orthogonal plane reflections containing  $O$  and hence could only have a single line of fixed points.

DEFINITION. A displacement which is a product of two orthogonal plane reflections whose planes of fixed points have an ordinary line  $l$  in common is called a *rotation about  $l$* , and  $l$  is called the *axis* of the rotation. If the axis is a minimal line the rotation is said to be *isotropic* or *minimal*.

THEOREM 14. *The product of two orthogonal reflections in perpendicular planes is a rotation of period two. It transforms every point  $P$  not on its axis to a point  $P'$  such that the axis is perpendicular to the line  $PP'$  at the mid-point of the pair  $PP'$ . It leaves invariant the points of its axis and the points in which any plane perpendicular to its axis meets the plane at infinity. Its axis cannot be a minimal line.*

*Proof.* Consider any plane  $\pi$  perpendicular to the planes of fixed points of the two orthogonal plane reflections  $\Lambda_1$  and  $\Lambda_2$ . By the first corollary of Theorem 5 the axis of  $\Lambda_2\Lambda_1$  is nonminimal and hence  $\pi$  is nonminimal. In  $\pi$  the transformations effected by  $\Lambda_1$  and  $\Lambda_2$  are orthogonal line reflections in the sense of Chap. IV, and their product is a point reflection (Theorem 5, Chap. IV) in the plane. From this the theorem follows in an obvious way.

DEFINITION. The product of two orthogonal reflections in perpendicular planes is called an *involutoric rotation* or an *orthogonal line reflection* or a *half turn*. If  $l$  is its axis and  $l'$  the polar with respect to  $\Sigma_\infty$  of the point at infinity of  $l$ , it may be denoted by  $\{U\}$ .\*

\* In the rest of this chapter this notation will be used in the sense here defined and not in the more general sense of § 101.

**THEOREM 15. DEFINITION.** *The product of the orthogonal plane reflections in three perpendicular planes is a transformation carrying each point  $P$  to a point  $P'$  such that the point  $O$  of intersection of the three planes is the mid-point of the pair  $PP'$ . A transformation of this sort is called a point reflection or symmetry with respect to the point  $O$  as center. It is not a displacement. The points  $P$  and  $P'$  are said to be symmetric with respect to  $O$ .*

*Proof.* In the plane at infinity the three orthogonal plane reflections effect the three harmonic homologies whose centers and axes are the vertices and respectively opposite sides of a triangle. The product therefore leaves all points at infinity invariant. It also leaves  $O$  invariant and is evidently of period two on the line of intersection of any two of the planes of fixed points of the orthogonal plane reflections. Hence it is a homology of period two with  $O$  as center and  $\pi_\infty$  as plane of fixed points. It is not a displacement, since by Theorem 13 a displacement leaving  $O$  invariant would have a line of fixed points passing through  $O$ .

**THEOREM 16.** *The transformations effected in a nonminimal plane  $\pi$  by the displacements leaving  $\pi$  invariant constitute the group of displacements and symmetries of the parabolic metric group whose absolute involution is that determined by  $\Sigma_\infty$  on the line at infinity of  $\pi$ .*

*Proof.* Let  $\Gamma$  be any displacement leaving  $\pi$  invariant,  $O$  an arbitrary point of  $\pi$ , and  $T$  the translation carrying  $O$  to  $\Gamma(O)$ . Then  $T^{-1}\Gamma(O) = O$ , and hence, by Theorem 13,  $T^{-1}\Gamma$  is a rotation. Moreover,  $T^{-1}\Gamma$  leaves  $\pi$  invariant.

It is obvious from the definition of a rotation that it can leave  $\pi$  invariant only in case its axis is perpendicular to  $\pi$  or in case it is of period two and its axis is a line of  $\pi$ . If  $T^{-1}\Gamma$  falls under the first of these cases, it effects a rotation in  $\pi$  according to the definition of rotation in Chap. IV, and thus  $\Gamma$  effects a displacement in  $\pi$ . If  $T^{-1}\Gamma$  falls under the second of these cases it effects, and therefore  $\Gamma$  also effects, a symmetry in  $\pi$  according to the definition in Chap. IV.

**COROLLARY 1.** *The transformations effected in a nonminimal plane  $\pi$  by the displacements and symmetries leaving  $\pi$  invariant constitute the group of displacements and symmetries of the parabolic metric group whose absolute involution is that determined by  $\Sigma_\infty$  on the line at infinity of  $\pi$ .*

COROLLARY 2. *If  $O$  is an arbitrary point, any displacement  $\Gamma$  is expressible in the forms*

$$\Gamma = TP \quad \text{and} \quad \Gamma = P'T',$$

*where  $T, T'$  are translations and  $P, P'$  rotations leaving  $O$  invariant.*

*Proof.* As in the proof of the theorem above, let  $T$  be the translation carrying  $O$  to  $\Gamma(O)$ . Then  $T^{-1}\Gamma(O) = O$  and hence, by Theorem 13,  $T^{-1}\Gamma$  is a rotation,  $P$ . Hence  $\Gamma = TP$ . If  $T'$  is the translation carrying  $O$  to  $\Gamma^{-1}(O)$ , it follows in like manner that  $\Gamma T'(O)$  is a rotation  $P'$  and hence that  $\Gamma = P'T'^{-1}$ .

COROLLARY 3. *The transformations effected on a nonminimal line  $p$  by the displacements leaving  $p$  invariant constitute the group composed of all parabolic transformations and involutions leaving the point at infinity of  $p$  invariant.*

### EXERCISES

1. Two point pairs are congruent if they are symmetric.
2. The set of all point reflections and translations forms a group which, unlike the analogous group in the plane (§ 45), is not a subgroup of the group of displacements. The product of two point reflections is a translation, and any translation is expressible as a product of two point reflections, one of which is arbitrary.
3. Study the theory of congruence in a minimal plane.
4. A rotation leaves no point invariant which is not on its axis. It leaves invariant all planes perpendicular to its axis and no others unless it is of period two, when it is an orthogonal line reflection.

**116. Euclidean geometry of three dimensions.** The last theorem may be regarded as the fundamental theorem of the parabolic metric geometry in space, for by means of it all the results of the two-dimensional parabolic metric geometry become immediately applicable.

Suppose now that we consider a three-space satisfying Assumptions A, E, H, C, R (or A, E, K), i.e. a real projective space. Suppose also that  $\Sigma_{\infty}$  be taken to be an elliptic polar system,\* i.e. the polar system of an imaginary ellipse (§ 79). Then in any plane the parabolic metric geometry reduces to the Euclidean geometry and the displacements which leave this plane invariant are Euclidean displacements.

\* The existence and properties of an elliptic polar system may be determined without recourse to imaginaries (in fact, on the basis A, E, P, S), as in § 89.

A set of assumptions for the Euclidean geometry of three dimensions is composed of I-XVI, given in §§ 29 and 66. We have seen in § 29 that I-IX are satisfied by a Euclidean space of three dimensions. Assumption XI is a consequence of Theorem 12, and Assumptions X, XII-XVI of Theorems 11 and 16. Hence *in a real three-space, if  $\Sigma_\infty$  is an elliptic polar system the parabolic metric geometry is the Euclidean geometry.*

The general remarks in § 66 are applicable to the three-dimensional case as well as to the two-dimensional one.

It was stated in § 66 that the congruence assumptions are no longer strictly independent when a full continuity assumption is added, because by introducing ideal elements and an arbitrary  $\Sigma_\infty$  (as in the present chapter) a relation of congruence may be defined for which the statements in X-XVI are theorems which can easily be proved. This view is not accepted by certain well-known mathematicians, who hold that the arbitrariness in the definition of the absolute involution somehow conceals a new assumption.\* It may, therefore, be well to restate the matter here.†

Assumptions I-IX, XVII are categorical for the Euclidean space; i.e. if two sets of objects  $[P]$  and  $[Q]$  satisfy the conditions laid down for points in the assumptions, there is a one-to-one reciprocal correspondence between  $[P]$  and  $[Q]$  such that the subsets called lines of  $[P]$  correspond to the subsets called lines of  $[Q]$ . Thus the internal structure of a Euclidean space is fully determined by Assumptions I-IX, XVII. The group leaving invariant the relations described in these assumptions is the affine group, and all the theorems of the affine geometry are consequences of these assumptions. The latter may therefore be characterized as the assumptions of affine geometry.

Among the theorems of the affine geometry is one which states that there is an infinity of subgroups, each one conjugate to all the rest and such that the set of theorems belonging to it constitutes the Euclidean geometry. Each of these groups is capable of being called the Euclidean group, and there is no theorem about one of them which is not true about all of them. The set of theorems stating relations invariant under any one of these groups is the Euclidean geometry. This set of theorems is the same whichever Euclidean group be selected, i.e. *the Euclidean geometry is a unique body of theorems.*

Each Euclidean group has a self-conjugate subgroup of displacements which defines a relation called congruence having the properties stated in

\* Cf. the remarks on a paper by the writer in the article by Enriques, *Encyclopédie des Sc. Math.* III 1, § 12.

† This discussion should be read in connection with the remarks on foundations of geometry in the introduction to Vol. I and in § 13 of this volume; also in connection with the remarks on the geometry corresponding to a group, §§ 84, 89, 110.

Assumptions X-XVI. Moreover, any relation which satisfies these assumptions is associated with a group of displacements which is self-conjugate under a Euclidean group.

Thus Assumptions X-XVI characterize the relation of congruence as completely as possible, i.e. any relation satisfying these assumptions must be that determined by one of the infinitely many groups of displacements. The set of theorems about congruence is unique and is the Euclidean geometry.

The relation between the affine geometry and the Euclidean geometry is analogous to that between the Euclidean geometry and the geometry belonging to any non-self-conjugate subgroup of a Euclidean group. Consider, for example, the subgroup obtained by leaving a particular point  $O$  invariant. A relation which is left invariant by this group may be defined as follows:

DEFINITION. A point  $P$  is *nearer* than a point  $Q$  if and only if  $\text{Dist}(OP) < \text{Dist}(OQ)$ .  $P$  and  $Q$  are *equally near* if  $\text{Dist}(OP) = \text{Dist}(OQ)$ .

There is an element of arbitrary choice in this definition, just as there is in the choice of an absolute involution to define the notion of congruence. Moreover, the geometry of *nearness* is just as truly a geometry as is the Euclidean geometry.\* It would be easy to put down a set of assumptions (XVIII-N) in terms of *near* regarded as an undefined relation, which would state the abstract properties of this relation, just as X-XVI state the abstract properties of congruence.

Another non-self-conjugate subgroup of the Euclidean group which gives rise to an interesting geometry is the group leaving invariant a line and a plane on this line. In terms of this group the notions of *forward* and *backward* and *up* and *down* can be defined, and the geometry corresponding to this group is a set of propositions embodying the abstract theory of this set of relations.

It is a theorem of Euclidean geometry that the Euclidean group has subgroups with the properties involved in these geometries, just as it is a theorem of affine geometry that the affine group has Euclidean subgroups and a theorem of projective geometry that the projective group has affine subgroups.

Assumptions I-IX, XVII have a different rôle from X-XVI or XVIII-N, in that they determine the set of objects (points and lines, etc.) which are presupposed by all the other assumptions. The choice of these assumptions is logically arbitrary. The choice of such sets of "assumptions" as X-XVI is not arbitrary; it must correspond to a properly chosen group of permutations of the objects determined by I-IX, XVII. When independence proofs are given for Assumptions X-XVI, it is done by giving new interpretations to the term "congruence," not to "point," or "line."

\* It is even possible to give a psychological significance to this geometry. The normal individual has a certain place, say home, in terms of nearness to which other places are thought of; here  $O$  is the central point of home. In astronomy stars are regarded as near or the contrary, according to their distance from the sun; here  $O$  is the center of the sun.

The point of view of the writer is that if X-XVI or XVIII-N are to be regarded as independent assumptions, their independence is of a lower grade than that of I-IX, XVII. They constitute a definition by postulates of a relation (congruence or nearness) among objects (points, lines, etc.) already fully determined. Their significance is that they characterize that subset of the theorems deducible from I-IX, XVII which corresponds to any Euclidean group and which therefore is the Euclidean geometry.

### EXERCISES

\*1. Develop the geometry corresponding to some non-self-conjugate subgroup of the Euclidean group. Determine a set of mutually independent assumptions characterizing this geometry.

2. The identity is the only transformation of the Euclidean group which leaves fixed two points  $A$  and  $B$  and two rays (cf. definition in § 16)  $AC$  and  $AD$  orthogonal to each other and to the line  $AB$ .

3. If  $a$  and  $b$  are any two rays having a common origin,  $O$ , and on different lines, there is a unique orthogonal line reflection and a unique orthogonal plane reflection transforming  $a$  into  $b$ .

4. If  $A, B, C, D$  are any four points no three of which are collinear, there exists a unique rotation leaving the line  $AB$  invariant and transforming  $C$  into a point of the plane  $ABD$  on the same side of  $AB$  with  $D$ .

5. Any transformation of the Euclidean group which leaves a line pointwise invariant and preserves sense is a rotation.

6. Any transformation of the Euclidean group which leaves a line pointwise invariant and alters sense is an orthogonal reflection in a plane containing this line.

7. There is one and only one displacement which transforms three mutually orthogonal rays  $OA, OB, OC$  into three mutually orthogonal rays  $O'A', O'B', O'C'$ , provided that  $S(OABC) = S(O'A'B'C')$ .

\*117. **Generalization to  $n$  dimensions.** The discussion of the Euclidean and affine geometries in §§ 111-116 is so arranged that it will generalize at once to any number of dimensions. It is recommended to the reader to carry out this generalization in detail, at least in the four-dimensional case.

The elementary theorems of alignment for four dimensions are given in § 12, Vol. I. The definition of a Euclidean four-space is given in § 28, Vol. II. The generalization of § 111 is obvious on comparing these two sections. A four-dimensional translation may be defined as a projective collineation leaving invariant all points of the three-space at infinity and also all lines through one of these points. The generalization of § 112 then follows at once.

A three-dimensional polar system may be defined as the polar system of a proper or improper regulus (Chap. XI, Vol. I; cf. also §§ 100–108, Vol. II), or it may be studied *ab initio* by generalizing Chap. X, Vol. I. The notion of perpendicular lines, planes, and three-spaces then follows at once and also the theorems generalizing those of § 113. An orthogonal reflection in an  $S_3$  is next defined as a projective collineation of period two, leaving invariant a point  $P$  at infinity and each point of a three-space whose plane at infinity is polar to  $P$  in the absolute polar system. All the theorems of §§ 114, 115 up to Theorem 13 then generalize at once. Theorems 13–15 must be modified, in view of the fact that there are more than one type of four-dimensional displacements leaving a point invariant. Theorem 16 holds unchanged.

Finally, it can be proved as in § 116 that in case of a real space and an elliptic polar system the parabolic metric geometry satisfies a set of axioms for Euclidean geometry of four dimensions. This set differs from the one used above, in that VIII is replaced by

VIII'. *If  $A, B, C, D$  are four noncoplanar points, there exists a point  $E$  not in the same  $S_3$  with  $A, B, C, D$ , and such that every point is in the same  $S_4$  with  $A, B, C, D, E$ .*

The introduction of nonhomogeneous coordinates in a space of  $n$  dimensions may be made by direct generalizations of § 69, Vol. I. The formulas for the affine group, the group of translations, the Euclidean group, and the group of displacements are then easily seen to be identical with those given in the sections below, except that the summations from 0 or 1 to 3 must in each case be replaced by summations from 0 or 1 to  $n$ .

**118. Equations of the affine and Euclidean groups.** With respect to a nonhomogeneous coordinate system in which  $\pi_\infty$  is the singular plane, the affine group is evidently the set of all projectivities of the form

$$(6) \quad \begin{aligned} x' &= a_{11}x + a_{12}y + a_{13}z + a_{10}, \\ y' &= a_{21}x + a_{22}y + a_{23}z + a_{20}, \\ z' &= a_{31}x + a_{32}y + a_{33}z + a_{30}, \end{aligned}$$

where

$$\Delta \equiv \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0,$$

and the variables and coefficients are elements of the geometric number system.

In the system of homogeneous plane coördinates in which the plane at infinity is represented by  $[1, 0, 0, 0]$ , this group takes the form

$$(7) \quad \begin{aligned} u'_0 &= b_{00}u_0 + b_{01}u_1 + b_{02}u_2 + b_{03}u_3, \\ u'_1 &= b_{11}u_1 + b_{12}u_2 + b_{13}u_3, \\ u'_2 &= b_{21}u_1 + b_{22}u_2 + b_{23}u_3, \\ u'_3 &= b_{31}u_1 + b_{32}u_2 + b_{33}u_3. \end{aligned}$$

In an ordered space the affine group has a subgroup consisting of all transformations for which  $\Delta$  is positive. This group has been considered in § 31. It also has obvious subgroups consisting of all transformations for which  $\Delta^2 = 1$  and for which  $\Delta = 1$ .

The equations of a translation parallel to the  $x$ -axis are evidently  $x' = x + a$ ,  $y' = y$ ,  $z' = z$ , and similar expressions represent a translation parallel to any other axis. Hence by the corollary of Theorem 3 the equations of the group of translations are

$$(8) \quad \begin{aligned} x' &= x + a, \\ y' &= y + b, \\ z' &= z + c. \end{aligned}$$

If the coördinates are so chosen that the planes  $\bar{x} = 0$ ,  $\bar{y} = 0$ ,  $\bar{z} = 0$  are mutually orthogonal, the equations of the circle at infinity in terms of the corresponding homogeneous coördinates are

$$a\bar{x}_1^2 + b\bar{x}_2^2 + c\bar{x}_3^2 = 0, \quad \bar{x}_0 = 0.$$

These are reducible by the transformation

$$(9) \quad x_0 = \bar{x}_0, \quad x_1 = \sqrt{a}\bar{x}_1, \quad x_2 = \sqrt{b}\bar{x}_2, \quad x_3 = \sqrt{c}\bar{x}_3$$

to

$$(10) \quad x_1^2 + x_2^2 + x_3^2 = 0, \quad x_0 = 0.$$

In the real geometry  $a, b, c$  are positive if the polar system is elliptic (§ 85), and the transformation (9) carries real points to real points. The formulas (9) are the only ones in the present section in which irrational expressions appear. Hence the rest of the discussion holds for any space satisfying Assumptions A, E, P,  $H_0$ . In any such space it is easily seen that (10) represents a conic whose polar system may be taken as  $\Sigma_\infty$ , but it does not follow, as in the real case, that any improper conic can be reduced to this form. The situation here is entirely analogous to that obtaining in § 62.

In the three-dimensional homogeneous plane coördinates,  $\pi_\infty$  and the planes tangent to the circle at infinity (10) satisfy the equation

$$(11) \quad u_1^2 + u_2^2 + u_3^2 = 0.$$

Any plane

$$(12) \quad u_0 + u_1 x' + u_2 y' + u_3 z' = 0$$

is the transform under a collineation of the form (6) of the plane

$$(13) \quad (u_0 + a_{10}u_1 + a_{20}u_2 + a_{30}u_3) + (a_{11}u_1 + a_{21}u_2 + a_{31}u_3)x \\ + (a_{12}u_1 + a_{22}u_2 + a_{32}u_3)y + (a_{13}u_1 + a_{23}u_2 + a_{33}u_3)z = 0.$$

Hence (11) is the transform of

$$(14) \quad (a_{11}^2 + a_{12}^2 + a_{13}^2)u_1^2 + (a_{21}^2 + a_{22}^2 + a_{23}^2)u_2^2 + (a_{31}^2 + a_{32}^2 + a_{33}^2)u_3^2 \\ + 2(a_{11}a_{21} + a_{12}a_{22} + a_{13}a_{23})u_1u_2 + 2(a_{11}a_{31} + a_{12}a_{32} + a_{13}a_{33})u_1u_3 \\ + 2(a_{21}a_{31} + a_{22}a_{32} + a_{23}a_{33})u_2u_3 = 0.$$

In order that (11) and (14) shall represent the same locus, we must have

$$(15) \quad a_{11}^2 + a_{12}^2 + a_{13}^2 = a_{21}^2 + a_{22}^2 + a_{23}^2 = a_{31}^2 + a_{32}^2 + a_{33}^2, \\ a_{11}a_{21} + a_{12}a_{22} + a_{13}a_{23} = a_{11}a_{31} + a_{12}a_{32} + a_{13}a_{33} \\ = a_{21}a_{31} + a_{22}a_{32} + a_{23}a_{33} = 0.$$

These conditions are equivalent to the equation (cf. § 95, Chap. X, Vol. I)

$$(16) \quad \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \cdot \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} = \begin{pmatrix} \rho & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \rho \end{pmatrix},$$

where  $\rho = a_{11}^2 + a_{12}^2 + a_{13}^2$ .

If the matrix  $(a_{11} a_{22} a_{33}) = A$  be interpreted as the matrix of a planar collineation, as in § 95, Vol. I, this states that the product of the collineation by the collineation represented by the transposed matrix is the identity. Hence the product of the two matrices in the reverse order is a matrix representing the identity. This means that

$$a_{11}^2 + a_{21}^2 + a_{31}^2 = a_{12}^2 + a_{22}^2 + a_{32}^2 = a_{13}^2 + a_{23}^2 + a_{33}^2,$$

and

$$a_{11}a_{12} + a_{21}a_{22} + a_{31}a_{32} = a_{11}a_{13} + a_{21}a_{23} + a_{31}a_{33} \\ = a_{12}a_{13} + a_{22}a_{23} + a_{32}a_{33} = 0.$$

Since the determinants of a matrix and of its transposed matrix are equal, we have

$$\Delta^2 = \rho^3 = (a_{11}^2 + a_{12}^2 + a_{13}^2)^3 = (a_{11}^2 + a_{21}^2 + a_{31}^2)^3.$$

DEFINITION. A matrix such that its product by a given matrix  $A$  is the identical matrix (§ 95, Vol. I) is called the *inverse* of  $A$  and is denoted by  $A^{-1}$ . A square matrix whose transposed matrix is equal to its inverse is called *orthogonal*. A linear transformation,

$$(17) \quad \begin{aligned} x' &= a_{11}x + a_{12}y + a_{13}z, \\ y' &= a_{21}x + a_{22}y + a_{23}z, \\ z' &= a_{31}x + a_{32}y + a_{33}z, \end{aligned}$$

whose matrix  $(a_{11} a_{22} a_{33})$  is orthogonal, is said to be *orthogonal*.

The results at which we have arrived may now be expressed in part as follows:

THEOREM 17. *The transformations of the parabolic metric group can be written in the form*

$$(18) \quad \begin{aligned} x' &= \rho(a_{11}x + a_{12}y + a_{13}z + k_1), \\ y' &= \rho(a_{21}x + a_{22}y + a_{23}z + k_2), \\ z' &= \rho(a_{31}x + a_{32}y + a_{33}z + k_3), \end{aligned}$$

where the matrix  $(a_{11} a_{22} a_{33})$  is orthogonal.

From the form of these equations we obtain the following corollaries:

COROLLARY 1. *Any transformation (18) of the Euclidean group is the product of an orthogonal transformation, a translation, and a homology of the form*

$$(19) \quad \begin{aligned} x' &= \rho x, \\ y' &= \rho y, \\ z' &= \rho z. \end{aligned}$$

COROLLARY 2. *A homology (19) is commutative with any collineation leaving the origin invariant.*

Since an orthogonal matrix is any matrix satisfying (16) with  $\rho = 1$ , we have

COROLLARY 3. *The product of two orthogonal transformations is orthogonal. The determinant of an orthogonal transformation is +1 or -1.*

In view of the formula for the inverse of a matrix (§ 95, Vol I), we have

COROLLARY 4. *A matrix  $(a_{11} a_{22} a_{33})$  is orthogonal if and only if*

$$(20) \quad A_{ij} = \Delta a_{ji}, \quad (i = 1, 2, 3; j = 1, 2, 3)$$

where  $\Delta$  is the determinant of the matrix and  $A_{ij}$  the cofactor of  $a_{ij}$ .

The matrix of an orthogonal transformation of period two is its own inverse and hence its own transposed. Hence

COROLLARY 5. *An orthogonal transformation is of period two if and only if  $a_{ij} = a_{ji}$ .*

The double points of any orthogonal transformation (17) must satisfy the equations

$$(21) \quad \begin{aligned} (a_{11} - 1)x + a_{12}y + a_{13}z &= 0, \\ a_{21}x + (a_{22} - 1)y + a_{23}z &= 0, \\ a_{31}x + a_{32}y + (a_{33} - 1)z &= 0. \end{aligned}$$

The determinant of the coefficients of these equations is

$$D_1 = \Delta - (A_{11} + A_{22} + A_{33}) + (a_{11} + a_{22} + a_{33}) - 1.$$

But since the transformation is orthogonal,  $A_{ii} = \Delta a_{ii}$ . Hence the determinant of (21) reduces to

$$D_1 = (1 - \Delta)(a_{11} + a_{22} + a_{33} - 1).$$

Another determinant which is of importance in the theory of orthogonal transformations is that of the equations

$$(22) \quad \begin{aligned} (a_{11} + 1)x + a_{12}y + a_{13}z &= 0, \\ a_{21}x + (a_{22} + 1)y + a_{23}z &= 0, \\ a_{31}x + a_{32}y + (a_{33} + 1)z &= 0. \end{aligned}$$

Any point satisfying these equations is transformed into its symmetric point with respect to the origin. The orthogonal transformation therefore transforms the line joining these points into itself and effects an involution with the origin as center on this line. The determinant of the equations (22) is

$$D_2 = \Delta + (A_{11} + A_{22} + A_{33}) + (a_{11} + a_{22} + a_{33}) + 1,$$

which reduces to

$$D_2 = (1 + \Delta)(a_{11} + a_{22} + a_{33} + 1).$$

Let us now consider an orthogonal transformation (17) which we shall denote by  $\Sigma$ . If  $\Delta = -1$  for  $\Sigma$ ,  $D_2 = 0$ , and hence there is at least one point which is carried by  $\Sigma$  into its symmetric point with respect to the origin. The plane through the origin perpendicular to the line joining these points is left invariant by  $\Sigma$ . On the other hand,  $D_1 \neq 0$  unless

$$(23) \quad a_{11} + a_{22} + a_{33} = 1,$$

and hence  $\Sigma$  leaves no other point than the origin invariant unless (23) is satisfied. Suppose now that (23) is satisfied. A cofactor of an element  $a_{ii}$  of the main diagonal of  $D_1$  is

$$A_{ii} - (a_{jj} + a_{kk}) + 1,$$

where  $i \neq j \neq k \neq i$ . By (20) this reduces to

$$-(a_{11} + a_{22} + a_{33}) + 1,$$

which vanishes. The cofactor of an element  $a_{ij}$  ( $i \neq j$ ) of  $D_1$  is

$$A_{ij} + a_{ji},$$

and by (20) this vanishes when  $\Delta = -1$ . Thus we have that if  $\Delta = -1$  and (23) is satisfied,  $\Sigma$  has a plane of fixed points. Since it transforms one point into its symmetric point with respect to the origin, it must be an orthogonal plane reflection. Thus we have proved

**THEOREM 18.** *An orthogonal transformation for which  $\Delta = -1$  always has an invariant plane. It either leaves no point except the origin invariant or it is an orthogonal plane reflection. The latter case occurs if and only if  $a_{11} + a_{22} + a_{33} = 1$ .*

By comparison with Corollary 5 above we have

**COROLLARY.** *An orthogonal transformation for which  $\Delta = -1$  is an orthogonal plane reflection if and only if  $a_{12} = a_{21}$ ,  $a_{23} = a_{32}$ , and  $a_{13} = a_{31}$ .*

Let us now consider an orthogonal transformation  $\Sigma$  for which  $\Delta = 1$ . In this case  $D_1 = 0$ , and hence there is always a line of fixed points passing through the origin. Let  $\Lambda_1$  be an orthogonal plane reflection containing a line of fixed points of  $\Sigma$ . Then  $\Sigma\Lambda_1$  is an orthogonal transformation for which  $\Delta = -1$  and for which there are other fixed points than the origin. By the last theorem, therefore, it is an orthogonal plane reflection  $\Lambda_2$ . From  $\Sigma\Lambda_1 = \Lambda_2$  follows  $\Sigma = \Lambda_2\Lambda_1$ . We therefore have

**THEOREM 19.** *An orthogonal transformation for which  $\Delta = 1$  is a rotation.*

**COROLLARY 1.** *An orthogonal transformation for which  $\Delta = -1$  is a symmetry.*

Any transformation (18) for which  $\rho = 1$  is a product of an orthogonal transformation and a translation. It is therefore either a displacement or a symmetry. By Theorem 16, Cor. 1, a homology (19) for which  $\rho^2 \neq 1$  is not a displacement or a symmetry. Hence we have

**COROLLARY 2.** *The subgroup of (18) for which  $\rho = 1$  and  $\Delta = 1$  is the group of displacements.*

**COROLLARY 3.** *The subgroup of (18) for which  $\rho = 1$  and  $\Delta^2 = 1$  is the group of displacements and symmetries.*

The coördinate system which has been employed above is such that the planes  $x = 0, y = 0, z = 0$  are mutually orthogonal. Moreover, the displacement

$$x' = y, \quad y' = z, \quad z' = x,$$

leaves  $(0, 0, 0)$  invariant and transforms  $(1, 0, 0)$  to  $(0, 1, 0)$  and  $(0, 1, 0)$  to  $(0, 0, 1)$ . Hence the pairs  $(0, 0, 0) (1, 0, 0)$ ,  $(0, 0, 0) (0, 1, 0)$ , and  $(0, 0, 0) (0, 0, 1)$  are congruent. Coördinates satisfying these conditions are said to be *rectangular*.

### EXERCISES

1. The group of displacements and symmetries leaves the quadratic form

$$u_1^2 + u_2^2 + u_3^2$$

absolutely invariant.

2. Two point pairs  $(a, b, c)(a', b', c')$  and  $(x, y, z)(x', y', z')$  are congruent if and only if  $(a - a')^2 + (b - b')^2 + (c - c')^2 = (x - x')^2 + (y - y')^2 + (z - z')^2$ .

3. Two planes

$$u_1x + u_2y + u_3z + u_0 = 0,$$

$$v_1x + v_2y + v_3z + v_0 = 0$$

are orthogonal if and only if  $u_1v_1 + u_2v_2 + u_3v_3 = 0$ .

4. Three planes

$$u_{i1}x + u_{i2}y + u_{i3}z + u_{i0} = 0, \quad (i = 1, 2, 3)$$

the coefficients being such that  $u_{i1}^2 + u_{i2}^2 + u_{i3}^2 = 1, \quad (i = 1, 2, 3)$

are mutually perpendicular if and only if the matrix  $(u_{i1}u_{i2}u_{i3})$  is orthogonal.

5. The three ordered triads of numbers  $(a_{i1}, a_{i2}, a_{i3}), i = 1, 2, 3$ , are direction cosines of mutually perpendicular vectors if and only if the matrix  $(a_{i1}a_{i2}a_{i3})$  is orthogonal.

**119. Distance, area, volume, angular measure.** The definition (§ 67) of distance between two points extends without modification to the three-dimensional case. The distance between a point  $O$  and a plane  $\pi$  is the distance between  $O$  and the point  $P$  in which

$\pi$  is met by the line through  $O$  perpendicular to  $\pi$ . The distance between two lines  $l_1, l_2$  is  $\text{Dist}(P_1, P_2)$ , where  $P_1$  and  $P_2$  are the points in which the common intersecting perpendicular line meets  $l_1$  and  $l_2$  respectively.

If the notion of equivalence of ordered point triads (§ 112) be extended by regarding two ordered triads as equivalent whenever they are congruent, it is obvious that any triad is equivalent to triads in any plane whatever and not merely, as in § 112, to triads in a system of parallel planes. Moreover, if  $ABC$  are noncollinear points such that  $AB$  is congruent to  $AC$ , the ordered triad  $ABC$  is congruent and therefore equivalent to the ordered triad  $ACB$ . Hence

$$ABC \asymp BCA \asymp CAB \asymp ACB \asymp CBA \asymp BAC,$$

i.e. according to the extended definition, any ordered triad is equivalent to any permutation of itself.

Since  $m(ABC) = -m(ACB)$ , the definition of measure (§ 49) cannot be extended to correspond to the new conception of equivalence. On the other hand, the notion of area (§ 68) of a triangle is directly applicable. The situation here is entirely analogous to that described in § 67 with regard to the measure of a vector and the distance between two points. The formal definition may be made as follows:

DEFINITION. Let  $OPQ$  be a triangle (called the *unit triangle*) which is such that the lines  $OP$  and  $OQ$  are orthogonal and the point pairs  $OP$  and  $OQ$  are congruent to the unit of distance. Then if  $A'B'C'$  is a triangle coplanar with  $OPQ$  and congruent to  $ABC$ , the positive number

$$\frac{1}{2} |m(A'B'C')| = a(ABC),$$

where  $m(A'B'C')$  is the measure (§ 49) of the ordered triad  $A'B'C'$  relative to the ordered triad  $OPQ$ , is called the *area of the triangle*  $ABC$ .

The definition of the measure of an ordered tetrad and of the volume of a tetrahedron may be taken from § 112, with the proviso that the unit tetrad  $OPQR$  is such that the lines  $OP, OQ, OR$  are mutually orthogonal and the point pairs  $OP, OQ, OR$  congruent to the unit of distance.

The definition of the measure of angle may be taken over literally from § 69. Since, however, any symmetry in a plane can be effected by a three-dimensional displacement, the indetermination in the measure of an angle is such that any angle whose measure is  $\beta$  also has the measure  $k\pi + \beta$ , where  $k$  is a positive or negative integer. The

measure of an angle may therefore be subjected to the condition  $0 \leq \beta < \pi$  or  $-\pi/2 < \beta \leq \pi/2$ .

DEFINITION. The *angular measure* of a pair of intersecting lines  $ab$  is the smallest value between 0 and  $2\pi$ , inclusive, of the measures of the four angles  $\angle a_1 b_1$  formed by a ray  $a_1$  of  $a$  and a ray  $b_1$  of  $b$ . It is denoted by  $m(ab)$ . If  $a$  and  $b$  do not intersect,  $m(ab)$  denotes  $m(a'b)$ , where  $a'$  is a line having a point in common with  $b$  and parallel to  $a$ . The *angular measure* of two planes  $\pi, \pi'$  is the angular measure of two lines  $l, l'$  perpendicular to  $\pi$  and  $\pi'$  respectively.

The following statements are easily proved and will be left to the reader as exercises (cf. § 72): In the case where  $a$  and  $b$  do not intersect, the value of  $m(ab)$  is independent of the choice of  $a'$ . Although in Euclidean plane geometry  $0 \leq m(ab) < \pi$ , in the three-dimensional case

$$0 \leq m(ab) < \frac{\pi}{2}.$$

If  $l_1$  and  $l_2$  are any two lines parallel to  $a$  and  $b$  respectively, and  $i_1$  and  $i_2$  are the minimal lines through the intersection of  $l_1$  and  $l_2$ ,  $m(ab)$  is the smaller of the two numbers

$$\theta_1 = -\frac{i}{2} \log R_x(l_1 l_2, i_1 i_2) \quad \text{and} \quad \theta_2 = -\frac{i}{2} \log R_x(l_1 l_2, i_2 i_1),$$

that determination of the logarithm in each case being chosen for which  $0 \leq \theta_1 < \pi$  and  $0 \leq \theta_2 < \pi$ .

The numbers which we have been defining in this section are some of the simplest absolute invariants of the group of displacements. The algebraic formulas for these invariants and some others are stated in the exercises below. In every case the radical sign indicates a *positive* root. By the angle between two vectors  $OA$  and  $OB$  is meant the measure of  $\angle AOB$ .

The *orthogonal projection* of a set of points  $[P]$  on a plane  $\pi$  is the set of points in which the lines perpendicular to  $\pi$  through the points  $P$  meet  $\pi$ . The *orthogonal projection* of a set of points  $[P]$  on a line  $l$  is the set of points in which the planes perpendicular to  $l$  through the points  $P$  meet  $l$ .

The exercises refer to four distinct noncoplanar points  $P_1 = (x_1, y_1, z_1)$ ,  $P_2 = (x_2, y_2, z_2)$ ,  $P_3 = (x_3, y_3, z_3)$ ,  $P_4 = (x_4, y_4, z_4)$ , no two of which are collinear with the origin. The coordinate system is rectangular, and  $O, P, Q, R$  denote the points  $(0, 0, 0)$ ,  $(1, 0, 0)$ ,  $(0, 1, 0)$ ,  $(0, 0, 1)$  respectively, as in § 112.

## EXERCISES

1.  $\text{Dist}(P_1P_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$ .

2. The cosines of the angles between a vector  $OP_1$  and the  $x$ ,  $y$ , and  $z$ -axes respectively are

$$\frac{x_1}{\sqrt{x_1^2 + y_1^2 + z_1^2}}, \quad \frac{y_1}{\sqrt{x_1^2 + y_1^2 + z_1^2}}, \quad \frac{z_1}{\sqrt{x_1^2 + y_1^2 + z_1^2}}.$$

These are referred to as the *direction cosines* of the vector  $OP_1$ . If  $r = \text{Dist}(P_1P_2)$ , the direction cosines of the vector  $P_1P_2$  are

$$\frac{x_2 - x_1}{r}, \quad \frac{y_2 - y_1}{r}, \quad \frac{z_2 - z_1}{r}.$$

3. The equation of a plane perpendicular to the line  $OP_1$  is

$$x_1x + y_1y + z_1z = k.$$

4. The distance from the point  $P_1$  to the plane  $ax + \beta y + \gamma z = \delta$  is

$$\left| \frac{ax_1 + \beta y_1 + \gamma z_1 - \delta}{\sqrt{a^2 + \beta^2 + \gamma^2}} \right|.$$

5. If  $Q_1$  is the orthogonal projection of  $P_2$  on the line  $OP_1$ , then

$$\frac{x_1x_2 + y_1y_2 + z_1z_2}{\sqrt{x_1^2 + y_1^2 + z_1^2}}$$

is  $\text{Dist}(OQ_1)$  in case  $Q_1$  and  $P_1$  are on the same side of  $O$ , and  $-\text{Dist}(OQ_1)$  in case  $Q_1$  and  $P_1$  are not on the same side of  $O$ .

$$x_1x_2 + y_1y_2 + z_1z_2 = \text{Dist } OP_1 \cdot \text{Dist } OP_2 \cdot \cos \angle P_1OP_2.$$

6.  $m(P_1P_2P_3P_4) = \text{Dist}(P_1P_2) \cdot \text{Dist}(P_3P_4) \cdot r \cdot \sin \theta$ , where  $r$  is the distance between the lines  $P_1P_2$  and  $P_3P_4$ , and  $\theta$  the angle between the vectors  $P_1P_2$  and  $P_3P_4$ .

7. If  $\theta$  denotes the measure of  $\angle P_1OP_2$ , and  $l, m, n$  the direction cosines of a vector  $OK$  perpendicular to the plane  $OP_1P_2$  and such that  $S(OP_1P_2K) = S(OPQR)$ ,

$$\begin{aligned} \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} &= \text{Dist}(OP_1) \cdot \text{Dist}(OP_2) \cdot \sin \theta \cdot l, \\ \begin{vmatrix} z_1 & x_1 \\ z_2 & x_2 \end{vmatrix} &= \text{Dist}(OP_1) \cdot \text{Dist}(OP_2) \cdot \sin \theta \cdot m, \\ \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} &= \text{Dist}(OP_1) \cdot \text{Dist}(OP_2) \cdot \sin \theta \cdot n. \end{aligned}$$

8. With respect to the coordinate system employed in § 118, the angle between two lines which meet  $\pi_\infty$  in  $(0, a_1, a_2, a_3)$  and  $(0, \beta_1, \beta_2, \beta_3)$  is

$$\theta = -\frac{i}{2} \log \frac{a_1\beta_1 + a_2\beta_2 + a_3\beta_3 + \sqrt{(a_1\beta_1 + a_2\beta_2 + a_3\beta_3)^2 - (a_1^2 + a_2^2 + a_3^2)(\beta_1^2 + \beta_2^2 + \beta_3^2)}}{a_1\beta_1 + a_2\beta_2 + a_3\beta_3 - \sqrt{(a_1\beta_1 + a_2\beta_2 + a_3\beta_3)^2 - (a_1^2 + a_2^2 + a_3^2)(\beta_1^2 + \beta_2^2 + \beta_3^2)}}.$$

9. If four planes  $\alpha, \beta, \gamma, \delta$  meet on a line,

$$\Re(\alpha\beta, \gamma\delta) = \frac{\sin(\alpha\gamma)}{\sin(\alpha\delta)} + \frac{\sin(\beta\gamma)}{\sin(\beta\delta)},$$

where  $(\alpha\gamma)$  denotes the angular measure of the ordered pair of planes  $\alpha\gamma$ .

**120. The sphere and other quadrics.** DEFINITION. A *sphere* is the set of all points  $[P]$  such that the point pairs  $OP$ , where  $O$  is a fixed point, are all congruent to a fixed point pair  $OP_0$ . In case the line  $OP_0$  is minimal, the sphere is said to be *degenerate*; otherwise it is *nondegenerate*. The point  $O$  is called the *center* of the sphere.

By comparison with the definition in § 60 it is clear that any section of a nondegenerate sphere by a nonminimal plane is a circle. In case the circle at infinity exists, two perpendicular sections  $C_1^2$  and  $C_2^2$  of a sphere  $S$  by nonminimal planes constitute with the circle at infinity three conic sections intersecting one another in pairs of distinct points. By § 105, Vol. I, there is one and but one quadric surface containing them. A nonminimal plane  $\pi$  through the center of the sphere meets this quadric in a conic section which contains at least two points of the circles  $C_1^2$  and  $C_2^2$  and two points of the circle at infinity. This conic is therefore a circle containing the points of the sphere  $S$  which are in  $\pi$ . Hence the sphere  $S$  is identical with the set of all ordinary points of the quadric surface containing  $C_1^2$ ,  $C_2^2$ , and the circle at infinity. Since  $O$  is the center of each circle in which  $S$  is met by a nonminimal plane through  $O$ ,  $O$  is the pole of the plane at infinity with regard to the quadric. Since a circle in a nonminimal plane contains the ordinary points of a nondegenerate conic, it follows that the quadric surface is nondegenerate, i.e. is a quadric which contains two proper or improper reguli.

In case the circle at infinity does not exist, improper elements may be adjoined as explained in § 85, Vol. I, so that the circle at infinity exists in the resulting improper space. The argument in the paragraph above thus applies to any space whatever which satisfies Assumptions A, E, P,  $H_0$ . Thus we have

**THEOREM 20.** *A nondegenerate sphere consists of the ordinary points of a nondegenerate quadric surface  $S^2$  such that all pairs of points in the plane at infinity conjugate with regard to  $S^2$  are conjugate with regard to the absolute polar system. The center of the sphere is the pole of the plane at infinity relative to this quadric.*

Comparing the definition above with Theorem 7, Chap. IV, we have

**COROLLARY.** *A degenerate sphere with a point  $O$  as center consists of all ordinary points on the cone of minimal lines through  $O$ , except  $O$  itself.*

Had a degenerate circle in the plane been defined in the same way that a degenerate sphere is defined above, it would have been found to consist of points on only one minimal line through  $O$ , since in the plane the group of displacements leaves each minimal line invariant.

The Euclidean classification of quadric surfaces may now be made in a manner entirely analogous to the Euclidean classification of conic sections in Chap. V. After completing the projective classification (§ 103) and the affine classification (§ 111, Ex. 2) and obtaining the properties of diameters and diametral planes, the principal remaining problem is that of determining the *axes*, an axis being defined as a line through the center of the quadric perpendicular to its conjugate planes.

A line  $l$  and a plane  $\pi$  meet the plane at infinity in a point  $L_\infty$  and a line  $p_\infty$  respectively. If  $l$  and  $\pi$  are perpendicular,  $L_\infty$  and  $p_\infty$  are polar with respect to  $\Sigma_\infty$ . If  $l$  and  $\pi$  are conjugate with regard to a quadric  $Q^2$ ,  $L_\infty$  and  $p_\infty$  are polar with respect to the conic (real, imaginary, or degenerate) in which  $Q^2$  meets  $\pi_\infty$ . Hence the problem of finding the axes is reduced to that of finding the points which have the same polar lines with respect to two conics. This problem has been treated in § 101, Vol. I, for the case where both conics are nondegenerate. In general the two conics have one and but one common self-polar triangle. Hence, in general, a quadric surface has three axes which are mutually orthogonal. The determination of the other cases which may arise is a problem (Ex. 5, below) requiring a comparatively simple application of methods and theorems which we have already explained.

The classification of point quadrics includes that of cones and conic sections, the properties of cones and conics in three-dimensional Euclidean geometry being by no means dual to each other. In connection with this it is of interest to prove the following theorem, which embodies perhaps the oldest definition of a conic.

**THEOREM 21.** *Any nondegenerate real conic is perspective with a circle.*

*Proof.* Let  $C^2$  be a given conic and  $K^2$  a circle in a different plane having a common tangent and point of contact with  $C^2$ . By Theorem 11, Chap. VIII, Vol. I,  $C^2$  and  $K^2$  are sections of the same cone.

**COROLLARY.** *Any cone of lines is a projection of a circle from a point.*

## EXERCISES

1. The equation of a sphere of center  $(a, b, c)$  in rectangular coördinates is

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = k.$$

2. The set of points on the lines of intersection of homologous planes in the corresponding pencils,

$$x_2 + \sqrt{-1}x_3 = \lambda(x_0 + x_1),$$

$$x_0 - x_1 = \lambda(x_2 - \sqrt{-1}x_3),$$

is a sphere.

3. A *right circular cone* is a projection of a circle from a point from which the extremities of any diameter are projected by a pair of perpendicular lines. Any conic may be regarded as the plane section of a right circular cone.

- \*4. Develop the theory of stereographic projection of a sphere on a plane (cf. § 100).

- \*5. Classify the quadric surfaces from the point of view of Euclidean geometry. Having made the classification geometrically, find normal forms for the equations of the quadrics of the different classes and the criteria to determine to which class a given quadric belongs. This is analogous to the work in Chap. V.

- \*6. Classify the linear complexes from the point of view of Euclidean geometry.

- \*7. Starting with a definition of an inversion with respect to a sphere analogous to that of an inversion with respect to a circle (§ 71), develop the theory of the inversion group of three-dimensions. This should be done both in the real and complex cases and the real and complex inversion spaces studied.

### 121. Resolution of a displacement into orthogonal line reflections.

The properties of the group of displacements are closely bound up with the theorem that any displacement is a product of two orthogonal line reflections. In proving this theorem we shall place no restriction on the absolute polar system  $\Sigma_\infty$ , except that it be nondegenerate, and shall base our reasoning on Assumptions A, E,  $H_0$  only. We are therefore obliged to consider transformations which do not exist in the Euclidean geometry, namely those with minimal lines as axes.

DEFINITION. The line at infinity polar in  $\Sigma_\infty$  to the center of a translation is called the *axis* of the translation. If the axis is tangent to the circle at infinity, the translation is said to be *isotropic* or *minimal*.

THEOREM 22. *A product of two orthogonal line reflections whose axes  $l$  and  $m$  are parallel is a translation whose axis is the line at infinity of any plane perpendicular to the plane of  $l$  and  $m$  and parallel to  $l$ . Conversely, let  $T$  be any translation and  $l$  any nonminimal line meeting its axis; then if  $m$  is the line containing the mid-points*

of every pair of points,  $L$  and  $T(L)$ , for which  $L$  is on  $l$ , and if  $l'$  is the pole in  $\Sigma_\infty$  of the point at infinity of  $l$ ,

$$T = \{ml'\} \cdot \{ll'\}.$$

*Proof.* If the axes  $l$  and  $m$  of two orthogonal line reflections  $\{ll'\}$  and  $\{mm'\}$  are parallel, they meet  $\pi_\infty$  in a point  $P_\infty$ . Each of the orthogonal line reflections effects in  $\pi_\infty$  a harmonic homology whose axis  $l$  is the polar of  $P_\infty$  in  $\Sigma_\infty$ . Hence the product leaves all points at infinity invariant. In the plane of  $l$  and  $m$  the product  $\{ml'\} \cdot \{ll'\}$  effects a planar translation parallel to any line perpendicular to  $l$ . Therefore the product  $\{ml'\} \cdot \{ll'\}$  is a translation in space parallel to this line. Its axis, therefore, is the line at infinity of any plane perpendicular to the plane of  $l$  and  $m$  and parallel to  $l$ .

The converse follows directly in the same manner as the analogous statement in Theorem 4, Chap. IV.

**THEOREM 23.** *Any displacement is a product of two orthogonal line reflections.*

*Proof.* In case the displacement, which we shall denote by  $\Delta$ , is a translation the theorem reduces to Theorem 22. In any other case  $\Delta$  is a product of a rotation and a translation (Theorem 16, Cor. 2), i.e.

$$\Delta = \{R_\infty \rho\} \cdot \{P_\infty \pi\} \cdot T,$$

where  $T$  is a translation which may be the identity. Thus  $\Delta$  effects in the plane at infinity a product of two harmonic homologies whose centers and axes are  $P_\infty, p_\infty$  and  $R_\infty, r_\infty$  respectively, where  $p_\infty$  is the line at infinity of  $\pi$  and  $r_\infty$  that of  $\rho$ .

Let  $Q$  be an arbitrary ordinary point and  $Q' = \Delta(Q)$ . Let  $l$  be the line of intersection of the planes joining  $Q$  to  $p_\infty$  and  $Q'$  to  $r_\infty$ . These planes cannot be parallel, because  $p_\infty$  and  $r_\infty$  do not coincide; and  $l$  cannot contain  $P_\infty$  or  $R_\infty$ , because  $P_\infty$  is not on  $p_\infty$  and  $R_\infty$  is not on  $r_\infty$ .

Let  $O$  be an ordinary point of  $l$  such that neither of the lines  $OQ$  and  $OQ'$  contains  $P_\infty$  or  $R_\infty$ . (If the lines  $OQ$  and  $OQ'$  coincide, they coincide with  $l$ .) Let  $P$  be the mid-point of  $OQ$ ,  $R$  the mid-point of  $OQ'$ , and let  $p$  and  $r$  be the lines  $PP_\infty$  and  $RR_\infty$  respectively. Then  $p$  and  $r$  are such that there exist orthogonal line reflections  $\{pp_\infty\}$  and  $\{rr_\infty\}$  such that

$$\{rr_\infty\}(Q') = O,$$

$$\{pp_\infty\}(O) = Q.$$

Hence

$$\{pp_\infty\} \cdot \{rr_\infty\} \cdot (Q') = Q.$$

Moreover,  $\{pp_x\} \cdot \{rr_x\}$  effects the inverse of the transformation effected in the plane at infinity by  $\Delta$ . Hence  $\{pp_x\} \cdot \{rr_x\} \cdot \Delta$  leaves invariant all points at infinity as well as  $Q$ , and hence

$$\{pp_x\} \cdot \{rr_x\} \cdot \Delta = 1,$$

or

$$\Delta = \{rr_x\} \cdot \{pp_x\}.$$

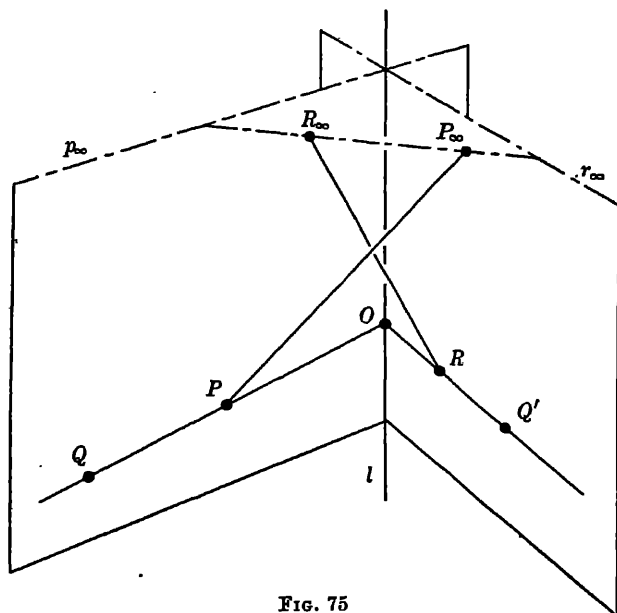


FIG. 75

It is now very easy to enumerate the possible types of displacements. A displacement  $\Delta$  being expressed in the form  $\{ll'\} \cdot \{mm'\}$ , the following cases can arise:\*

I. The lines  $l$  and  $m$  intersect in an ordinary point  $O$ .  $\Delta$  is a rotation which is the product of the orthogonal reflections in the planes perpendicular to  $l$  and  $m$  respectively at  $O$ . Two subcases must be distinguished:

(a) The plane containing  $l$  and  $m$  is not minimal.  $\Delta$  is a rotation about the common intersecting perpendicular of  $l$  and  $m$ .

(b) The plane containing  $l$  and  $m$  is minimal.  $\Delta$  is an isotropic rotation about the line joining  $O$  to the point in which the plane of

\* It is to be remembered that neither  $l$  nor  $m$  can be minimal.

$l$  and  $m$  touches the circle at infinity. It evidently effects a parabolic transformation in the pencil of planes meeting its axis and also effects an elation in the fixed plane on the axis.

II. The lines  $l$  and  $m$  are parallel. If we denote their common point at infinity by  $P_\infty$  and its polar line with respect to  $\Sigma_\infty$  by  $p_\infty$ , Theorem 22 states that  $\Delta$  is a translation whose axis is the line polar in  $\Sigma_\infty$  to the point in which the plane of  $l$  and  $m$  meets  $p_\infty$ . The latter point is the center of the translation. Two cases arise:

- (a) The axis of the translation is not tangent to the circle at infinity.
- (b) The axis of the translation is tangent to the circle at infinity, and the translation is isotropic.

III. The lines  $l$  and  $m$  do not intersect. Again two cases arise:

- (a) The lines  $l$  and  $m$  have a common intersecting perpendicular line  $a$  (Theorem 6) which is not minimal. Let  $p$  be the line parallel to  $m$  and passing through the point of intersection of  $l$  with  $a$ . Then

$$\Delta = \{ll'\} \cdot \{pp'\} \cdot \{pp'\} \cdot \{mm'\}.$$

Thus  $\Delta$  is the product of a rotation  $\{ll'\} \cdot \{pp'\}$  about  $a$  by a translation  $\{pp'\} \cdot \{mm'\}$  parallel to  $a$ .

- (b) The lines  $l$  and  $m$  have no common intersecting perpendicular. In this case they are (Theorem 6) both parallel to the same minimal plane  $\alpha$ . Let  $a_\infty$  be the line at infinity of  $\alpha$ , and  $A_\infty$  its point of contact with the circle at infinity. Then  $l$  and  $m$  pass through points of  $a_\infty$  distinct from each other and from  $A_\infty$ , and  $l'$  and  $m'$  pass through  $A_\infty$ . Therefore  $\Delta$  effects a transformation of Type III (§ 40, Vol. I) in the plane at infinity, with  $A_\infty$  as its fixed point and  $a_\infty$  as its fixed line. It also effects a parabolic transformation in the pencil of planes with  $a_\infty$  as axis. Thus its only fixed point is  $A_\infty$ , its only fixed line  $a_\infty$ , and its only fixed plane  $\pi_\infty$ .

DEFINITION. A displacement of Type IIIa, i.e. a product of a non-isotropic rotation by a translation parallel to its axis, is called a *twist* or *screw motion*. The axis of the rotation is called the *axis* of the twist.

THEOREM 24. *A displacement which interchanges two distinct ordinary points is an orthogonal line reflection.*

*Proof.* Denote the given points by  $A$  and  $B$ . The given displacement  $\Delta$  cannot be a translation, because a translation carrying a point  $A$  to a point  $B$  would carry  $B$  to a point  $C$  such that  $B$  is the

mid-point of the pair  $AC$ . Nor can  $\Delta$  be a twist or a transformation of Type IIIb, because either of these types effects the same transformation as a translation on a certain system of parallel planes, and hence no point can be transformed involutorically. And  $\Delta$  cannot be an isotropic rotation, because in this case it would effect a parabolic transformation in the planes on its axis and an elation in the one fixed plane on the axis. Hence  $\Delta$  is a nonisotropic rotation. By reference to § 115 it follows that  $\Delta$  must be an orthogonal line reflection.

**THEOREM 25.** *If  $\Lambda_1, \Lambda_2, \Lambda_3$  are three orthogonal line reflections whose axes are parallel or have a common intersecting perpendicular  $l$ , the product  $\Lambda_3\Lambda_2\Lambda_1$  is an orthogonal line reflection whose axis is parallel to the other three axes in the first case and is an intersecting perpendicular of  $l$  in the second case.*

*Proof.* In case the three axes are parallel, by Theorem 22,  $\Lambda_2\Lambda_1$  is a translation which is also expressible as the product of  $\Lambda_3$  by another orthogonal line reflection  $\Lambda_4$ , so that

$$\Lambda_2\Lambda_1 = \Lambda_3\Lambda_4,$$

and hence

$$\Lambda_3\Lambda_2\Lambda_1 = \Lambda_4.$$

In case the three axes have a common intersecting perpendicular  $l$ , the orthogonal line reflections effect involutions on  $l$  having the point at infinity of  $l$  as a common double point. Hence (§ 108, Theorem 42) the product  $\Lambda_3\Lambda_2\Lambda_1$  effects an involution on  $l$  whose double points are the point at infinity and an ordinary point  $P$ . Hence, by Theorem 24,  $\Lambda_3\Lambda_2\Lambda_1$  is an orthogonal line reflection  $\Lambda_4$ . Since  $P$  is left invariant by  $\Lambda_4$ , it is on the axis of  $\Lambda_4$ ; and this axis is perpendicular to  $l$  because  $\Lambda_4$  leaves  $l$  invariant.

#### EXERCISE

The product of an isotropic rotation by a translation parallel to its axis is an isotropic rotation about an axis in the same minimal plane.

**122. Rotation, translation, twist.** Let us now require the absolute polar system to be elliptic, as in the real Euclidean geometry. In this case there are no minimal lines, and hence the possible types of displacement are reduced to Ia, IIa, IIIa. Thus we have

**THEOREM 26.** *In case the absolute polar system is elliptic any displacement is a rotation or a translation or a twist.*

With this assumption about the absolute polar system we have a particularly simple method for the combination of displacements which depends on Theorem 25. Suppose that we wish to combine two displacements  $\{l_2l'_2\} \cdot \{l_1l'_1\}$  and  $\{l_4l'_4\} \cdot \{l_3l'_3\}$ . Let  $a$  be a common intersecting perpendicular of  $l_1$  and  $l_2$ , and  $b$  of  $l_3$  and  $l_4$ , and let  $m$  be a common intersecting perpendicular of  $a$  and  $b$ . Then the product  $\Delta$  of the two displacements satisfies the following conditions:

$$\begin{aligned}\Delta &= \{l_4l'_4\} \cdot \{l_3l'_3\} \cdot \{l_2l'_2\} \cdot \{l_1l'_1\} \\ &= \{l_4l'_4\} \cdot \{l_3l'_3\} \cdot \{mm'\} \cdot \{mm'\} \cdot \{l_2l'_2\} \cdot \{l_1l'_1\}.\end{aligned}$$

By the theorem just proved there exist two orthogonal line reflections  $\{pp'\}$ ,  $\{qq'\}$  such that

$$(24) \quad \{l_4l'_4\} \cdot \{l_3l'_3\} \cdot \{mm'\} = \{qq'\}$$

and

$$(25) \quad \{mm'\} \cdot \{l_2l'_2\} \cdot \{l_1l'_1\} = \{pp'\}.$$

Hence

$$\Delta = \{qq'\} \cdot \{pp'\}.$$

Another way of phrasing this argument is as follows:

$$\text{By (24),} \quad \{l_4l'_4\} \cdot \{l_3l'_3\} = \{qq'\} \cdot \{mm'\},$$

$$\text{and, by (25),} \quad \{l_3l'_3\} \cdot \{l_1l'_1\} = \{mm'\} \cdot \{pp'\}.$$

$$\text{Hence} \quad \Delta = \{qq'\} \cdot \{mm'\} \cdot \{mm'\} \cdot \{pp'\} = \{qq'\} \cdot \{pp'\}.$$

The analogy of this process with that of the composition of vectors is very striking. A vector is denoted by two points. A displacement is denoted by  $\Delta_2 \cdot \Delta_1$  where  $\Delta_2$  and  $\Delta_1$  are the orthogonal line reflections of which it is the product. In order to add two vectors  $AB$  and  $CD$  we choose an arbitrary point  $O$  and determine points  $P$  and  $Q$  such that

$$AB = PO \text{ and } CD = OQ.$$

Then we have

$$AB + CD = PO + OQ = PQ.$$

In the case of two displacements  $\Delta_2 \cdot \Delta_1$  and  $\Delta_4 \cdot \Delta_3$  we find an orthogonal line reflection  $\Delta$  (which is not arbitrary but is determined according to Theorem 25), for which there are two others,  $\Delta_5$  and  $\Delta_6$ , such that

$$\Delta_2 \cdot \Delta_1 = \Delta \cdot \Delta_5 \text{ and } \Delta_4 \cdot \Delta_3 = \Delta_6 \cdot \Delta.$$

Hence

$$\Delta_2 \cdot \Delta_3 \cdot \Delta_1 = \Delta_6 \cdot \Delta \cdot \Delta \cdot \Delta_5 = \Delta_6 \cdot \Delta_5.$$

Similar remarks can be made with regard to any group of transformations which are products of pairs of involutonic transformations. See § 108 and, particularly, the series of articles by H. Wiener which are there referred to.

The resolution of a general displacement into a product of two rotations of period two is a special solution of the problem to express

a given displacement  $\Delta$  as a product  $P\Lambda$  where  $P$  and  $\Lambda$  are rotations,  $\Lambda$  being of period two. The general solution of this problem may be found very simply in terms of the special one as follows:

Let  $P$  be any point of space, and let  $a$  be any line through  $P$  such that

$$\Delta = \{bb'\} \cdot \{aa'\}.$$

Let  $p$  be the line through  $P$  perpendicular to  $a$  and intersecting  $b$ , and let  $\pi$  be the plane through  $P$  perpendicular to  $p$ . Then any line  $l$  on  $P$  and  $\pi$  may be taken as the axis of  $\Lambda$ . This is obvious if  $l = a$ . If  $l \neq a$ , the product  $\{aa'\} \cdot \{ll'\}$  is a rotation about  $p$ , because  $l$  and  $a$  are perpendicular to  $p$  at  $P$ . Hence

$$\Delta \cdot \{ll'\} = \{bb'\} \cdot \{aa'\} \cdot \{ll'\} = P$$

is a rotation about an axis through the point of intersection of  $b$  and  $p$ . Hence

$$(26) \quad \Delta = P\Lambda$$

where

$$\Lambda = \{ll'\}.$$

Moreover, if  $l$  be any line through  $P$  and not in  $\pi$ ,  $\{aa'\} \cdot \{ll'\}$  is a rotation about a line  $q$  perpendicular to  $a$  and  $l$  and hence distinct from  $p$ . Since  $q$  is perpendicular to  $a$  and not identical with  $p$ , it does not meet  $b$ . Hence the displacement

$$\Delta \cdot \{ll'\} = \{bb'\} \cdot \{aa'\} \cdot \{ll'\}$$

is not a rotation. Hence the pencil of lines on  $P$  and  $\pi$  is the set of all lines on  $P$  which are axes of the rotations  $\Lambda$  of period two such that  $\Delta = P\Lambda$  where  $P$  is a rotation.

This argument applies to any ordinary point  $P$ . There is no difficulty in seeing that any point at infinity is also the center of a flat pencil of lines any one of which may be chosen as the axis of  $\Lambda$  in (26). From this it follows by Theorem 24, Chap. XI, Vol. I, that the set of all lines which are axes of  $\Lambda$ 's satisfying (26) form a linear complex. The argument for the case when  $P$  is at infinity is left as an exercise for the reader (Ex. 7). By another application of Theorem 24, Chap. XI, Vol. I, it is easy to prove that the axes of the rotations  $P$  which satisfy (26) are the lines of another linear complex. This is also left as an exercise (Ex. 8). Other instances of the resolution of a general displacement into displacements of special types are given in Exs. 9-11. These exercises all connect closely with those given in the next section.

DEFINITION. A twist  $\Gamma$  such that  $\Gamma^2$  is a translation is called a *half twist*.

An orthogonal line reflection is a special case of a half twist, and any half twist is a product of two orthogonal line reflections whose axes are perpendicular.

### EXERCISES

1. If the three common intersecting perpendiculars of the pairs of opposite edges of a simple hexagon are also the lines joining the mid-points of the pairs of vertices on opposite edges, they have a common intersecting perpendicular.

2. If the product of three orthogonal line reflections is another line reflection, the three axes are parallel or are all met by a common perpendicular.

3. For any three congruent figures  $F_1, F_2, F_3$  there exists a figure  $F$  and three lines  $l_1, l_2, l_3$  such that

$$F_1 = \{l_1 l'_1\} F, \quad F_2 = \{l_2 l'_2\} F, \quad F_3 = \{l_3 l'_3\} F.$$

(See the note by G. Darboux on p. 351 of *Leçons de Cinématique*, Paris, 1897, by G. Koenigs, where the theorem is credited in part to Stéphanos.)

4. The axes of two harmonic orthogonal line reflections meet and are perpendicular.

5. For any pair of orthogonal line reflections there is a third which is harmonic to both.

6. Under what conditions are two displacements commutative?

7. For any displacement  $\Delta$  there exists a linear complex  $C$  of lines such that every ordinary line of  $C$  is an axis of a rotation  $\Lambda$  of period two such that

$$\Delta = P\Lambda$$

where  $P$  is a rotation. No line not in  $C$  is an axis of such a  $\Lambda$ .

8. If  $\Delta$  is a displacement which is not of period two, the axes of the rotations  $P$  determined in Ex. 7 form a linear complex  $C_1$  which has in common with  $C$  all the lines perpendicular to the axis of  $\Delta$ .

9. Any displacement  $\Delta$  can be put in the form

$$\Delta = \Lambda P$$

where  $\Lambda$  and  $P$  are rotations and  $\Lambda$  is of period two. The axes of the  $\Lambda$ 's satisfying this condition constitute the ordinary lines of the complex  $C$  (Ex. 7) and those of the  $P$ 's the ordinary lines of  $C_1$  (Ex. 8).

10. Any displacement  $\Delta$  can be put in the form

$$(27) \quad \Delta = P_2 \cdot P_1$$

where  $P_1$  and  $P_2$  are rotations or translations. If  $\Delta$  is not a rotation or translation, the axis of  $P_1$  or of  $P_2$  can be chosen arbitrarily. The axes of the  $P_1$ 's which satisfy (27) are carried into the axes of the corresponding  $P_2$ 's by a correlation  $\Gamma$ .

11. Any displacement  $\Delta$  can be put in the form

$$(28) \quad \Delta = PH$$

where  $P$  is a rotation or translation and  $H$  a half twist. The axis either of  $P$  or of  $H$  can be chosen arbitrarily. For any  $P$  and  $H$  satisfying (28) there exists a rotation or translation  $P'$  and a half twist  $H'$  such that

$$\Delta = HP' \text{ and } \Delta = H'P.$$

12. Every symmetry is expressible as a product in either order of an orthogonal reflection in a plane  $\pi$  and a rotation about a line  $l$  perpendicular to  $\pi$ .

13. The mid-points of pairs of points which correspond under a symmetry are the points of the plane  $\pi$  (Ex. 12) or else coincide with the point  $l\pi$ . The planes perpendicular to the lines joining these pairs at their mid-points pass through the point  $l\pi$ .

14. Every symmetry transformation is expressible as a product in either order of an orthogonal plane reflection and an orthogonal line reflection.

15. Determine the types of symmetry transformations which are distinct under the Euclidean group.

**123. Properties of displacements.** The main properties of displacements which we have found may be stated as follows for the real Euclidean geometry:

Any displacement  $\Delta$  has a unique axis  $a$  which is a line at infinity only in case  $\Delta$  is a translation. The displacement is a product of two orthogonal line reflections, i.e.

$$\Delta = \{l_2 l_1'\} \cdot \{l_1 l_1'\}.$$

The lines  $l_1$  and  $l_2$  meet  $a$  in two points  $A_1$  and  $A_2$  and are perpendicular to it. Let the measure of the angle between  $l_1$  and  $l_2$  be  $\theta$  and the distance between  $A_1$  and  $A_2$  be  $d$ . Then  $\Delta$  is the resultant of a translation  $T$  parallel to  $a$  which carries every point  $X$  to a point  $X'$  such that

$$\text{Dist}(XX') = 2d,$$

and a rotation  $P$  with  $a$  as axis which carries each plane  $\pi$  on  $d$  to a plane  $\pi'$  such that the angular measure of  $\pi$  and  $\pi'$  is  $2\theta$ .

DEFINITION. The numbers  $2\theta$  and  $2d$  respectively are called the *angle of rotation* and *distance of translation* respectively of  $\Delta$ .

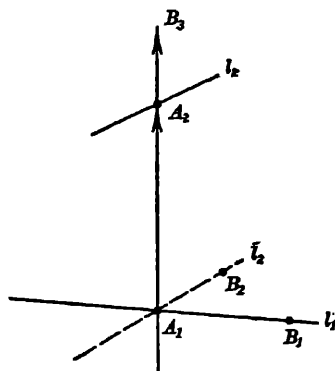


FIG. 76

The rotation  $P$  such that  $\Delta = TP = PT$  is

$$P = \{\bar{l}_2 \bar{l}'_2\} \cdot \{l_1 l'_1\},$$

where  $\bar{l}_2$  is the line through  $A_1$  parallel to  $l_2$ . Let  $B_1$  and  $B_2$  be two points of  $l_1$  and  $\bar{l}_2$  respectively, so chosen that the measure of  $\angle B_1 A_1 B_2$  is  $\theta$  (and not  $\pi - \theta$ ).<sup>\*</sup> Let one of the two sense-classes (§ 31) in the Euclidean space be designated as positive.

If  $0 \neq \theta \neq \frac{\pi}{2}$ , there are two points  $B_3, B'_3$  on  $a$  such that

$$\text{Dist}(A_1 B_3) = \text{Dist}(A_1 B'_3) = \tan \theta.$$

These points are on opposite sides of the plane  $A_1 B_1 B_2$  and hence  $S(A_1 B_1 B_2 B_3) \neq S(A_1 B_1 B_2 B'_3)$ . Let  $B_3$  be that one of these points for which  $S(A_1 B_1 B_2 B_3)$  is positive. If  $\theta = 0$ , let  $B_3 = A_1$ . It is easily seen that this determination of  $B_3$  is the same for any choice of  $B_1$  and  $B_2$  subject to the conditions imposed above. Hence *any displacement  $\Delta$  for which  $\theta \neq \frac{\pi}{2}$  determines uniquely a line  $a$  and two vectors  $A_1 A_2$  and  $A_1 B_3$ , which are parallel to  $a$  if  $a$  is ordinary. If  $a$  is ideal,  $\Delta$  is a translation and  $A_1 B_3$  zero.*

Conversely, an ordinary line  $a$  and two vectors parallel to  $a$  determine a unique displacement  $\Delta$ . For let  $A_1$  be any point of  $a$ , and  $l_1$  any line through  $A_1$  and perpendicular to  $a$ . Then the first vector determines a unique point  $A_2$  and the second a unique point  $B_3$ . There are two lines  $\bar{l}_2, \bar{\bar{l}}_2$  through  $A_1$  perpendicular to  $a$  and such that  $m(l_1 \bar{l}_2) = m(l_1 \bar{\bar{l}}_2) = \theta$  where  $\tan \theta = \text{Dist } B_1 B_3$ . Let  $B_1$  be an arbitrary point of  $l_1$ , and  $\bar{B}_2, \bar{\bar{B}}_2$  points of  $\bar{l}_2, \bar{\bar{l}}_2$  respectively, such that  $\theta$  is the measure of  $\angle B_1 A_1 \bar{B}_2$  and  $\angle B_1 A_1 \bar{\bar{B}}_2$ . Then let  $B_2$  be that one of  $\bar{B}_2$  and  $\bar{\bar{B}}_2$  such that  $S(A_1 B_1 B_2 B_3)$  is positive, and let  $l_2$  be the line through  $A_2$  parallel to  $A_1 B_2$ . The displacement determined is

$$\Delta = \{l_2 l'_2\} \cdot \{l_1 l'_1\}.$$

Hence *any displacement  $\Delta$  which is not a half twist determines and is determined by a line  $a$  and two vectors  $A_1 A_2$  and  $A_1 B_3$ . From this it is plain that if it be desired to specify a displacement by means of parameters or coördinates, it is necessary to give a set of numbers which will determine the line  $a$  (e.g. the Plücker coördinates of*

<sup>\*</sup> The measure of any pair of lines in three-dimensional Euclidean geometry satisfies the condition  $0 \leq \theta \leq \frac{\pi}{2}$ . Cf. § 119.

the line) and two additional numbers which will specify the vectors  $A_1A_2$  and  $A_1B_3$ . This question is considered from various points of view in the following sections.

For a treatment of the general problem of parameter representations of displacements and, indeed, of the whole theory of displacements, see the articles by E. Study, *Mathematische Annalen*, Vol. XXXIX (1891), p. 441, and *Sitzungsberichte der Berliner Mathematischen Gesellschaft*, Vol. XII (1913), p. 36. The exercises in this section and the last one are largely drawn from the first of these articles and from the articles by Wiener, referred to above.

### EXERCISES

1. Let  $l$  be the axis of a twist,  $a$  any ray perpendicular to and intersecting  $l$ , and  $b$  the ray into which  $a$  is displaced. Let  $c$  be the ray with origin at the mid-point of the segment joining the origin of  $a$  and  $b$  and bisecting the angle between the rays through this point parallel to  $a$  and  $b$  respectively. (Two rays are parallel if they are on parallel lines and on the same side of the line joining their origins.) The given twist is the product of the line reflection whose axis contains  $a$  by the line reflection whose axis contains  $c$ .

2. The product of three rotations whose axes have a point in common and whose angles of rotation are respectively double the angles between the ordered pairs of planes determined by the pairs of axes in a definite order, is the identity.

3. The rotations  $P$  and  $P'$  described in Ex. 11, § 122, have the same *angle of rotation*, and the half twists  $H$  and  $H'$  described in the same exercise have the same *distance of translation*.

4. There exists an orthogonal line reflection interchanging two congruent ordered pairs of points  $A_1B_1$  and  $A_2B_2$  if and only if  $A_1B_2$  is congruent to  $A_2B_1$ .

5. There is a unique orthogonal line reflection carrying a given sense-class on a line  $l$  to a given sense-class on a line  $l'$ . The axes of the two orthogonal line reflections carrying a line  $l$  to a line  $l'$  are perpendicular to each other and to the common intersecting perpendicular of  $l$  and  $l'$  at the mid-point of the pair of points in which the latter meets  $l$  and  $l'$ .

6. If an ordered triad of noncollinear points  $A_1B_1C_1$  is congruent to an ordered triad  $A_2B_2C_2$ , the axis of the displacement carrying  $A_1, B_1, C_1$  to  $A_2, B_2, C_2$  respectively meets orthogonally the axis of the orthogonal line reflection which carries  $A_1$  and  $B_1$  to two points  $A'_1$  and  $B'_1$  of the line  $A_2B_2$  such that  $S(A'_1B'_1) \neq S(A_2B_2)$ .

7. If three noncollinear points  $A_1, A_2, A_3$  are displaced into  $A_2, A_3, A_4$  respectively, the axis of the displacement is the common intersecting perpendicular of the line joining  $A_2$  to the mid-point of  $A_1A_3$  and the line joining  $A_3$  to the mid-point of  $A_2A_4$ .

8. Show how to construct the axis of the displacement carrying an ordered point triad  $A_1B_1C_1$  to a congruent ordered triad  $A_2B_2C_2$ .

9. If a line  $l$  be displaced to a line  $l'$ , the mid-points of pairs of congruent points are the points of a line  $\bar{l}$  or are identical; the planes perpendicular to the lines joining the pairs of congruent points at their mid-points meet on a line  $\bar{l}$  or are parallel or coincide. Under what circumstances do the different cases arise?

10. If a plane  $\alpha$  be displaced to a plane  $\alpha'$ , the mid-points of the pairs of congruent points are the points of a plane  $\bar{\alpha}$  or the points of a line or coincide; the planes perpendicular to the lines joining the pairs of congruent points at their mid-points pass through a point  $\bar{A}$  or all meet on a line or coincide. Under what circumstances do the different cases arise?

11. Let  $\Delta$  be a displacement,  $P$  a variable point of space,  $P' = \Delta(P)$ ,  $\bar{P}$  the mid-point of the pair  $PP'$ , and  $\pi$  the plane through  $\bar{P}$  perpendicular to the line  $PP'$  if  $P \neq P'$ . Then if  $\Delta$  is not a half twist, the transformations  $T_1$  such that  $T_1(P) = \bar{P}$  and  $T_2$  such that  $T_2(\bar{P}) = P'$  are affine collineations and

$$T_2 T_1 = \Delta = T_1 T_2.$$

If  $\Delta$  is not a rotation, the transformation  $\Gamma$  such that  $\Gamma(P) = \pi$  is a projective correlation such that  $\Gamma(\pi) = P'$ ; i.e. such that

$$\Gamma^2 = \Delta.$$

If  $\Delta$  is not a rotation or a half twist, the transformation  $N$  such that  $N(\bar{P}) = \pi$  is a projective correlation, and in fact is the null-system of the complex  $C$  referred to in Ex. 7, § 122. These transformations also satisfy the equations

$$T_1 = N\Gamma, \quad T_2 = \Gamma N, \quad N\Delta = \Delta N.$$

12. Using the notations of Ex. 11, if  $\alpha$  is any plane,  $\Delta(\alpha) = \alpha'$ , and  $T_2(\alpha) = \bar{\alpha}$ , then  $\bar{\alpha}$  bisects the pair of planes  $\alpha$  and  $\alpha'$ , and  $T_1(\bar{\alpha}) = \alpha'$ .

13. In the correlation  $N$  the lines  $\bar{l}$  and  $\bar{l}$  defined by Ex. 9 correspond. The plane  $\bar{\alpha}$  and the point  $\bar{A}$  defined in Ex. 10 also correspond in  $N$ .

14. The linear complex  $C$  (Ex. 7, § 122) contains every line  $\bar{l}$  which coincides with the line  $\bar{l}$  determined by the same line  $l$  (Ex. 9). Hence it is the set of those lines  $\bar{l}$  which are perpendicular to the lines joining corresponding points of  $l$  and  $l'$ , and it is also the set of lines  $\bar{l}$  which intersect the lines joining corresponding points of  $l$  and  $l'$ .

15. The affine collineation  $T_1$  (Ex. 11) carries the axis of  $P$  (Ex. 11, § 122) to that of  $H'$ .

16. The correlation  $\Gamma$  (Ex. 11) carries the axis of  $P_1$  (Ex. 10, § 122) to that of  $P_2$ .

17. The transformations  $T^{-1}$ ,  $T_2$ ,  $\Gamma$ ,  $\Gamma^{-1}$  all carry  $C$  (Ex. 7, § 122) into  $C_1$  (Ex. 8, § 122).

**124. Correspondence between the rotations and the points of space.** If we confine attention to the rotations leaving a point  $O$  invariant,\*

\* By the reasoning in § 90 it is clear that this amounts to considering the effect of all displacements on the field of vectors.

the considerations of the last section simplify considerably. The points  $A_1$  and  $A_2$  may be taken as coincident with  $O$ , and the point  $B_3$  shall be denoted by  $R$ . Then every noninvolutoric rotation  $P$  corresponds to a definite point  $R$  on its axis. An involutoric rotation (orthogonal line reflection) may be taken to correspond to the point at infinity of its axis. Hence the rotations leaving  $O$  invariant correspond in a one-to-one and reciprocal way to the points of the real projective space consisting of the given Euclidean space and its points at infinity.

Let  $OX, OY, OZ$  be axes of a rectangular coördinate system with  $O$  as center such that  $S(OXYZ)$  is the positive sense-class. Whenever  $R$  is distinct from the origin, denote the measures of  $\angle ROX, \angle ROY, \angle ROZ$  by  $\alpha, \beta, \gamma$  respectively. Then the coördinates of  $R$  are

$$x = \tan \theta \cos \alpha,$$

$$y = \tan \theta \cos \beta,$$

$$z = \tan \theta \cos \gamma.$$

Let  $(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$  be the homogeneous coördinates of  $R$ , so chosen that if  $R$  is ordinary,

$$X = \frac{\alpha_1}{\alpha_0}, \quad Y = \frac{\alpha_2}{\alpha_0}, \quad Z = \frac{\alpha_3}{\alpha_0};$$

and if  $R$  is at infinity,  $\alpha_0 = 0$ . In either case we may take

$$\alpha_0 = \cos \theta, \quad \alpha_1 = \sin \theta \cos \alpha, \quad \alpha_2 = \sin \theta \cos \beta, \quad \alpha_3 = \sin \theta \cos \gamma.$$

According to Theorem 23 any rotation  $(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$  is expressible as a product of two involutoric rotations  $(0, \lambda_1, \lambda_2, \lambda_3)$  and  $(0, \mu_1, \mu_2, \mu_3)$ . According to the convention just introduced, the  $\lambda$ 's and  $\mu$ 's may be regarded as direction cosines. Hence, by Exs. 5 and 7, § 119,

$$(29) \quad \alpha_0 = \lambda_1 \mu_1 + \lambda_2 \mu_2 + \lambda_3 \mu_3, \quad \alpha_1 = \begin{vmatrix} \lambda_2 & \lambda_3 \\ \mu_2 & \mu_3 \end{vmatrix}, \quad \alpha_2 = \begin{vmatrix} \lambda_3 & \lambda_1 \\ \mu_3 & \mu_1 \end{vmatrix}, \quad \alpha_3 = \begin{vmatrix} \lambda_1 & \lambda_2 \\ \mu_1 & \mu_2 \end{vmatrix}.$$

Two fundamental problems now arise: (1) to express the coördinates of the point representing the resultant of two rotations in terms of the coördinates of the points representing the rotations, and (2) to write the equations of a rotation in terms of the parameters  $(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ .

The formulas (29) are a special case of the formulas which furnish the solution of the first of these problems. The formulas for the general case may be found by an application of the method for

compounding rotations described in § 122. Let the two rotations correspond to  $A = (\alpha_0, \alpha_1, \alpha_2, \alpha_3)$  and  $B = (\beta_0, \beta_1, \beta_2, \beta_3)$  respectively. Let  $\mu_1, \mu_2, \mu_3$  be direction cosines of a line perpendicular to  $OA$  and  $OB$ . Then the rotation  $(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$  is expressible by means of the formulas (29) and  $(\beta_0, \beta_1, \beta_2, \beta_3)$  by the following:

$$(30) \quad \beta_0 = \mu_1\nu_1 + \mu_2\nu_2 + \mu_3\nu_3, \quad \beta_1 = \begin{vmatrix} \mu_2 & \mu_3 \\ \nu_2 & \nu_3 \end{vmatrix}, \quad \beta_2 = \begin{vmatrix} \mu_3 & \mu_1 \\ \nu_3 & \nu_1 \end{vmatrix}, \quad \beta_3 = \begin{vmatrix} \mu_1 & \mu_2 \\ \nu_1 & \nu_2 \end{vmatrix}.$$

According to the principle explained in § 122, the point  $(\gamma_0, \gamma_1, \gamma_2, \gamma_3)$  which represents the product of  $(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$  followed by  $(\beta_0, \beta_1, \beta_2, \beta_3)$  is

$$(31) \quad \gamma_0 = \lambda_1\nu_1 + \lambda_2\nu_2 + \lambda_3\nu_3, \quad \gamma_1 = \begin{vmatrix} \lambda_2 & \lambda_3 \\ \nu_2 & \nu_3 \end{vmatrix}, \quad \gamma_2 = \begin{vmatrix} \lambda_3 & \lambda_1 \\ \nu_3 & \nu_1 \end{vmatrix}, \quad \gamma_3 = \begin{vmatrix} \lambda_1 & \lambda_2 \\ \nu_1 & \nu_2 \end{vmatrix}.$$

The result of eliminating the  $\lambda$ 's,  $\mu$ 's, and  $\nu$ 's from these equations is

$$(32) \quad \begin{aligned} \gamma_0 &= \alpha_0\beta_0 - \alpha_1\beta_1 - \alpha_2\beta_2 - \alpha_3\beta_3, \\ \gamma_1 &= \alpha_1\beta_0 + \alpha_0\beta_1 + \alpha_2\beta_2 - \alpha_3\beta_3, \\ \gamma_2 &= \alpha_2\beta_0 - \alpha_3\beta_1 + \alpha_0\beta_2 + \alpha_1\beta_3, \\ \gamma_3 &= \alpha_3\beta_0 + \alpha_2\beta_1 - \alpha_1\beta_2 + \alpha_0\beta_3. \end{aligned}$$

This is most easily verified by substituting (29), (30), and (31) in (32). The rotation  $(\gamma_0, \gamma_1, \gamma_2, \gamma_3)$  which is the product of  $(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$  and  $(\beta_0, \beta_1, \beta_2, \beta_3)$  must be that given by (32); for if not, there would be some case in which (32) would not be satisfied by the values of  $\alpha_0, \beta_0, \gamma_0$ , etc. given by (29), (30), and (31).

The formulas (32), which are due to O. Rodrigues, *Journal de Mathématiques*, Vol. V (1840), p. 380, are the same as those for the multiplication of quaternions. Cf. § 127.

The problem (2) of expressing the coefficients of the equations (17) of a rotation in terms of the coördinates of the corresponding point  $(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$  may be solved very easily by the formulas and theorems of § 118, in the case of rotations of period two. The involutoric rotation corresponding to  $(0, \lambda_1, \lambda_2, \lambda_3)$  is, in fact,

$$(33) \quad \begin{aligned} x' &= (2\lambda_1^2 - 1)x + 2\lambda_1\lambda_2y + 2\lambda_1\lambda_3z, \\ y' &= 2\lambda_1\lambda_2x + (2\lambda_2^2 - 1)y + 2\lambda_2\lambda_3z, \\ z' &= 2\lambda_1\lambda_3x + 2\lambda_2\lambda_3y + (2\lambda_3^2 - 1)z. \end{aligned}$$

This is easily verified, because (1) the matrix is orthogonal and its determinant is +1, (2) the transformation leaves the point  $(\lambda_1, \lambda_2, \lambda_3)$

invariant, (3) the matrix is symmetric and hence corresponds to a transformation of period two.

To obtain the equations of the transformation corresponding to  $(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$  it would be sufficient to take the product of (33) and the corresponding transformation in terms of  $\mu_1, \mu_2, \mu_3$  and compare with equations (29). The algebraic computations involved would, however, be more complicated than in the following method, which is based on a simple observation with regard to collineations whose equations are of the form

$$(34) \quad \begin{aligned} \alpha_0 x &= \alpha_0 \bar{x} + \alpha_1 \bar{y} - \alpha_2 \bar{z}, \\ \alpha_0 y &= -\alpha_3 \bar{x} + \alpha_0 \bar{y} + \alpha_1 \bar{z}, \\ \alpha_0 z &= \alpha_2 \bar{x} - \alpha_1 \bar{y} + \alpha_0 \bar{z}. \end{aligned}$$

If  $\bar{P} = (\bar{x}, \bar{y}, \bar{z})$  and  $P = (x, y, z)$ , then the vector  $O\bar{P}$  is perpendicular to the vector  $\bar{P}P$ , because

$$(35) \quad \bar{x}(x - \bar{x}) + \bar{y}(y - \bar{y}) + \bar{z}(z - \bar{z}) = 0.$$

The transformation (34) also has the obvious property of leaving invariant all points on the line joining the origin to  $(\alpha_1, \alpha_2, \alpha_3)$ .

Conversely, if a collineation

$$(36) \quad \begin{aligned} \rho x &= \alpha_{11} \bar{x} + \alpha_{12} \bar{y} + \alpha_{13} \bar{z}, \\ \rho y &= \alpha_{21} \bar{x} + \alpha_{22} \bar{y} + \alpha_{23} \bar{z}, \\ \rho z &= \alpha_{31} \bar{x} + \alpha_{32} \bar{y} + \alpha_{33} \bar{z}, \end{aligned}$$

has the property that whenever  $\bar{P} = (\bar{x}, \bar{y}, \bar{z})$  is distinct from  $P = (x, y, z)$ ,  $OP$  is perpendicular to  $P\bar{P}$ , the relation (35) requires that  $\alpha_{ij} = -\alpha_{ji}$  whenever  $i \neq j$  and that  $\rho = \alpha_{11} = \alpha_{22} = \alpha_{33}$ . If, moreover, (36) leaves all points of the line joining the origin to  $(\alpha_1, \alpha_2, \alpha_3)$  invariant, it must be either of the form (34) or of the form

$$(34') \quad \begin{aligned} \alpha_0 x &= \alpha_0 \bar{x} - \alpha_3 \bar{y} + \alpha_2 \bar{z}, \\ \alpha_0 y &= \alpha_3 \bar{x} + \alpha_0 \bar{y} - \alpha_1 \bar{z}, \\ \alpha_0 z &= -\alpha_2 \bar{x} + \alpha_1 \bar{y} + \alpha_0 \bar{z}. \end{aligned}$$

It is also to be observed that the determinant of Transformations (34) and (34') is  $\alpha_0 A$ , where

$$(37) \quad A = \alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2.$$

This determinant can vanish for real  $\alpha$ 's only if  $\alpha_0 = 0$ .

Now consider an orthogonal transformation (17) representing a rotation  $P$  which is not of period two. Let  $P$  be an arbitrary point,  $P' = P(P)$ , and  $\bar{P}$  the mid-point of  $P$  and  $P'$ . The relation between  $P$  and  $\bar{P}$  is given by the equations\*

$$(38) \quad \begin{aligned} 2\bar{x} &= (a_{11} + 1)x + a_{12}y + a_{13}z, \\ 2\bar{y} &= a_{21}x + (a_{22} + 1)y + a_{23}z, \\ 2\bar{z} &= a_{31}x + a_{32}y + (a_{33} + 1)z. \end{aligned}$$

The line  $\bar{P}P$  is perpendicular to  $O\bar{P}$  and (38) must have the same invariant points as  $P$ . Hence if  $P$  is the rotation corresponding to  $(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ , the equations of the transformation from  $\bar{P}$  to  $P$  must be of the form (34) or (34').

Forming the determinants analogous to (19) in § 31, we see that  $S(OP\bar{P}R)$ , where  $R = (\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ , is positive if  $\bar{P}$  is given by (34) and negative if  $\bar{P}$  is given by (34'). Hence (38) must be the inverse of (34). Solving the equations (34) we have

$$(39) \quad \begin{aligned} \bar{x} &= \frac{\alpha_0^2 + \alpha_1^2}{A}x + \frac{\alpha_1\alpha_2 - \alpha_0\alpha_3}{A}y + \frac{\alpha_1\alpha_3 + \alpha_0\alpha_2}{A}z, \\ \bar{y} &= \frac{\alpha_1\alpha_2 + \alpha_0\alpha_3}{A}x + \frac{\alpha_0^2 + \alpha_2^2}{A}y + \frac{\alpha_2\alpha_3 - \alpha_0\alpha_1}{A}z, \\ \bar{z} &= \frac{\alpha_1\alpha_3 - \alpha_0\alpha_2}{A}x + \frac{\alpha_2\alpha_3 + \alpha_0\alpha_1}{A}y + \frac{\alpha_0^2 + \alpha_3^2}{A}z. \end{aligned}$$

Since (38) and (39) must be the same transformation, we have

$$(40) \quad \begin{aligned} a_{11} &= 2 \frac{\alpha_0^2 + \alpha_1^2}{A} - 1, & a_{12} &= 2 \frac{\alpha_1\alpha_2 - \alpha_0\alpha_3}{A}, & a_{13} &= 2 \frac{\alpha_1\alpha_3 + \alpha_0\alpha_2}{A}, \\ a_{21} &= 2 \frac{\alpha_1\alpha_2 + \alpha_0\alpha_3}{A}, & a_{22} &= 2 \frac{\alpha_0^2 + \alpha_2^2}{A} - 1, & a_{23} &= 2 \frac{\alpha_2\alpha_3 - \alpha_0\alpha_1}{A}, \\ a_{31} &= 2 \frac{\alpha_1\alpha_3 - \alpha_0\alpha_2}{A}, & a_{32} &= 2 \frac{\alpha_2\alpha_3 + \alpha_0\alpha_1}{A}, & a_{33} &= 2 \frac{\alpha_0^2 + \alpha_3^2}{A} - 1. \end{aligned}$$

These are the formulas, due to Euler, for expressing the coefficients of an orthogonal transformation in terms of the homogeneous parameters  $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ .

\* The transformation from  $P$  to  $\bar{P}$  is that denoted by  $T_1$  in Ex. 11, § 123.

The formulas for the  $\alpha$ 's in terms of the  $\alpha_0$ 's may be obtained by taking linear combinations of Equations (40):

$$1 + \alpha_{11} + \alpha_{22} + \alpha_{33} = \frac{4\alpha_0^2}{A}, \quad \alpha_{21} - \alpha_{12} = 4\frac{\alpha_0\alpha_2}{A}, \quad \alpha_{13} - \alpha_{31} = \frac{4\alpha_0\alpha_2}{A},$$

$$\alpha_{32} - \alpha_{23} = \frac{4\alpha_0\alpha_1}{A}.$$

From this it follows that

$$(41) \quad \alpha_0 : \alpha_1 : \alpha_2 : \alpha_3 = 1 + \alpha_{11} + \alpha_{22} + \alpha_{33} : \alpha_{22} - \alpha_{33} : \alpha_{13} - \alpha_{31} : \alpha_{21} - \alpha_{12}.$$

**125. Algebra of matrices.** The algebra of the last section may be put in a most compact form by means of matrix notation. This requires one or two new definitions. The *sum* of two matrices is defined by means of the following equation:

$$(42) \quad \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} & a_{13} + b_{13} \\ a_{21} + b_{21} & a_{22} + b_{22} & a_{23} + b_{23} \\ a_{31} + b_{31} & a_{32} + b_{32} & a_{33} + b_{33} \end{pmatrix}.$$

This operation obviously satisfies the associative and commutative laws, namely

$$A + (B + C) = (A + B) + C,$$

$$A + B = B + A,$$

where  $A, B, C$  stand for matrices.

Multiplication of matrices has been defined in § 95, Vol. I, i.e.

$$(43) \quad (a_{ij}) \cdot (b_{ij}) = (c_{ij}),$$

where  $c_{ij} = \sum_{k=1}^3 a_{ik}b_{kj}$ . Under this definition it is clear that

$$A(B + C) = AB + AC$$

and

$$(B + C)A = BA + CA.$$

Also it has already been proved that

$$(AB)C = A(BC).$$

It is now easy to see that, under these definitions, matrices have most of the properties of a noncommutative number system in the sense of Chap. VI, Vol. I, the matrices

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

taking the rôles of 0 and 1 respectively. The matrices of the form

$$\begin{pmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{pmatrix}$$

form by themselves a number system which is isomorphic with the number system of the geometry. Such a matrix may be called a *scalar* and be denoted by  $x$ .

Now let us denote the orthogonal matrix of the equations of a rotation (17) by  $R$ , and let the skew symmetric matrix

$$\begin{pmatrix} 0 & \frac{\alpha_2}{\alpha_0} & -\frac{\alpha_1}{\alpha_0} \\ -\frac{\alpha_2}{\alpha_0} & 0 & \frac{\alpha_1}{\alpha_0} \\ \frac{\alpha_1}{\alpha_0} & -\frac{\alpha_1}{\alpha_0} & 0 \end{pmatrix}$$

be denoted by  $S$ . Then the matrix of the transformation (34) is  $1 + S$  and the matrix of the transformation (38) is  $\frac{1}{2}(1 + R)$ . The comparing of coefficients of (38) and of (39) amounts to writing

$$1 + R = 2(1 + S)^{-1}.$$

This equation may be transformed as follows:

$$R = 2(1 + S)^{-1} - 1,$$

$$R = 2(1 + S)^{-1} - (1 + S)(1 + S)^{-1},$$

$$R = (1 - S)(1 + S)^{-1}.$$

The last equation, however, states a relation which is obvious from the point of view of matrices. For if  $S$  be any skew symmetric matrix, the transposed of  $S$  is  $-S$ . Since the product of the transposed matrices of the two given matrices is the transposed of the product, the transposed of

$$(1 - S)(1 + S)^{-1}$$

is

$$(1 + S)(1 - S)^{-1},$$

which is also its inverse. Hence, whenever

$$R = (1 - S)(1 + S)^{-1},$$

$R$  is orthogonal.

This equation may be solved as follows:

$$\begin{aligned} 1 + R &= (1 + S)(1 + S)^{-1} + (1 - S)(1 + S)^{-1} \\ &= 2(1 + S)^{-1}, \end{aligned}$$

$$(1 + R)^{-1} = \frac{1}{2}(1 + S),$$

$$2(1 + R)^{-1} - 1 = S,$$

$$2(1 + R)^{-1} - (1 + R)(1 + R)^{-1} = S,$$

$$(1 - R)(1 + R)^{-1} = S,$$

which gives the formula for a skew symmetric matrix in terms of an orthogonal matrix.

The operation of taking the inverse of a matrix is defined (cf. § 95, Vol. I) in case the determinant of the matrix is distinct from zero. In the operations above, this is a restriction on the matrix  $1 + R$  and, by comparison with Equations (22), is seen to mean that no point must be transformed by the rotation corresponding to  $R$  into its symmetric point with respect to the origin.

The generalization from three-rowed to  $n$ -rowed matrices is obvious, and we thus have the skew symmetric and orthogonal matrices of  $n$  rows connected by the relations

$$(44) \quad R = (1 - S)(1 + S)^{-1},$$

$$(45) \quad S = (1 - R)(1 + R)^{-1}.$$

The equations between the corresponding elements in the matrices which enter in the first of these two matrix equations are the formulas given by Cayley (Collected Works, Cambridge, 1889, Vol. I, p. 332), expressing the  $n^2$  coefficients of an orthogonal transformation as rational functions of  $\frac{n(n-1)}{2}$  parameters.

**126. Rotations of an imaginary sphere.** The group of rotations leaving a point invariant may be regarded as a subgroup of the collineations of a sphere having this point as center. Let us consider the imaginary sphere

$$(46) \quad x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0$$

and apply some of the results obtained in § 102. If a collineation

$$(47) \quad \begin{aligned} x'_0 &= c_{00}x_0 + c_{01}x_1 + c_{02}x_2 + c_{03}x_3, \\ x'_1 &= c_{10}x_0 + c_{11}x_1 + c_{12}x_2 + c_{13}x_3, \\ x'_2 &= c_{20}x_0 + c_{21}x_1 + c_{22}x_2 + c_{23}x_3, \\ x'_3 &= c_{30}x_0 + c_{31}x_1 + c_{32}x_2 + c_{33}x_3, \end{aligned}$$

carries each line of one regulus on the sphere into itself, any point  $(x_0, x_1, x_2, x_3)$  satisfying the condition (46) must be carried into a point  $(x'_0, x'_1, x'_2, x'_3)$  satisfying the condition

$$(48) \quad x'^2_0 + x'^2_1 + x'^2_2 + x'^2_3 = 0,$$

which states that it is on the sphere, and the condition

$$(49) \quad x_0x'_0 + x_1x'_1 + x_2x'_2 + x_3x'_3 = 0,$$

which states that it is on the plane tangent at  $(x_0, x_1, x_2, x_3)$ . Substituting (47) in (48) we have, as in § 118,

$$\sum_{i=0}^3 c_{ij}^2 = \sum_{i=0}^3 c_{i1}^2 = \sum_{i=0}^3 c_{i2}^2 = \sum_{i=0}^3 c_{i3}^2,$$

$$c_{0i}c_{0j} + c_{1i}c_{1j} + c_{2i}c_{2j} + c_{3i}c_{3j} = 0 \quad \text{if } i \neq j.$$

Substituting (47) in (49) we have

$$c_{ij} = -c_{ji} \quad \text{if } i \neq j,$$

$$c_{00} = c_{11} = c_{22} = c_{33}.$$

The matrix of the equations (47) must therefore be of the form

$$(50) \quad \begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 \\ -\alpha_1 & \alpha_0 & -\alpha_3 & \alpha_2 \\ -\alpha_2 & \alpha_3 & \alpha_0 & -\alpha_1 \\ -\alpha_3 & -\alpha_2 & \alpha_1 & \alpha_0 \end{pmatrix}$$

or

$$(51) \quad \begin{pmatrix} \beta_0 & \beta_1 & \beta_2 & \beta_3 \\ -\beta_1 & \beta_0 & \beta_3 & -\beta_2 \\ -\beta_2 & -\beta_3 & \beta_0 & \beta_1 \\ -\beta_3 & \beta_2 & -\beta_1 & \beta_0 \end{pmatrix}.$$

On multiplying together two matrices of one of these forms, the product is seen to be of the same form; whereas if two matrices of different forms are multiplied together, the product does not satisfy the condition  $c_{ij} = -c_{ji}$ ,  $i \neq j$ . Hence the matrices of the form (50) must represent the projective collineations leaving all lines of one regulus on (46) invariant, and those of the form (51) must represent the projective collineations leaving all lines of the other regulus invariant. Hence, by § 102, any direct projective collineation leaving the sphere invariant is represented by a product of a matrix of type (50) by one of type (51).

A rotation is a direct collineation leaving invariant both the sphere and the plane at infinity  $x_0 = 0$ . A collineation (47) leaves  $x_0 = 0$  invariant if and only if  $c_{01} = c_{02} = c_{03} = 0$ . But on multiplying (50) and (51) it is clear that this can happen only if  $\alpha_0 = \rho\beta_0$ ,  $\alpha_1 = -\rho\beta_1$ ,  $\alpha_2 = -\rho\beta_2$ ,  $\alpha_3 = -\rho\beta_3$ ,  $\rho$  being any number except zero. Hence the matrix representing a rotation is  $A\bar{A}$ , where

$$A = \begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 \\ -\alpha_1 & \alpha_0 & -\alpha_3 & \alpha_2 \\ -\alpha_2 & \alpha_3 & \alpha_0 & -\alpha_1 \\ -\alpha_3 & -\alpha_2 & \alpha_1 & \alpha_0 \end{pmatrix} \quad \text{and} \quad \bar{A} = \begin{pmatrix} \alpha_0 & -\alpha_1 & -\alpha_2 & -\alpha_3 \\ \alpha_1 & \alpha_0 & -\alpha_3 & \alpha_2 \\ \alpha_2 & \alpha_3 & \alpha_0 & -\alpha_1 \\ \alpha_3 & -\alpha_2 & \alpha_1 & \alpha_0 \end{pmatrix}.$$

The matrix of the product  $A\bar{A}$  is

$$\begin{pmatrix} \alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2 & 0 & 0 & 0 \\ 0 & \alpha_0^2 + \alpha_1^2 - \alpha_2^2 - \alpha_3^2 & 2(\alpha_1\alpha_2 - \alpha_0\alpha_3) & 2(\alpha_1\alpha_3 + \alpha_0\alpha_2) \\ 0 & 2(\alpha_1\alpha_2 + \alpha_0\alpha_3) & \alpha_0^2 + \alpha_2^2 - \alpha_1^2 - \alpha_3^2 & 2(\alpha_2\alpha_3 - \alpha_0\alpha_1) \\ 0 & 2(\alpha_1\alpha_3 - \alpha_0\alpha_2) & 2(\alpha_2\alpha_3 + \alpha_0\alpha_1) & \alpha_0^2 + \alpha_3^2 - \alpha_1^2 - \alpha_2^2 \end{pmatrix},$$

which agrees with (40) of § 124.

Hence the parameters  $(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$  in the Euler formulas may be regarded as the elements of a matrix of the form (50) which represents the projectivity effected on one of the reguli of (46) by the rotation.

If two rotations effect projectivities A and B respectively on a regulus, the product of the rotations effects the projectivity BA on the regulus (§ 102). Hence the product of two rotations whose parameters are  $(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$  and  $(\beta_0, \beta_1, \beta_2, \beta_3)$  respectively has the parameters  $(\gamma_0, \gamma_1, \gamma_2, \gamma_3)$ , where

$$\begin{pmatrix} \gamma_0 & \gamma_1 & \gamma_2 & \gamma_3 \\ -\gamma_1 & \gamma_0 & \gamma_3 & -\gamma_2 \\ -\gamma_3 & -\gamma_2 & \gamma_0 & \gamma_1 \\ -\gamma_2 & \gamma_3 & -\gamma_1 & \gamma_0 \end{pmatrix} = \begin{pmatrix} \beta_0 & \beta_1 & \beta_2 & \beta_3 \\ -\beta_1 & \beta_0 & -\beta_3 & \beta_2 \\ -\beta_3 & \beta_2 & \beta_0 & -\beta_1 \\ -\beta_2 & -\beta_3 & \beta_1 & \beta_0 \end{pmatrix} \cdot \begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 \\ -\alpha_1 & \alpha_0 & -\alpha_3 & \alpha_2 \\ -\alpha_2 & \alpha_3 & \alpha_0 & -\alpha_1 \\ -\alpha_3 & -\alpha_2 & \alpha_1 & \alpha_0 \end{pmatrix}.$$

This yields the same formulas as (32) in § 124.

### EXERCISE

A parameter representation for the sphere (46) is

$$x_0 = i(\lambda_1\mu_1 + \lambda_0\mu_0),$$

$$x_1 = \lambda_1\mu_1 - \lambda_0\mu_0,$$

$$x_2 = \lambda_1\mu_0 + \lambda_0\mu_1,$$

$$x_3 = i(\lambda_1\mu_0 - \lambda_0\mu_1),$$

where  $i^2 = -1$ . The two reguli on the sphere are the sets of lines for which  $\lambda_1/\lambda_0$  and  $\mu_1/\mu_0$  respectively are constant. The transformation whose matrix is (50) is given by the projectivity

$$\lambda'_0 = (\alpha_0 + i\alpha_1)\lambda_0 + (\alpha_3 - i\alpha_2)\lambda_1,$$

$$\lambda'_1 = -(\alpha_3 + i\alpha_2)\lambda_0 + (\alpha_0 - i\alpha_1)\lambda_1.$$

**127. Quaternions.** The definitions of sum and product of matrices in § 125 for three-rowed matrices clearly apply to matrices of any number of rows. With this understanding the sum of two matrices of the form (50) is obviously a matrix of the same form. The same has been seen in the last section to be true of the products of two

such matrices. Hence the set of all such matrices is carried into itself by the operations of addition and multiplication of matrices defined in § 125.

Let us introduce the notation

$$1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad i = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$j = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad k = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

Then any matrix of the sort we are considering is expressible in the form

$$\alpha_0 1 + \alpha_1 i + \alpha_2 j + \alpha_3 k.$$

The matrices  $i, j, k$  satisfy the following multiplication table:

$$(52) \quad \begin{array}{c|ccc} & i & j & k \\ \hline i & -1 & k-j & \\ j & -k-1 & i & \\ k & j-i & -1 & \end{array}$$

It has been seen in § 125 that matrices satisfy the associative and commutative laws of addition, the associative laws of multiplication, and the distributive laws. They obviously do not, in the present case, satisfy the commutative law of multiplication. Addition is performed by the rule

$$(53) \quad (\alpha_0 1 + \alpha_1 i + \alpha_2 j + \alpha_3 k) + (\beta_0 1 + \beta_1 i + \beta_2 j + \beta_3 k) \\ = (\alpha_0 + \beta_0) 1 + (\alpha_1 + \beta_1) i + (\alpha_2 + \beta_2) j + (\alpha_3 + \beta_3) k,$$

and multiplication by the rule

$$(54) \quad (\alpha_0 1 + \alpha_1 i + \alpha_2 j + \alpha_3 k) \cdot (\beta_0 1 + \beta_1 i + \beta_2 j + \beta_3 k) \\ = \gamma_0 1 + \gamma_1 i + \gamma_2 j + \gamma_3 k,$$

where

$$(55) \quad \begin{aligned} \gamma_0 &= \alpha_0 \beta_0 - \alpha_1 \beta_1 - \alpha_2 \beta_2 - \alpha_3 \beta_3, \\ \gamma_1 &= \alpha_0 \beta_1 + \alpha_1 \beta_0 + \alpha_2 \beta_3 - \alpha_3 \beta_2, \\ \gamma_2 &= \alpha_0 \beta_2 - \alpha_1 \beta_3 + \alpha_2 \beta_0 + \alpha_3 \beta_1, \\ \gamma_3 &= \alpha_0 \beta_3 + \alpha_1 \beta_2 - \alpha_2 \beta_1 + \alpha_3 \beta_0. \end{aligned}$$

From (53) it is clear that the operation of subtraction can be performed on any two matrices of this form. From (55) it is clear that  $(\beta_0 1 + \beta_1 i + \beta_2 j + \beta_3 k)^{-1}$  exists whenever the determinant

$$\begin{vmatrix} \beta_0 & -\beta_1 & -\beta_2 & -\beta_3 \\ \beta_1 & \beta_0 & \beta_3 & -\beta_2 \\ \beta_2 & -\beta_3 & \beta_0 & \beta_1 \\ \beta_3 & \beta_2 & -\beta_1 & \beta_0 \end{vmatrix} \equiv (\beta_0^2 + \beta_1^2 + \beta_2^2 + \beta_3^2)^2$$

is different from zero. This condition is satisfied whenever  $\beta_0, \beta_1, \beta_2, \beta_3$  are real.

Hence when  $\alpha_0, \alpha_1, \alpha_2, \alpha_3$  are real, the matrices of the form (50) constitute a noncommutative number system in the sense of Chap. VI, Vol. I. This number system is, in fact, the Hamiltonian system of quaternions. Compare the references at the end of the next section, particularly p. 178 of the article in the *Encyclopädie* and the article by Dickson in the *Bulletin of the American Mathematical Society*.

### EXERCISE

A system of quaternions may be defined as a set of objects  $[q]$  such that (1) for every ordered pair of vectors  $a, b$  there is a  $q$ , which we shall denote by  $\begin{pmatrix} a \\ b \end{pmatrix}$ ; (2) for every  $q$  there is at least one pair of vectors; (3) two pairs of vectors  $OA, OB$  and  $OA', OB'$  correspond to the same  $q$  if and only if the ordered triads  $OAB$  and  $OA'B'$  are coplanar and directly similar in their common plane; (4) the  $q$ 's are subject to operations of addition and multiplication defined by the equations

$$\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ b \end{pmatrix} = \begin{pmatrix} a + c \\ b \end{pmatrix}, \quad (b \neq 0)$$

$$\begin{pmatrix} a \\ b \end{pmatrix} \times \begin{pmatrix} b \\ c \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix}. \quad (b \neq 0 \neq c)$$

Prove that a system of  $q$ 's satisfies the fundamental theorems of a number system with the exception of the commutative law of multiplication. See G. Koenigs, *Leçons de Cinématique* (Paris, 1897), p. 464.

**128. Quaternions and the one-dimensional projective group.** On comparing (32) and (55) it is clear that there is a correspondence between quaternions, taken homogeneously, and the rotations leaving a point invariant in which if two quaternions  $q_1, q_2$  correspond to the rotations  $P_1, P_2$  respectively, the product  $q_2 q_1$  corresponds to  $P_2 P_1$ . The group of rotations is isomorphic with the group of projective transformations of the circle at infinity and hence with the projective group

of any complex one-dimensional form. There must, therefore, be a relation between quaternions and the one-dimensional projectivities,

$$x' = \frac{\alpha x + \beta}{\gamma x + \delta}.$$

The simplest way to obtain a number system corresponding to these transformations is to apply the operations of addition and multiplication as defined above to two-rowed matrices, i.e.

$$\begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{pmatrix} + \begin{pmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & \delta_2 \end{pmatrix} = \begin{pmatrix} \alpha_1 + \alpha_2 & \beta_1 + \beta_2 \\ \gamma_1 + \gamma_2 & \delta_1 + \delta_2 \end{pmatrix},$$

$$\begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{pmatrix} \cdot \begin{pmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & \delta_2 \end{pmatrix} = \begin{pmatrix} \alpha_1 \alpha_2 + \beta_1 \gamma_2 & \alpha_1 \beta_2 + \beta_1 \delta_2 \\ \gamma_1 \alpha_2 + \delta_1 \gamma_2 & \gamma_1 \beta_2 + \delta_1 \delta_2 \end{pmatrix}.$$

If we write

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad e_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

we have

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \alpha e_1 + \beta e_2 + \gamma e_3 + \delta e_4.$$

The units  $e_1, e_2, e_3, e_4$  satisfy the multiplication table

	$e_1$	$e_2$	$e_3$	$e_4$
$e_1$	$e_1$	$e_2$	0	0
$e_2$	0	0	$e_1$	$e_2$
$e_3$	$e_3$	$e_4$	0	0
$e_4$	0	0	$e_3$	$e_4$

Although these matrices satisfy the associative and distributive laws of addition and multiplication and the commutative law of addition, it is clear that they do not constitute a number system, because it is possible to have  $ab = 0$  when  $a \neq 0$  and  $b \neq 0$ . Nevertheless, if we write

$$1 = e_1 + e_4, \quad i = \sqrt{-1}(e_1 - e_4), \quad j = e_2 - e_3, \quad k = \sqrt{-1}(e_2 + e_3),$$

any matrix  $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$  is expressible linearly in  $1, i, j, k$ ; and

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.$$

Hence the system of two-rowed matrices

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

where  $\alpha, \beta, \gamma, \delta$  are complex numbers, is equivalent to the set of elements

$$(56) \quad a1 + bi + cj + dk,$$

where  $1, i, j, k$  satisfy the multiplication table (52) of quaternions. The elements (56) are quaternions, properly so called, only when  $a, b, c, d$  are real. When  $a, b, c, d$  are ordinary complex numbers, the elements (56) do not form a number system in the sense of Chap. VI, Vol. I, because there can be elements  $x, y$  both different from 0 such that  $xy = 0$ .

It is interesting to note that  $1, i, j, k$  are the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix},$$

which represent the identity, and three mutually harmonic involutions

$$x' = -x, \quad x' = -\frac{1}{x}, \quad x' = \frac{1}{x}.$$

If the projectivities are represented on a conic, these three involutions have the vertices of a self-polar triangle as centers.

The matrix represented by

$$\begin{pmatrix} \alpha_0 1 + \alpha_1 i + \alpha_2 j + \alpha_3 k \\ \alpha_0 + \sqrt{-1} \alpha_1 & \alpha_2 + \sqrt{-1} \alpha_3 \\ -\alpha_2 + \sqrt{-1} \alpha_3 & \alpha_0 - \sqrt{-1} \alpha_1 \end{pmatrix},$$

and its determinant is

$$\alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2.$$

The geometric significance of this remark is obvious on comparison with the exercise in § 126.

The relation between quaternions and the one-dimensional projective group was discovered by B. Peirce (cf. Chap. VI by A. Cayley in Tait's Quaternions, 3d edition, Cambridge, 1890). It is an instance of a general relation, noted by H. Poincaré, between any linear associative algebra and a corresponding linear group. On this subject see E. Study, Mathematical Papers from the Chicago Congress (New York, 1896), p. 376, and Encyclopädie der Math. Wiss., I A 4, § 12; Lie-Sheffers, Kontinuierliche Gruppen (Leipzig, 1893), Chap. XXI; and L. E. Dickson, Bulletin of the American Mathematical Society, Vol. XXII (1915), p. 53. On the general subject of linear associative algebra see L. E. Dickson, Linear Algebras, Cambridge Tracts in Mathematics, No. 16, 1914; and the article by E. Study and E. Cartan in the Encyclopédie des Sciences Mathématiques, I 5.

\*129. Representation of rotations and one-dimensional projectivities by points. The parameter representation of the rotations about a point which we based in § 124 on a Euclidean construction has now been seen to be connected in the closest way with the theory of the one-dimensional projective group. It is therefore of interest to set up the correspondence between the points of space and the rotations about a point in a form which puts in evidence also the correspondence between the points of space and the one-dimensional projectivities. This has been studied in detail in the memoir by Stéphanos referred to in Ex. 3, § 110. It will be merely outlined here, because the proofs are all simple applications of theorems which should by this time be familiar to the reader. The construction given below has the advantage over the one given in § 123 of being valid in a general projective space.

Let  $S^2$  be an arbitrary sphere. (In order to connect with our previous work  $S^2$  may be taken as the imaginary sphere  $x^2 + y^2 + z^2 + 1 = 0$ ). Let  $R_1^2$  and  $R_2^2$  be the two reguli on  $S^2$ ,  $O$  the center of  $S^2$ , and  $C_\infty^2$  the circle at infinity.

An arbitrary rotation  $P$  leaving  $O$  invariant determines and is fully determined by a projectivity  $\Gamma$  of  $C_\infty^2$ , and hence is fully determined by its effect on three points  $P_1, P_2, P_3$  of  $C_\infty^2$ . If  $l_1, l_2, l_3$  are the lines of  $R_1^2$  on  $P_1, P_2, P_3$  respectively, and  $m_1, m_2, m_3$  the lines of  $R_2^2$  on the points  $P(P_1), P(P_2), P(P_3)$  respectively, the planes  $l_1 m_1, l_2 m_2, l_3 m_3$  meet in a point  $R$ . Let  $R$  correspond to  $P$  and to  $\Gamma$  (cf. Ex. 2, § 110).

The following propositions are now easily established by reference to theorems on one-dimensional forms:

The point  $R$  is on the axis of  $P$  and is independent of the choice of  $P_1, P_2, P_3$ .

If the line  $OR$  meets  $S^2$  in two points  $Q_1, Q_2$ ,  $R(Q_1, Q_2, O, R)$  is the cross ratio of  $\Gamma$ .

The involutions correspond to points of the plane at infinity.

Pairs of inverse projectivities correspond to pairs of points having  $O$  as mid-point.

Harmonic projectivities (§ 80, Vol. I) of  $C_\infty^2$  correspond to points which are conjugate with respect to  $S^2$ .

The projectivities of  $C_\infty^2$  harmonic to a given projectivity correspond to the points of a plane. Such a set of projectivities may be called a *bundle of projectivities*.

The projectivities common to two bundles correspond to the points of a line and may be called a *pencil* of projectivities.

A pencil of involutions according to this definition is the same as a pencil of involutions according to the definition in § 78, Vol. I.

The product of the projectivities corresponding to points  $R_1$  and  $R_2$ , not collinear with  $O$ , corresponds to a point  $R_3$  obtained by the following construction: Let  $l', l''$  be the lines of  $R_1^2$  through the points in which  $OR_1$  meets  $S^2$ , and let  $m', m''$  be the lines of  $R_2^2$  through the points in which  $OR_2$  meets  $S^2$ . The line through  $R_1$  meeting  $m'$  and  $m''$  intersects the line through  $R_2$  meeting  $l'$  and  $l''$  in the point  $R_3$ . If  $l'$  and  $l''$  coincide, the line meeting them is understood to be tangent to  $S^2$ , and a similar convention is adopted in case  $m'$  and  $m''$  coincide.

If  $R_1$  be regarded as fixed and  $R_2$  as variable,  $R_3$  is connected with  $R_2$  by the relation

$$R_3 = \Lambda(R_2),$$

where  $\Lambda$  is a projective collineation leaving the lines  $l', l''$  pointwise invariant. In case  $l' = l''$ ,  $\Lambda$  is a collineation of the type in which all points and planes on  $l'$  are invariant and each plane on  $l'$  is transformed by an elation whose center is the point of contact of this plane with  $S^2$ .

If  $R_2$  be regarded as fixed and  $R_1$  as variable, the transformation defined by the relation

$$R_3 = \Delta(R_1)$$

is a collineation interchanging the reguli  $R_1^2$  and  $R_2^2$ , and carrying each line  $l$  of  $R_1^2$  into the line  $m$  of  $R_2^2$  in the plane  $R_3l$ , and each line  $m$  of  $R_2^2$  into the line  $l$  of  $R_1^2$  in the plane  $Om$ .

The propositions above are derivable from Assumptions *A, E, P*. In a real space we have

The rotations represented by points of a line all carry a certain ray with  $O$  as origin to a certain other ray with  $O$  as origin. Conversely, all rotations carrying a given ray with  $O$  as origin to a second ray with  $O$  as origin are represented by points of a line.

The necessary and sufficient condition that two rotations  $P_1, P_2$  be harmonic is that there exists a ray  $r$  such that  $P_1(r)$  is opposite to  $P_2(r)$ .

The representation of rotations by points given in § 124 is identical with the one given in this section, in case  $S^2$  is imaginary. In case  $S^2$  is real, the real points of space represent imaginary rotations.

If  $S^2$  is a ruled quadric and  $C^2$  a real conic, the construction above gives a representation of the real projectivities of a one-dimensional form by the points of space not on  $S^2$ . The sets of points  $[D]$  and  $[O]$  representing the direct and opposite projectivities respectively are such that any two points of the same set can be joined by a segment consisting of points of this set, whereas any segment joining a  $D$  to an  $O$  contains a point of  $S^2$ . The sets  $[D]$  and  $[O]$  are called the two *sides* of  $S^2$ .

### EXERCISES

1. Study the configuration formed by the points representing the rotations which carry into itself (a) a regular tetrahedron; (b) a cube; (c) a regular icosahedron. (Cf. Stéphanos, loc. cit., p. 348.)

2. A real quadric (ruled or not) determines two sets of points, its *sides*, such that two points of the same side can be joined by a segment consisting entirely of points of this side and such that any segment joining two points of different sides contains one point of the quadric. If the quadric is not ruled, one and only one of its sides contains all points of a plane. This side is called the *outside* or *exterior*, and the other the *inside* or *interior*.

**130. Parameter representation of displacements.** Simple algebraic considerations will enable us to extend the parameter representation of rotations considered in the sections above so as to cover the case of displacements in general. We will suppose the general displacement given in the form

$$(57) \quad \begin{aligned} x'_0 &= a_{00}x_0, \\ x'_1 &= a_{10}x_0 + a_{11}x_1 + a_{12}x_2 + a_{13}x_3, \\ x'_2 &= a_{20}x_0 + a_{21}x_1 + a_{22}x_2 + a_{23}x_3, \\ x'_3 &= a_{30}x_0 + a_{31}x_1 + a_{32}x_2 + a_{33}x_3, \end{aligned}$$

where the matrix  $(a_{11} a_{22} a_{33})$  is orthogonal. According to § 126, if  $a_{10} = a_{20} = a_{30} = 0$ , the matrix of (57) is expressible in the form  $A\bar{A}$ ,  $A$  and  $\bar{A}$  being defined at the bottom of page 336.

Now observe that if

$$(58) \quad B = \begin{pmatrix} 2\beta_0 & 0 & 0 & 0 \\ 2\beta_1 & 0 & 0 & 0 \\ 2\beta_2 & 0 & 0 & 0 \\ 2\beta_3 & 0 & 0 & 0 \end{pmatrix},$$

and  $C$  is any four-rowed matrix,  $C \cdot B$  is a matrix in which all elements except those of the first column are zero. From this it

follows that  $A(\bar{A}-B)$  will be of the form (57). In fact, if we require also that

$$(59) \quad \alpha_0\beta_0 + \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 = 0,$$

$$\text{we have } \begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 \\ -\alpha_1 & \alpha_0 & -\alpha_3 & \alpha_2 \\ -\alpha_2 & \alpha_3 & \alpha_0 & -\alpha_1 \\ -\alpha_3 & -\alpha_2 & \alpha_1 & \alpha_0 \end{pmatrix} \cdot \begin{pmatrix} \alpha_0 - 2\beta_0 - \alpha_1 - \alpha_2 - \alpha_3 \\ \alpha_1 - 2\beta_1 - \alpha_0 - \alpha_3 - \alpha_2 \\ \alpha_2 - 2\beta_2 - \alpha_3 - \alpha_0 - \alpha_1 \\ \alpha_3 - 2\beta_3 - \alpha_2 - \alpha_1 - \alpha_0 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2 & 0 \\ 2(\alpha_1\beta_0 - \alpha_0\beta_1 + \alpha_3\beta_3 - \alpha_2\beta_2) & \alpha_0^2 + \alpha_1^2 - \alpha_2^2 - \alpha_3^2 \\ 2(\alpha_2\beta_0 - \alpha_3\beta_1 - \alpha_0\beta_3 + \alpha_1\beta_2) & 2(\alpha_1\alpha_2 + \alpha_0\alpha_3) \\ 2(\alpha_3\beta_0 + \alpha_2\beta_1 - \alpha_1\beta_3 - \alpha_0\beta_2) & 2(\alpha_1\alpha_3 - \alpha_0\alpha_2) \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 \\ 2(\alpha_1\alpha_3 - \alpha_0\alpha_2) & 2(\alpha_1\alpha_3 + \alpha_0\alpha_2) \\ \alpha_0^2 + \alpha_2^2 - \alpha_1^2 - \alpha_3^2 & 2(\alpha_2\alpha_3 - \alpha_0\alpha_1) \\ 2(\alpha_2\alpha_3 + \alpha_0\alpha_1) & \alpha_0^2 + \alpha_3^2 - \alpha_1^2 - \alpha_2^2 \end{pmatrix}.$$

Hence the coefficients of (57) are given in terms of two sets of homogeneous parameters  $\alpha_0, \alpha_1, \alpha_2, \alpha_3; \beta_0, \beta_1, \beta_2, \beta_3$  by the equations (40), together with  $\alpha_{00}=1$  and

$$(60) \quad \begin{aligned} a_{10} &= 2(\alpha_1\beta_0 - \alpha_0\beta_1 + \alpha_3\beta_2 - \alpha_2\beta_3)A, \\ a_{20} &= 2(\alpha_2\beta_0 - \alpha_3\beta_1 - \alpha_0\beta_2 + \alpha_1\beta_3)A, \\ a_{30} &= 2(\alpha_3\beta_0 + \alpha_2\beta_1 - \alpha_1\beta_2 - \alpha_0\beta_3)A, \end{aligned}$$

provided that the  $\alpha$ 's and  $\beta$ 's are connected by the relation (59). Conversely, the  $\alpha$ 's and  $\beta$ 's are determined by the coefficients of (57) according to the equations (41) and the following:

$$(61) \quad \begin{aligned} \beta_0:\beta_1:\beta_2:\beta_3 &= a_{10}(a_{23} - a_{23}) + a_{20}(a_{13} - a_{31}) + a_{30}(a_{21} - a_{12}): \\ &- a_{10}(1 + a_{11} + a_{22} + a_{33}) - a_{20}(a_{21} - a_{12}) + a_{30}(a_{13} - a_{31}): \\ &a_{10}(a_{21} - a_{12}) - a_{20}(1 + a_{11} + a_{22} + a_{33}) - a_{30}(a_{32} - a_{23}): \\ &- a_{10}(a_{13} - a_{31}) + a_{20}(a_{23} - a_{32}) - a_{30}(1 + a_{11} + a_{22} + a_{33}). \end{aligned}$$

The last equations are obtained by solving (59) and (60) simultaneously for the  $\beta$ 's and substituting the values of the  $\alpha$ 's given by (41).

It remains to find the formulas for the parameters  $(\alpha_0'', \alpha_1'', \alpha_2'', \alpha_3''; \beta_0'', \beta_1'', \beta_2'', \beta_3'')$  of a displacement  $\Delta''$  which is such that  $\Delta'' = \Delta' \cdot \Delta$ , where  $\Delta$  has the parameters  $(\alpha_0, \alpha_1, \alpha_2, \alpha_3; \beta_0, \beta_1, \beta_2, \beta_3)$  and  $\Delta'$  the parameters  $(\alpha_0', \alpha_1', \alpha_2', \alpha_3'; \beta_0', \beta_1', \beta_2', \beta_3')$ .

We have seen that the matrix of  $\Delta$  is of the form  $A(\bar{A} - B)$ , where  $A$  and  $\bar{A}$  are of the form given at the bottom of page 336 and  $B$  is given by (58). In like manner  $\Delta'$  can be expressed in the analogous form  $A'(\bar{A}' - B')$  and  $\Delta''$  in the form  $A''(\bar{A}'' - B'')$ . Since the  $\beta$ 's do not enter into any coefficients of (57) except  $\alpha_{10}$ ,  $\alpha_{20}$ ,  $\alpha_{30}$ , it is clear that  $\alpha''_0$ ,  $\alpha''_1$ ,  $\alpha''_2$ ,  $\alpha''_3$  are given by the formulas (32), or, in other words, that  $A'' = A'A$ . By definition,

$$\begin{aligned} A''(\bar{A}'' - B'') &= A'(\bar{A}' - B')A(\bar{A} - B) \\ &= A'\bar{A}'A\bar{A} - A'\bar{A}'AB - A'B'\bar{A}\bar{A} + A'B'AB. \end{aligned}$$

In view of (59), the elements of the first row of  $AB$  are all zero. Hence all the elements of  $B'AB$  are zeros. Hence

$$A'B'AB = 0.$$

Since  $A$  and  $\bar{A}'$  are the matrices of transformations of two conjugate reguli, each transformation leaving all the lines of the other regulus invariant, they are commutative. Hence

$$A''(\bar{A}'' - B'') = A'A\bar{A}'\bar{A} - A'A\bar{A}'B - A'AA^{-1}B'\bar{A}.$$

But 
$$A^{-1} = A^* \cdot \frac{1}{\alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2},$$

where 
$$A^* = \begin{pmatrix} \alpha_0 & -\alpha_1 & -\alpha_2 & -\alpha_3 \\ \alpha_1 & \alpha_0 & \alpha_3 & -\alpha_2 \\ \alpha_2 & -\alpha_3 & \alpha_0 & \alpha_1 \\ \alpha_3 & \alpha_2 & -\alpha_1 & \alpha_0 \end{pmatrix},$$

and 
$$B'\bar{A}\bar{A} = B' \cdot (\alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2).$$

Hence

$$(62) \quad A''(\bar{A}'' + B'') = A'A(\bar{A}'\bar{A} - \bar{A}'B - A^*B').$$

Since  $A'' = A'A$  and  $\bar{A}'' = \bar{A}'\bar{A}$ , it follows that

$$(63) \quad B'' = \bar{A}'B + A^*B'.$$

Hence

$$\begin{aligned} \beta''_0 &= \alpha'_0\beta_0 - \alpha'_1\beta_1 - \alpha'_2\beta_2 - \alpha'_3\beta_3 + \alpha_0\beta'_0 - \alpha_1\beta'_1 - \alpha_2\beta'_2 - \alpha_3\beta'_3, \\ \beta''_1 &= \alpha'_1\beta_0 + \alpha'_0\beta_1 - \alpha'_2\beta_2 + \alpha'_3\beta_3 + \alpha_1\beta'_0 + \alpha_0\beta'_1 + \alpha_3\beta'_2 - \alpha_2\beta'_3, \\ \beta''_2 &= \alpha'_2\beta_0 + \alpha'_3\beta_1 + \alpha'_0\beta_2 - \alpha'_1\beta_3 + \alpha_2\beta'_0 - \alpha_3\beta'_1 + \alpha_0\beta'_2 + \alpha_1\beta'_3, \\ \beta''_3 &= \alpha'_3\beta_0 - \alpha'_2\beta_1 + \alpha'_1\beta_2 + \alpha'_0\beta_3 + \alpha_3\beta'_0 + \alpha_2\beta'_1 - \alpha_1\beta'_2 + \alpha_0\beta'_3. \end{aligned} \quad (64)$$

Rewriting (32) in our present notation, we also have

$$\begin{aligned}
 (65) \quad \alpha_0'' &= \alpha_0' \alpha_0 - \alpha_1' \alpha_1 - \alpha_2' \alpha_2 - \alpha_3' \alpha_3, \\
 \alpha_1'' &= \alpha_0' \alpha_1 + \alpha_1' \alpha_0 + \alpha_2' \alpha_3 - \alpha_3' \alpha_2, \\
 \alpha_2'' &= \alpha_0' \alpha_2 - \alpha_1' \alpha_3 + \alpha_2' \alpha_0 + \alpha_3' \alpha_1, \\
 \alpha_3'' &= \alpha_0' \alpha_3 + \alpha_1' \alpha_2 - \alpha_2' \alpha_1 + \alpha_3' \alpha_0.
 \end{aligned}$$

The formulas (64) and (65) can be put into a very convenient form by means of the notation of biquaternions.\* Let us define a biquaternion as any element of a number system whose elements are expressions of the form

$$(66) \quad s = (\alpha_0 + \alpha_1 i + \alpha_2 j + \alpha_3 k) + \epsilon (\beta_0 + \beta_1 i + \beta_2 j + \beta_3 k),$$

where the  $\alpha$ 's and  $\beta$ 's are numbers of the geometric number system,  $i, j, k$  are subject to the multiplication table (52), and  $\epsilon$  is subject to the rules

$$\epsilon^2 = 0, \quad \epsilon x = x \epsilon,$$

where  $x$  is any other element, and where the elements (66) are added and multiplied according to the usual rules for addition and multiplication of polynomials.

If the product of  $s$  and  $s'$ , where

$$s' = (\alpha_0' + \alpha_1' i + \alpha_2' j + \alpha_3' k) + \epsilon (\beta_0' + \beta_1' i + \beta_2' j + \beta_3' k),$$

be denoted by

$$s'' = s' \cdot s = (\alpha_0'' + \alpha_1'' i + \alpha_2'' j + \alpha_3'' k) + \epsilon (\beta_0'' + \beta_1'' i + \beta_2'' j + \beta_3'' k),$$

the  $\alpha_0'', \dots, \beta_3''$  are given by the formulas (64) and (65).

For a more complete study of the parameter representation of displacements, see E. Study, *Geometrie der Dynamen* (particularly II, § 21), Leipzig, 1903.

### EXERCISES

1. The parameters of a twist may be taken so that  $\alpha_1, \alpha_2, \alpha_3$  are the direction cosines of the axis of the twist;  $\alpha_0 = \cot \theta$ , where  $2\theta$  is the angle of rotation; and  $\beta_0 = d$ , where  $2d$  is the distance of translation.

2. Find the equations of  $T_1, T_2, \Gamma, N$ , etc. as defined in the exercises of § 123.

\*3. Find a parameter representation for the displacements in a plane which is analogous to the one studied above (cf. Study, *Leipziger Berichte*, Vol. XLI (1889), p. 222).

\* W. K. Clifford, *Preliminary Sketch of Biquaternions*, *Mathematical Papers* (London, 1882), p. 181. The system of biquaternions here used is one of the three systems of hypercomplex numbers known by this name. See § 146, below.

## GENERAL EXERCISES

Classify each theorem in this list of exercises according to the type of projective space in which it may be valid and according to the geometry to which it belongs.

1. A homology whose plane of fixed points is ideal is called a *dilation* or *expansion*. Any transformation of the Euclidean group is either a displacement or a dilation or the product of a rotation by a dilation.

2. Any transformation of the Euclidean group leaves at least one line invariant.

3. Any transformation of the Euclidean group is either a displacement or a dilation or the product of a displacement by a dilation whose center is on a fixed line of the displacement.

4. Let  $l$  be a line which is invariant under a transformation  $\Gamma$  of the Euclidean group, and let  $k$  be the characteristic cross ratio (§ 73, Vol. I) of the projectivity effected by  $\Gamma$  on  $l$ .  $\Gamma$  is a displacement or symmetry if and only if  $k = \pm 1$ .

5. Any transformation of the Euclidean group which alters sense can be expressed as a product  $\Delta P \Lambda$ , where  $\Delta$  is a dilation or the identity,  $P$  an orthogonal plane reflection,  $\Lambda$  an orthogonal line reflection or the identity.

6. If two triangles in different planes are perspective, and the plane of one be rotated about the axis of perspectivity, the center of perspectivity will describe a circle in a plane perpendicular to the axis of perspectivity (Cremona, Projective Geometry, Chap. XI).

7. The planes tangent to the circle at infinity constitute a degenerate plane quadric. With any real nondegenerate quadric this determines a range of quadrics, i.e. a family of quadrics of the form

$$f(u_1, u_2, u_3) + \lambda(u_1^2 + u_2^2 + u_3^2) = 0,$$

where  $f(u_1, u_2, u_3)$  is the equation in plane coordinates of the given quadric. This is called a *confocal system of quadrics*. Besides the circle at infinity this range contains three other degenerate quadrics, an imaginary ellipse, a real ellipse, and a hyperbola. There is one quadric of the range tangent to any plane of space. There are three quadrics of the range through any point of space, and their tangent planes at this point are mutually orthogonal.

8. Let  $[l]$  and  $[m]$  be two bundles of lines related by a projective transformation  $\Gamma$ . There is one and, in general, only one set of three mutually perpendicular lines  $l_1, l_2, l_3$  transformed by  $\Gamma$  to three mutually perpendicular lines  $m_1, m_2, m_3$ . There are two real pencils of lines in  $[l]$  which are transformed by  $\Gamma$  into congruent pencils of  $[m]$ . What special cases arise? Cf. Encyclopédie des Sc. Math., III, 8, § 9.

9. Let  $\Gamma$  be a collineation of space. The planes  $\Gamma(\pi_\infty)$  and  $\Gamma^{-1}(\pi_\infty)$  are called the *vanishing planes* of  $\Gamma$ . Through each point of space there is a pair of lines each of which is transformed by  $\Gamma$  into a congruent line (i.e. pairs of points go into congruent pairs). These lines are all parallel to  $\Gamma^{-1}(\pi_\infty)$ .

10. A collineation  $\Gamma$  which does not leave the plane at infinity invariant determines two systems of confocal quadrics such that the one system is carried by  $\Gamma$  into the other. Cf. § 84 and the references given there.

11. Let  $T$  be a direct-similarity transformation of a plane,  $A_1$  a variable point of this plane,  $A_2 = T(A_1)$ , and  $A_3$  a point such that the variable triangle  $A_1A_2A_3$  is directly similar to a fixed triangle  $B_1B_2B_3$ . Then the transformations from  $A_1$  to  $A_3$  and from  $A_2$  to  $A_3$  are direct-similarity transformations. Both of these transformations have the same finite fixed elements as  $T$ .\*

12. Let  $T$  be an affine transformation,  $A_1$  a variable point,  $A_2 = T(A_1)$ , and  $A_0$  a point such that the ratio  $A_0A_1/A_0A_2$  is constant. The transformation  $P$  from  $A_1$  to  $A_0$  is directly similar and has the same fixed elements as  $T$ . If  $T$  is a similarity transformation, so is  $P$ .

13. If  $T_1$  and  $T_2$  are affine transformations,  $A_0$  a variable point,  $A_1 = T_1(A_0)$ ,  $A_2 = T_2(A_0)$ , and  $A_3$  a point such that  $A_1A_0A_2A_3$  is a parallelogram, the transformation from  $A_0$  to  $A_3$  is affine.

\* On this and the following exercises cf. *Encyclopädie der Math. Wiss.* IIIAB 9, pp. 914-915.

## CHAPTER VIII

### NON-EUCLIDEAN GEOMETRIES

131. **Hyperbolic metric geometry in the plane.** According to the point of view explained in § 34 there must be a geometry corresponding to the projective group of a conic section. The case of a real conic in a real plane is one of extreme interest because of its close analogy with the Euclidean geometry, as will be seen at once.

DEFINITION. An arbitrary but fixed conic of a plane  $\pi$  is called *the absolute conic* or *the absolute*. The interior of this conic is called *the hyperbolic plane*. Points interior to the conic are called *ordinary points* or *hyperbolic points*, and those on the conic or exterior to it are called *ideal points*. A line consisting entirely of ideal points is called an *ideal line*, and the set of ordinary points on any other line is called an *ordinary line* or a *hyperbolic line*. The group of all projective collineations leaving the absolute conic invariant is called *the hyperbolic (metric) group of the plane*, and the corresponding geometry is called *the hyperbolic plane geometry*.

Let us at first assume only that the plane  $\pi$  is ordered  $(A, E, S, P)$ . On this basis we have as a consequence the theorems in §§ 74, 75 on the interior of a conic, that the points of an ordinary line satisfy the definition in § 23 of a linear convex region. This determines the meaning of the terms "segment," "ray," "between," "precede," etc. as applied to collinear ordinary points and sets of points in the hyperbolic plane. The ordinal properties of the hyperbolic plane may be summarized as follows:

THEOREM 1. *The hyperbolic plane satisfies Assumptions I–VI given for the Euclidean plane in § 29.*

*Proof.* Assumptions I, II, III, V are direct consequences of the proposition that the points of an ordinary line constitute a linear convex region. Assumption VI, that the interior of a conic contains at least three noncollinear points, is an obvious consequence of §§ 74, 75.

The hypothesis of Assumption IV is that three points  $A, B, C$  are noncollinear and that two other points  $D$  and  $E$  satisfy the order relations  $\{BCD\}$  and  $\{CEA\}$ . The conclusion is that there exists a point  $F$  on the line  $DE$  and between  $A$  and  $B$ . To prove this it is necessary to show (1) that the point of intersection  $F$  of the projective lines  $DE$  and  $AB$  is interior to the absolute conic and (2) that  $F$  is between  $A$  and  $B$ . Let  $l$  be a line exterior to the conic, and let its points of intersection with the lines  $AB, BC, CA$  respectively be  $F_\infty, D_\infty, E_\infty$ . By hypothesis and § 75, the pair  $DD_\infty$  is not separated by  $BC$  and the pair  $EE_\infty$  is separated by  $AC$ . Hence, by § 26, the pair  $FF_\infty$  is separated by  $AB$ . Since  $F_\infty$  is exterior to the conic,  $F$  is interior (§ 75) and between  $A$  and  $B$ .

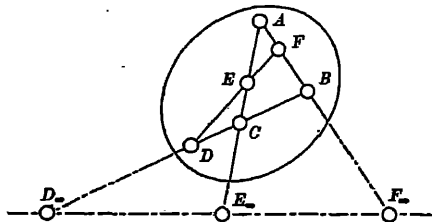


FIG. 77

**THEOREM 2.** *The hyperbolic plane does not satisfy Assumption IX, § 29. On the contrary, if  $a$  is any line and  $A$  any point not on  $a$  there are infinitely many lines on  $A$  and coplanar with  $a$  which do not meet  $a$ .*

*Proof.* By § 75 the projective line containing  $a$  also contains an infinity of points exterior to the absolute. Any line of the hyperbolic plane contained in the projective line joining  $A$  to one of these points fails to meet  $a$ .

**DEFINITION.** If a projective line containing a line  $a$  of a hyperbolic plane meets the absolute conic in two points  $B_\infty, C_\infty$ , and  $A$  is any ordinary point not on  $a$ , the ordinary lines contained in the projective lines  $AB_\infty$  and  $AC_\infty$  are said to be *parallel* to  $a$ . The segments  $AB_\infty$  and  $AC_\infty$ , consisting entirely of points interior to the absolute, constitute, together with  $A$ , two rays which are also said to be *parallel* to  $a$ .

If the projective plane  $\pi$  be supposed real, the points  $B_\infty$  and  $C_\infty$  exist for every line  $a$ , and hence we have

**THEOREM 3.** *In the real hyperbolic plane there are two and only two lines which pass through any point  $A$  and which are parallel to a line  $a$  not on  $A$ . There are two and only two rays with  $A$  as end parallel to  $a$ .*

This theorem of course does not require full use of continuity assumptions. It would also be valid if we assumed merely that any line through an interior point of a conic meets the conic (cf. § 76).

DEFINITION. The points on the absolute are sometimes called *points at infinity* or *infinite points*; and the points exterior to the absolute, *ultra-infinite points*.

### 132. Orthogonal lines, displacements, and congruence.

DEFINITION. Two lines (or two points) are said to be *orthogonal* or *perpendicular* to each other if they are conjugate with respect to the absolute.

Of two perpendicular points one is, of course, always ultra-infinite, but no analogous statement holds for perpendicular lines. From the corresponding theorems on conics we deduce at once

THEOREM 4. *The pairs of perpendicular lines on an ordinary point are pairs of a direct involution. Through an ordinary point there is one and but one line perpendicular to a given ordinary line.*

DEFINITION. A transformation of  $\pi$  which effects an involution on the absolute conic whose axis contains ordinary points is called an *orthogonal line reflection*. A transformation of  $\pi$  which effects an involution on the absolute conic whose center is an ordinary point is called a *point reflection*. A product of two orthogonal line reflections is called a *displacement*. A product of an odd number of orthogonal line reflections is called a *symmetry*. Two figures such that one can be carried to the other by a displacement are said to be *congruent*, and two figures such that one can be carried to the other by a symmetry are said to be *symmetric*.

An orthogonal line reflection is a harmonic homology whose center and axis are pole and polar with respect to the absolute conic. Since the axis contains an interior point, the center is exterior and the involution effected on the absolute alters sense (§ 74). Conversely, it follows from § 74 that an involution on the absolute conic which alters sense is effected by a harmonic homology whose center is exterior to the absolute conic, — i.e. by an orthogonal line reflection.

Since any direct projectivity is a product of two opposite involutions (§ 74), the displacements as defined above are identical with the projective collineations which transform the absolute conic into itself with preservation of sense. In particular, a point reflection is a

displacement. On the other hand, the symmetries are the projective collineations which carry the absolute into itself and interchange the two sense-classes on the absolute.

From these remarks it is evident that the theory of displacements can be obtained from the theorems on projectivities of a conic in Chap. VIII, Vol. I, and in Chap. V, Vol. II. Some of the theorems may also be obtained very easily as projective generalizations of simple Euclidean theorems.

In proving these theorems we shall suppose that we are dealing with the real projective plane and not merely with an ordered plane as in Theorem I. It would be sufficient, however, to assume merely that every opposite involution is hyperbolic (i.e. that every line through an interior point of a conic meets it), for this proposition is the only consequence of the continuity of the real plane which we use in our arguments.

Let us first prove that Assumption X (§ 66) of the Euclidean geometry holds for the hyperbolic geometry. It is to be shown that if  $A, B$  are two distinct points, then on any ray  $c$  with an end  $C$  there is a unique point  $D$  such that  $AB$  is congruent to  $CD$ . The points  $A$  and  $C$  are the centers of elliptic involutions on the absolute. It is shown in § 76 that one such involution can be transformed into any other by either a direct or an opposite involution. Hence there is a displacement  $\Delta$  carrying  $A$  to  $C$ .

The absolute conic may be regarded as a circle  $C^2$  in a Euclidean plane whose line at infinity is the pole of  $C$  with regard to the absolute. In this case  $C$  is the center of the Euclidean circle, and the hyperbolic displacements are the Euclidean rotations leaving  $C$  invariant. The required theorem now follows from the Euclidean proposition that there is one and only one rotation carrying  $B$  to a point  $D$  of a ray having  $C$  as end. The point  $D$  is interior to  $C^2$  because  $B$  is.

Assumption XI, § 66, holds good in the hyperbolic geometry because the displacements form a group. Assumption XII may be proved for the hyperbolic geometry by the argument used in § 66 for the Euclidean case. The same is true of Assumption XIII if we understand by the mid-point of a pair  $AB$  the ordinary point which is harmonically separated by the pair  $AB$  from a point conjugate to it with respect to the absolute.

DEFINITION. A circle is the set  $[P]$  of all points such that the point pairs  $OP$  where  $O$  is a fixed point are all congruent to a fixed point pair  $OP_0$ .

If the absolute be identified, as in the proof of Assumption X above, with a Euclidean circle  $C^2$ , and  $O$  with its center, it is obvious that the circles of the hyperbolic plane having  $O$  as center are identical with the Euclidean circles interior to and concentric with  $C^2$ . Hence we obtain from the properties of a pencil of concentric Euclidean circles (§ 71)

THEOREM 5. DEFINITION. A circle in the hyperbolic plane is a conic entirely interior to the absolute. It touches the absolute in two conjugate imaginary points  $A, B$ , and the tangents at these points pass through the center of the circle. The polar of the center passes through  $A$  and  $B$  and is called the axis of the circle. All its real points are exterior to the absolute conic.

It will be proved in § 134 (Theorem 7, Cor. 1) that two circles can have at most two real points in common. Once this is established, the proof of Assumption XIV in § 66 applies without change to the hyperbolic geometry.

Assumption XV is proved in § 134 as Cor. 2 of Theorem 7.

Assumption XVI may be proved as follows: Let  $A, B, C$  be three points in the order  $\{ABC\}$ , and let  $P_\infty$  and  $Q_\infty$  be the points in which the line  $AB$  meets the absolute conic, the notation being assigned so that we have  $\{P_\infty ABC Q_\infty\}$ . Let  $B_1, B_2, B_3, \dots$  be points in the order  $\{P_\infty ABB_1B_2B_3 \dots\}$  such that  $AB$  is congruent to each of the pairs  $BB_1, B_1B_2$ , etc. Choose a scale (Chap. VI, Vol. I) in which  $P_\infty A Q_\infty$  correspond to 0, 1,  $\infty$  respectively, and let  $b$  be the coördinate of  $B$ . By the hypothesis about the order relations,  $b > 1$ . The displacement carrying  $AB$  to  $BB_1$  is a projectivity of the line  $AB$  which leaves  $P_\infty$  and  $Q_\infty$  respectively invariant and transforms  $A$  to  $B$ . Hence it has the equation

$$x' = bx$$

with respect to the scale  $P_\infty, A, Q_\infty$ . The coördinates of  $B_1, B_2, B_3, \dots$  are therefore  $b^2, b^3, b^4, \dots$  respectively. The coördinate of  $C$  is, by the hypothesis that  $\{ABC\}$ , some positive number  $c$  greater than  $b$ . There are at most a finite number of values of

$b^n$  ( $n = 1, 2, \dots$ ) between  $b$  and  $c$ . Hence there are at most a finite number of the points  $B_1, B_2, \dots$  between  $B$  and  $C$ . This is what is stated in Assumption XVI.

We have now seen, taking for granted two results which will be proved in § 134, that all the assumptions (cf. §§ 29 and 66) of Euclidean plane geometry except the assumption about parallel lines are satisfied in the real hyperbolic plane, and that the parallel-line assumption is not satisfied.

### EXERCISES

1. If corresponding angles of two triangles are congruent, the corresponding sides are congruent.
2. The absence of a theory of similar triangles in hyperbolic geometry is due to what fact about the group of the geometry?
3. The perpendiculars at the mid-points of the sides of a triangle meet in a point (which may be ideal).
- \*4. Classify the conic sections from the point of view of hyperbolic geometry.

**133. Types of hyperbolic displacements.** According to § 77, Vol I, any displacement has a center and an axis which it leaves invariant. If the center is interior, the axis meets the absolute in two conjugate imaginary points, and the displacement effects an elliptic transformation on the absolute. If the center is exterior, the axis meets the absolute in two real points, and the displacement effects a hyperbolic transformation on the absolute. If the center is on the absolute, the axis is tangent, and the displacement effects a parabolic transformation on the absolute.

In the first case, the points into which a displacement and its powers carry a point distinct from its center are, by definition, on a circle which is transformed into itself by the given rotation.

In the second case, since the displacement is a product of two orthogonal line reflections whose axes pass through the center, it is obvious that the displacement leaves invariant any conic  $C^2$  which touches the absolute in the two points in which it is met by the axis of the displacement. Such a conic is obtained from the absolute by a homology whose center and axis are the center and axis of the displacement in question. From this it follows in an obvious way that  $C^2$  is entirely interior or entirely exterior to the absolute. We are interested in the case in which  $C^2$  is interior.

Let the points of contact of  $C^2$  with the absolute  $K^2$  be  $P$  and  $Q$  respectively. Since the center of the displacement  $O$  and the line  $PQ$  are polar with respect to  $C^2$ ,  $P$  and  $Q$  are the ends of two segments  $\sigma, \tau$  of points of  $C^2$  which are (in the hyperbolic plane) on opposite sides of the line  $PQ$ . Any line through  $O$  and a point of the hyperbolic plane is perpendicular to  $PQ$  and meets  $\sigma, PQ$ , and  $\tau$  in three points  $S, M, T$  respectively. If  $S', M', T'$  are the points analogously determined by another line through  $O$ , let  $\bar{M}$  be the mid-point of the pair  $MM'$ . Then the displacement which is the product of the orthogonal line reflection with  $O\bar{M}$  as axis by that with  $OM'$  as axis carries  $S, M, T$  to  $S', M', T'$  respectively.

This result may be expressed by saying that  $\sigma$  is the locus of a point  $S'$ , on a given side of  $PQ$ , such that if  $M'$  is the foot of the perpendicular from  $S'$  to  $PQ$ ,  $S'M'$  is congruent to  $SM$ . For this reason  $\sigma$  and  $\tau$  are called *equidistantal curves* of  $PQ$ .

A point  $A$  can be carried into a point  $B$  by a displacement leaving a given line  $l$ , not on  $A$ , invariant, if and only if the two points are on the same equidistantal curve of  $l$ . The equidistantal curves have some of the properties of parallel lines in the Euclidean geometry.

A displacement which effects a parabolic transformation on the absolute is a product of two orthogonal line reflections whose axes intersect in the center  $O$  of the displacement. Hence the displacement leaves invariant any conic which has contact of the third order (see § 47, Vol. I) with the absolute at  $O$ . And by the same reasoning as employed in the second case, a point  $P$  can be transformed into a point  $P'$  by a displacement which is parabolic on the absolute with a fixed point at  $O$  if and only if  $P$  and  $P'$  are on a conic having contact of the third order with the absolute at  $O$ .

**DEFINITION.** A conic interior to the absolute and having contact of the third order with it is called a *horocycle*.

The circles, equidistantal curves, and horocycles are all *path curves* of one-parameter groups of rotations.

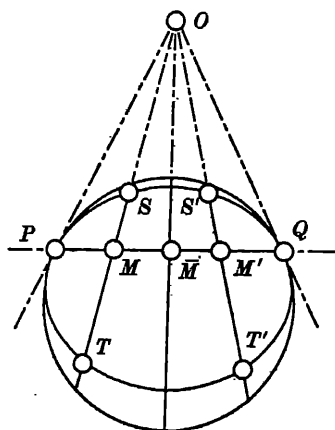


FIG. 78

### 134. Interpretation of hyperbolic geometry in the inversion plane.

Although the theory of conics touching a fixed conic in pairs of points has not been taken up explicitly in this book, we have in the inversion geometry a body of theorems from which the part of it needed for our present purpose can be obtained by the principle of transference. •

It has been seen in § 94, Theorem 16, that any transformation of the inversion group which carries a circle  $K^2$  into itself effects a projective transformation of this circle into itself. Moreover, there is one and only one direct circular transformation which effects a given projectivity on  $K^2$ . Hence *the group of direct circular transformations leaving a circle of the inversion plane invariant is simply isomorphic with the hyperbolic metric group, and the geometry of this subgroup of the inversion group is the hyperbolic geometry.*

The circles orthogonal to  $K^2$  have the property that there is one and only one such circle through each pair of distinct points interior to  $K^2$ . Since they also are transformed into themselves by the group which is here in question, it is to be expected that they correspond to the lines of the hyperbolic plane. This may be proved as follows:

Let the inversion plane be represented by a sphere  $S^2$  in a Euclidean three-space. Let  $K^2$  be the circle in which  $S^2$  is met by a plane  $\pi$  through its center, and let us regard the points of  $\pi$  interior to  $K^2$  as a hyperbolic plane. The circles of  $S^2$  orthogonal to  $K^2$  are those in which  $S^2$  is met by planes perpendicular to  $\pi$ . Hence if we let each point  $P$  of  $S^2$  on one side of  $K^2$  correspond to the point  $P'$  of  $\pi$  such that the line  $PP'$  is perpendicular to  $\pi$ , a correspondence  $\Gamma$  is established between the hyperbolic plane and the points on one side of a circle  $K^2$  in the inversion plane in such a way that the lines of the hyperbolic plane correspond to the circles orthogonal to  $K^2$ . Moreover, since the direct circular transformations of the inversion plane are effected by three-dimensional collineations leaving  $S^2$  invariant, the direct circular transformations leaving  $K^2$  invariant correspond under  $\Gamma$  to displacements and symmetries of the hyperbolic plane. Thus we have

**THEOREM 6.** *There is a one-to-one reciprocal correspondence  $\Gamma$  between the points of a hyperbolic plane as defined in § 131 and the points on one side of a circle  $K^2$  in an inversion plane (or inside a circle of the Euclidean plane) in which sets of collinear points of the*

*hyperbolic plane correspond to sets of points on circles orthogonal to  $K^2$ , and in which displacements and symmetries of the hyperbolic plane correspond to direct circular transformations leaving  $K^2$  invariant.*

**THEOREM 7.** *In the correspondence  $\Gamma$  the circles of the hyperbolic plane correspond to circles of the inversion plane which are entirely on one side of  $K^2$ .*

*Proof.* Let  $C^2$  be any circle entirely on one side of  $K^2$ , and let  $O$  and  $O'$  be the two points which are inverse with respect to both  $K^2$  and  $C^2$ , i.e. the limiting points of the pencil of circles containing  $C^2$  and  $K^2$  (§§ 71, 96). In the Euclidean plane obtained by omitting  $O'$  from the inversion plane,  $O$  is the center of both  $K^2$  and  $C^2$ , and hence the direct circular transformations leaving  $K^2$  and  $C^2$  invariant are the rotations about  $O$  and the orthogonal line reflections whose axes are on  $O$ . These correspond under  $\Gamma$  to the displacements and symmetries of the hyperbolic plane which leave  $O$  invariant. Hence the points of  $C^2$  correspond to a circle of the hyperbolic plane.

Since any circle of the hyperbolic plane may be displaced into one whose center corresponds under  $\Gamma$  to  $O$ , the argument just made shows that every circle of the hyperbolic plane may be obtained as the correspondent under  $\Gamma$  of a circle of the inversion plane which is interior to  $K^2$ .

This theorem enables us to carry over a large body of theorems on circles from the Euclidean geometry to the hyperbolic. For example, we have at once the following corollaries:

**COROLLARY 1.** *Two circles in the hyperbolic plane can have at most two real points in common.*

**COROLLARY 2.** *If the line joining the centers of two circles in the hyperbolic plane meets them in pairs of points which separate each other, the circles meet in two points, one on each side of the line.*

The first of these corollaries, on comparison with Theorem 5, yields the following projective theorem: *Two conics interior to a real conic and touching it in pairs of conjugate imaginary points can have at most two real points in common, and always have two conjugate imaginary points in common.*

**THEOREM 8.** *In the correspondence  $\Gamma$  equidistancial curves of the hyperbolic plane correspond to those portions of circles intersecting  $K^2$ , not orthogonally, which are on one side of  $K^2$ . Two equidistancial*

*curves which are parts of one conic in the hyperbolic plane are parts of circles inverse to each other with respect to  $K^2$ .*

*Proof.* A circle  $K_1^2$  of  $S^2$  which intersects  $K^2$  in two points  $P, Q$  without being perpendicular to it is a section of  $S^2$  by a plane not perpendicular to  $\pi$ . The correspondence  $\Gamma$  transforms this circle into a conic section  $C^2$  in  $\pi$  which is the projection of  $K_1^2$  from the point at infinity of a line perpendicular to  $\pi$ . The tangents to  $K_1^2$  at  $P$  and  $Q$  are transformed into tangents to  $K^2$ . Hence  $C^2$  touches  $K^2$  at  $P$  and  $Q$ .

The portions of  $K_1^2$  on the two sides of  $K^2$  on  $S^2$  correspond to the two segments of  $C^2$  having  $P$  and  $Q$  as ends; but only one of these portions of  $K_1^2$  is on the side of  $K^2$  which is in correspondence with the hyperbolic plane by means of  $\Gamma$ . The segment of  $C^2$  which is not in correspondence with this portion of  $K_1^2$  is evidently in correspondence with a portion of the circle into which  $K_1^2$  is transformed by the three-dimensional orthogonal reflection with  $\pi$  as plane of fixed points.

This proves that the part of any circle  $K_1^2$  of the inversion plane which is on one side of  $K^2$  corresponds under  $\Gamma$  to an equidistantal curve  $E_1$ , and that that part of the circle inverse to  $K_1^2$  with respect to  $K^2$  which is on the same side of  $K^2$  corresponds to the equidistantal curve  $E_2$  which is part of the same conic with  $E_1$ . That any equidistantal curve is in correspondence with a portion of some circle of the inversion plane is easily proved by an argument like that used in the last theorem.

**COROLLARY 1.** *In the correspondence  $\Gamma$  a circle touching  $K^2$  corresponds to a horocycle of the hyperbolic plane.*

Since each equidistantal curve corresponds to a portion of a circle of the inversion plane, it follows that two equidistantal curves can have at most two real points in common. It must be noted that two conics containing each an equidistantal curve can have four real points in common, since each conic accounts for two equidistantal curves.

In like manner two horocycles can have at most two real points in common, and, still more generally,

**COROLLARY 2.** *Two loci each of which is a circle, horocycle, or equidistantal curve can have at most two points in common.*

## EXERCISES

1. Show that  $\Gamma$  may be extended so that the ultra-infinite lines of the hyperbolic plane correspond to imaginary circles of the inversion plane which are orthogonal to  $K^2$ .

2. Study the theory of pencils of circles, equidistancial curves, and horocycles in the hyperbolic plane by means of the correspondence  $\Gamma$ . (A list of the theorems will be found in an article by E. Ricordi, *Giornale di Matematiche*, Vol. XVIII (1880), p. 255, and in Chap. XI of *Non-Euclidean Geometry* by J. L. Coolidge, Oxford, 1909.)

3. Develop the theory of conics touching a fixed conic in pairs of points.

**135. Significance and history of non-Euclidean geometry.** In proving the two corollaries of Theorem 7 we have completed the proof (§ 132) that the congruence assumptions of § 66 are satisfied in the hyperbolic plane. Combining this result with Theorems 1 and 2, we have

**THEOREM 9.** *In the real hyperbolic plane geometry, Assumptions I-VI, VII, X-XVI of the assumptions for Euclidean plane geometry in §§ 29 and 66 are true, and Assumption IX is false.*

**COROLLARY.** *Assumption XVII of § 29 is true in the hyperbolic plane geometry.*

The existence of the hyperbolic geometry therefore furnishes a proof of the independence\* of Assumption IX as an assumption of Euclidean geometry. This assumption is equivalent to, though not identical in form with, Euclid's parallel postulate.† And it is the interest in the parallel postulate which has been the chief historical reason for the development of the hyperbolic geometry.

The question whether the postulate of Euclid was independent or not was raised very early. In fact, the arrangement of propositions in Euclid's *Elements* shows that he had worked on the question himself. The effort to prove the postulate as a theorem continued for centuries, and in the course of time a considerable number of theorems were shown to be independent of this assumption. Eventually the question arose, what sort of theorems could be proved by taking the contrary of Euclid's assumption as a new assumption.

\*Cf. § 2, Vol. I, and § 18, Vol. II.

†Cf. Vol. I, p. 202, of Heath, *The Thirteen Books of Euclid's Elements*, Cambridge, 1908.

This question seems to have been taken up systematically for the first time by G. Saccheri,\* who obtained a large body of theorems on this basis, but seems to have been restrained from drawing, or at least publishing, more radical conclusions by the weight of religious disapproval. The credit for having propounded the body of theorems based on a contradiction of the parallel postulate as a self-consistent mathematical science, i.e. as a non-Euclidean geometry, belongs to J. Bolyai† (1832) and N. I. Lobachevski‡ (1829), although many of the ideas involved seem to have been already in the possession of C. F. Gauss.§ It was not, however, until it had been shown by Beltrami|| that the hyperbolic plane geometry could be regarded as the geometry of a pseudospherical surface in Euclidean space, that an independence proof (cf. Introduction, Vol. I) for the parallel assumption could be said to have been given. The work of Beltrami depends on the investigation by Riemann¶ of the differential geometry ideas at the basis of geometry (1854). Riemann seems to deserve the credit for the discovery of the elliptic geometry (§§ 141-143 below), though it is not clear that he distinguished between the two types of elliptic geometry.\*\*

The proof of the existence of a non-Euclidean geometry was made capable of a simpler form by the discovery of A. Cayley†† (1859) that a metric geometry can be built up, using a conic as absolute. The relation of Cayley's work to other branches of geometry and the previous studies of non-Euclidean geometry was made plain by F. Klein‡‡ in connection with his elucidation of the rôle of groups in geometry. The representation of the hyperbolic plane by means of the interior

\* *Euclides ab omni naevo vindicatus*, Milan, 1733. German translation in "Die Theorie der Parallellinien von Euklid bis auf Gauss," by F. Engel and P. Stäckel, Leipzig, 1896.

† English translation by G. B. Halsted, under the title "The Science Absolute of Space," 4th ed., Austin, Texas, 1896.

‡ German translation by Engel, under the title "Zwei geometrische Abhandlungen," Leipzig, 1898. Cf. also a translation by Halsted of another work entitled "The Theory of Parallels," Austin, Texas, 1892.

§ *Werke*, Vol. VIII, pp. 157-268.

|| *Saggio di interpretazione della geometria non-euclidea*, *Giornale di Matematiche*, Vol. VI (1868), p. 284.

¶ English translation by W. K. Clifford, in *Nature*, Vol. VIII (1873), and in Clifford's "Mathematical Papers" (London, 1882), p. 55.

\*\* Cf. F. Klein, *Autographierte Vorlesungen über nicht-euklidische Geometrie*, Vol. I (Göttingen, 1892), p. 287.

†† *Collected Works*, Vol. II (Cambridge, 1889), p. 583.

‡‡ *Mathematische Annalen*, Vol. IV (1871), p. 573.

of a circle (§ 134), and the representation of the elliptic plane given in Ex. 12, § 141, are due to R. De Paolis\* and H. Poincaré.†

For the history of non-Euclidean geometry and an exposition of parts of it, the reader is referred to R. Bonola, *Non-Euclidean Geometry*, English translation by H. S. Carslaw, Chicago, 1912. Other texts in English are J. L. Coolidge, *Non-Euclidean Geometry*, Oxford, 1909; Manning, *Non-Euclidean Geometry*, Boston, 1901; D. M. Y. Sommerville, *The Elements of Non-Euclidean Geometry*, London, 1914; H. S. Carslaw, *The Elements of Non-Euclidean Plane Geometry and Trigonometry*, London, 1916. Besides these we may mention D. M. Y. Sommerville's *Bibliography of Non-Euclidean Geometry*, London, 1911.

There are numerous other geometries closely related to the non-Euclidean geometries touched on in this chapter. Of particular interest are the geometries associated with Hermitian forms investigated by G. Fubini (*Atti del Reale Istituto Veneto*, Vol. LXIII (1904), p. 501) and E. Study,‡ and the geometry of the Physical Theory of Relativity.§

**136. Angular measure.** The measure of angles may be defined precisely as in the Euclidean geometry, and we carry over the definitions and theorems of § 69 without modification. If we represent the absolute and an arbitrary point  $O$  by a Euclidean circle  $C^2$  and its center, the Euclidean rotations about  $O$  are identical with the hyperbolic rotations about  $O$ , and hence the two angular measures as determined by the method of § 69 are identical. By § 72, if  $a$  and  $b$  are two lines intersecting in  $O$ , and  $\theta$  is the measure of the smallest angle  $\angle AOB$  for which  $A$  is a point of  $a$  and  $B$  a point of  $b$ ,

$$(1) \quad \theta = -\frac{i}{2} \log R(ab, i_1 i_2),$$

where  $i_1$  and  $i_2$  are the minimal lines through  $O$ . Since  $i_1$  and  $i_2$  are the tangents to  $C^2$  through  $O$ , it follows that (1) may be taken as the formula for the measure of any ordered pair of lines  $a, b$  in the

\* *Atti della R. Accademia dei Lincei*, Ser. 3, Vol. II (1877-1878), p. 31.

† *Acta Mathematica*, Vol. I (1882), p. 8, and *Bulletin de la Société mathématique de France*, Vol. XV (1887), p. 208.

‡ *Mathematische Annalen*, Vol. LX (1905), p. 321.

§ Cf. F. Klein, *Jahresbericht der Deutschen Mathematiker-Vereinigung*, Vol. XIX (1910), p. 281, and the article by Wilson and Lewis referred to in § 48 above.

hyperbolic plane if  $i_1$  and  $i_2$  are understood to be the tangents to the absolute through the point of intersection of  $a$  and  $b$ .  $\clubsuit$

If the hyperbolic plane is represented as in § 134 by the interior of a circle  $C^2$ , the angular measure of any two hyperbolic lines is identical with the Euclidean measure of the angle (§ 93) between the two circles orthogonal to  $C^2$  which represent them. This has just been seen for the case where the two circles are lines through the center of  $C^2$ . In the general case a point  $A$  of intersection of the two circles orthogonal to  $C^2$  may be transformed to the center of  $C^2$  by a direct circular transformation  $\Lambda$ . The transformation  $\Lambda$  as a direct circular transformation leaves Euclidean angular measure invariant (§ 93), and as a displacement of the hyperbolic plane leaves hyperbolic angular measure invariant. Since the two measures are identical at the center of  $C^2$ , they must also be identical at  $A$ .

As an application of this result we may prove the following remarkable theorem:

**THEOREM 10.** *The sum of the angles of a triangle is less than  $\pi$ .*

*Proof.* Let the triangle be  $ABC$ , and let the absolute and the point  $A$  be represented by a Euclidean circle  $C^2$  and its center. Then the hyperbolic lines  $AB$  and  $AC$  are represented by Euclidean lines through the center of  $C^2$ , and the hyperbolic line  $BC$  is represented by a circle  $K^2$  through  $B$  and  $C$  orthogonal to  $C^2$  (fig. 79).

The hyperbolic measures of the angles at  $A$ ,  $B$ , and  $C$  respectively are equal to the Euclidean measures of  $\angle BAC$  and two angles formed by  $AB$  and  $AC$  with the tangents to  $K^2$  at  $B$  and  $C$  respectively. The sum of these three angles is easily seen to be less than that of the angles of the Euclidean (rectilinear) triangle  $ABC$ . Hence it is less than  $\pi$ .

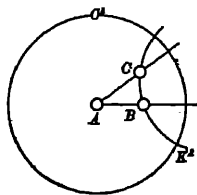


FIG. 79

The theorem that the sum of the angles of a triangle is  $\pi$  may be substituted for Assumption IX as an assumption of Euclidean geometry;\* the proposition just proved can be taken as the corresponding assumption of hyperbolic geometry; and the proposition that the sum of the angles of a triangle is greater than  $\pi$  can be taken as an assumption for elliptic geometry.

\* On the history of this theorem cf. Bonola, loc. cit., Chap. II. This reference will also be found useful in connection with the exercises.

## EXERCISES

\*1. Prove from Assumptions I-VI, X-XVI that if the sum of the angles of one triangle is greater than, equal to, or less than  $\pi$ , the corresponding statement also holds for all other triangles.

\*2. Prove from Assumptions I-VI, X-XVI that the sum of the angles of a triangle is less than or equal to  $\pi$ .

**137. Distance.** Since the conic section is a self-dual figure, it is to be expected that the formula for the measure of point-pairs is analogous to (1). As a matter of fact, we shall only modify the factor  $-i/2$ . If  $A$  and  $B$  are two ordinary points, let  $A_\infty, B_\infty$  be the points in which the line  $AB$  meets the absolute, the notation being assigned so that the points are in the order  $\{A_\infty ABB_\infty\}$ . Then  $\Re(AB, A_\infty B_\infty)$  is positive (§ 24), and hence  $\log \Re(AB, A_\infty B_\infty)$  has a real value. We define the distance between  $A$  and  $B$  by means of the equation

$$(2) \quad \text{Dist}(AB) = \gamma \log \Re(AB, A_\infty B_\infty),$$

where  $\gamma$  is an arbitrary constant and the real determination of the logarithm is taken.

It is seen at once that

$$\text{Dist}(AB) = \text{Dist}(BA),$$

because  $\Re(AB, A_\infty B_\infty) = \Re(BA, B_\infty A_\infty)$ ,

and that if  $A, B, C$  are collinear points in the order  $\{ABC\}$ ,

$$\text{Dist}(AB) + \text{Dist}(BC) = \text{Dist}(AC),$$

because  $\Re(AB, A_\infty B_\infty) \cdot \Re(BC, A_\infty B_\infty) = \Re(AC, A_\infty B_\infty)$ .

Moreover, it is evident from the properties of the collineations transforming a conic into itself that a necessary and sufficient condition for the congruence of two point-pairs  $AB, CD$  is

$$\Re(AB, A_\infty B_\infty) = \Re(CD, C_\infty D_\infty),$$

where  $A_\infty, B_\infty$  are chosen as above and  $C_\infty, D_\infty$  are chosen analogously. Hence a necessary and sufficient condition for the congruence of  $AB$  and  $CD$  is

$$\text{Dist}(AB) = \text{Dist}(CD).$$

Hence the distance function defined above is fully analogous to that used in Euclidean geometry (§ 67). The constant  $\gamma$  may be determined by choosing a fixed point-pair  $OP$  as the unit of distance. We then have

$$(3) \quad \frac{1}{\gamma} = \log \Re(OP, O_\infty P_\infty).$$

138. Algebraic formulas for distance and angle. Let us consider the symmetric bilinear form

$$\begin{aligned} f(X, X') = & a_{00}x_0x'_0 + a_{01}x_0x'_1 + a_{02}x_0x'_2 \\ & + a_{01}x_1x'_0 + a_{11}x_1x'_1 + a_{12}x_1x'_2 \\ & + a_{02}x_2x'_0 + a_{12}x_2x'_1 + a_{22}x_2x'_2 \end{aligned}$$

and the covariant form

$$\begin{aligned} F(u, u') = & A_{00}u_0u'_0 + A_{01}u_0u'_1 + A_{02}u_0u'_2 \\ & + A_{01}u_1u'_0 + A_{11}u_1u'_1 + A_{12}u_1u'_2 \\ & + A_{02}u_2u'_0 + A_{12}u_2u'_1 + A_{22}u_2u'_2, \end{aligned}$$

where the  $A_{ij}$ 's are defined as in § 85. With respect to homogeneous coördinates,  $f(X, X) = 0$  is the equation of a point conic, and  $F(u, u) = 0$  of the line conic composed of the tangents to  $f(X, X) = 0$ . Let us take this conic as the absolute and derive the formulas for the measure of distance and of angle.

Let  $Y = (y_0, y_1, y_2)$  and  $Z = (z_0, z_1, z_2)$  be two distinct points. The points of the line joining them are

$$\lambda Y + \mu Z = (\lambda y_0 + \mu z_0, \lambda y_1 + \mu z_1, \lambda y_2 + \mu z_2),$$

and the points in which this line meets  $f(X, X) = 0$  are determined by the values of  $\lambda/\mu$  satisfying the equation

$$0 = f(\lambda Y + \mu Z, \lambda Y + \mu Z) = \lambda^2 f(Y, Y) + 2\lambda\mu f(Y, Z) + \mu^2 f(Z, Z).$$

These values are

$$\begin{aligned} \frac{\lambda_1}{\mu_1} &= \frac{-f(Y, Z) + \sqrt{f^2(Y, Z) - f(Y, Y)f(Z, Z)}}{f(Y, Y)}, \\ \frac{\lambda_2}{\mu_2} &= \frac{-f(Y, Z) - \sqrt{f^2(Y, Z) - f(Y, Y)f(Z, Z)}}{f(Y, Y)}. \end{aligned}$$

Let us denote the two points of the absolute corresponding to  $(\lambda_1, \mu_1)$  and  $(\lambda_2, \mu_2)$  by  $I_1$  and  $I_2$  respectively. Then

$$\text{Dist}(YZ) = \gamma \log R(YZ, I_1 I_2).$$

Since  $(\lambda, \mu)$  is  $(1, 0)$  for  $Y$  and  $(0, 1)$  for  $Z$ , we have (§ 65, Vol I)

$$R(YZ, I_1 I_2) = \frac{\mu_1 \lambda_2}{\lambda_1 \mu_2}.$$

Hence

$$\begin{aligned} (4) \quad \text{Dist}(YZ) &= \gamma \log \frac{f(Y, Z) + \sqrt{f^2(Y, Z) - f(Y, Y)f(Z, Z)}}{f(Y, Z) - \sqrt{f^2(Y, Z) - f(Y, Y)f(Z, Z)}} \\ &= \gamma \log \frac{(f(Y, Z) + \sqrt{f^2(Y, Z) - f(Y, Y)f(Z, Z)})^2}{f(Y, Y)f(Z, Z)}. \end{aligned}$$

By precisely the same reasoning applied to the dual case we have for the measure of a pair of lines  $u = (u_0, u_1, u_2)$ ,  $v = (v_0, v_1, v_2)$ .

$$(5) \quad m(u, v) = -\frac{i}{2} \log \frac{F(u, v) + \sqrt{F^2(u, v) - F(u, u)F(v, v)}}{F(u, v) - \sqrt{F^2(u, v) - F(u, u)F(v, v)}} \\ = -\frac{i}{2} \log \frac{(F(u, v) + \sqrt{F^2(u, v) - F(u, u)F(v, v)})^2}{F(u, u)F(v, v)}.$$

Denoting  $\text{Dist}(Y, Z)$  by  $d$ , we obtain

$$(6) \quad e^{\frac{d}{2\gamma}} = \frac{f(Y, Z) + \sqrt{f^2(Y, Z) - f(Y, Y)f(Z, Z)}}{\sqrt{f(Y, Y)f(Z, Z)}},$$

and hence

$$(7) \quad \cosh \frac{d}{2\gamma} = \frac{e^{\frac{d}{2\gamma}} + e^{-\frac{d}{2\gamma}}}{2} = \sqrt{\frac{f^2(Y, Z)}{f(Y, Y)f(Z, Z)}},$$

and

$$(8) \quad \sinh \frac{d}{2\gamma} = \frac{e^{\frac{d}{2\gamma}} - e^{-\frac{d}{2\gamma}}}{2} = \sqrt{\frac{f^2(Y, Z) - f(Y, Y)f(Z, Z)}{f(Y, Y)f(Z, Z)}}.$$

In like manner, if  $\theta = m(uv)$ ,

$$(9) \quad e^{i\theta} = \frac{F(u, v) + \sqrt{F^2(u, v) - F(u, u)F(v, v)}}{\sqrt{F(u, u)F(v, v)}}.$$

$$(10) \quad \cos \theta = \sqrt{\frac{F^2(u, v)}{F(u, u)F(v, v)}}.$$

$$(11) \quad \sin \theta = i \sqrt{\frac{F^2(u, v) - F(u, u)F(v, v)}{F(u, u)F(v, v)}} \\ = \sqrt{\frac{F(u, u)F(v, v) - F^2(u, v)}{F(u, u)F(v, v)}}.$$

For a further discussion of these formulas see Clebsch-Lindemann, *Vorlesungen über Geometrie*, Vol. II, Part III, Leipzig, 1891.

\*139. **Differential of arc.** The homogeneous coördinates of all points not on the absolute,

$$(12) \quad f(X, X) = 0,$$

may be subjected to the relation

$$(13) \quad f(X, X) = C,$$

where  $C$  is a constant. Since  $f(X, X)$  is quadratic, this determines two sets of coördinates  $(x_0, x_1, x_2)$  for each point of the hyperbolic

plane instead of an infinity of sets as in unrestricted homogeneous coördinates.\*

Some definite determination of the values of each of the homogeneous coördinates is manifestly necessary in order to apply the processes of differential calculus to formulas in homogeneous coördinates. The particular relation  $f(X, X) = C$  has the advantage, among others, of not being singular for any point not on the absolute.

Suppose now that  $(x_0, x_1, x_2)$  describes a locus determined by the condition that  $x_0, x_1, x_2$  are functions of a parameter  $t$ . Then, in the familiar notation,†

$$\begin{aligned}\frac{ds}{dt} &= L \frac{\text{Dist}(X, X + \Delta X)}{\Delta t} \\ &= L \frac{2\gamma \sinh \frac{1}{2\gamma} \text{Dist}(X, X + \Delta X)}{\Delta t} \\ &= L \frac{2\gamma}{\Delta t} \sqrt{\frac{f^2(X, X + \Delta X) - f(X, X)f(X + \Delta X, X + \Delta X)}{f(X, X)f(X + \Delta X, X + \Delta X)}},\end{aligned}$$

by (8). Since  $f(Y + Y', Z) = f(Y, Z) + f(Y', Z)$ , this reduces to

$$\begin{aligned}\left(\frac{ds}{dt}\right)^2 &= L \frac{4\gamma^2}{(\Delta t)^2} \\ &= L \frac{4\gamma^2}{(\Delta t)^2} \frac{(f(X, X) + f(X, \Delta X))^2 - f(X, X)(f(X, X) + 2f(X, \Delta X) + f(\Delta X, \Delta X))}{f(X, X)(f(X, X) + 2f(X, \Delta X) + f(\Delta X, \Delta X))} \\ &= L \frac{4\gamma^2}{(\Delta t)^2} \frac{f^2\left(X, \frac{\Delta X}{\Delta t}\right) - f(X, X)f\left(\frac{\Delta X}{\Delta t}, \frac{\Delta X}{\Delta t}\right)}{f(X, X)(f(X, X) + 2f(X, \Delta X) + f(\Delta X, \Delta X))} \\ &= 4\gamma^2 \frac{f^2\left(X, \frac{dX}{dt}\right) - f(X, X)f\left(\frac{dX}{dt}, \frac{dX}{dt}\right)}{f^2(X, X)},\end{aligned}$$

\* If  $(x_0, x_1, x_2)$  are interpreted as rectangular coördinates in a Euclidean space of three dimensions,  $f(X, X) = C$  is the equation of a quadric surface, and we have a correspondence in which each point of the hyperbolic plane corresponds to a pair of points of the quadric surface. By properly choosing  $f(X, X)$ , this correspondence can be reduced to that given in § 134 between the hyperbolic plane and the surface of a sphere.

† We are applying theorems of calculus here on the same basis that we have employed algebraic theorems in other parts of the work.

in which  $\frac{dX}{dt}$  represents  $\left(\frac{dx_0}{dt}, \frac{dx_1}{dt}, \frac{dx_2}{dt}\right)$ . In differential notation this formula is

$$(14) \quad ds^2 = 4\gamma^2 \frac{f^2(X, dX) - f(X, X)f(dX, dX)}{f^2(X, X)}.$$

By duality we have a corresponding formula for the differential of angle,

$$(15) \quad d\theta^2 = \frac{F^2(u, du) - F(u, u)F(du, du)}{F^2(u, u)}.$$

These formulas are independent of the particular determination of our coördinates by means of the relation (13). If we differentiate (13) we obtain

$$f(X, dX) = 0,$$

so that for this particular determination of coördinates

$$(16) \quad ds^2 = -4\gamma^2 \frac{f(dX, dX)}{f(X, X)} = -4\gamma^2 \frac{f(dX, dX)}{C}.$$

Let us now choose the homogeneous coördinate system so that

$$f(X, X) = x_1^2 + x_2^2 - 4\gamma^2 x_0^2,$$

and choose  $C = -4\gamma^2$  so that, for points not on the absolute,

$$(17) \quad x_1^2 + x_2^2 - 4\gamma^2 x_0^2 = -4\gamma^2.$$

If  $\gamma$  is real and not zero, we are dealing with hyperbolic geometry, and

$$(18) \quad \begin{aligned} ds^2 &= f(dX, dX) \\ &= dx_1^2 + dx_2^2 - 4\gamma^2 dx_0^2. \end{aligned}$$

If we substitute  $u = \frac{2x_1}{1+x_0}, \quad v = \frac{2x_2}{1+x_0}$

in the value for  $ds^2$  given in (18), we obtain

$$(19) \quad ds^2 = \frac{du^2 + dv^2}{\left[1 - \frac{u^2 + v^2}{16\gamma^2}\right]^2}.$$

Regarding  $u$  and  $v$  as parameters of a surface in a Euclidean space, (19) gives the linear element of the surface (cf. Eisenhart, Differential Geometry, § 30). This is a surface for which, in the usual notation of differential geometry,  $E = G$  and  $F = 0$ . The curvature of this surface is constant and equal to  $-1/4\gamma^2$  (cf. Clebsch-Lindemann, loc. cit., Vol. II, p. 525). From this it follows that the hyperbolic plane

geometry in the neighborhood of any point is equivalent to the geometry on a portion of a surface of constant negative curvature.

If we substitute  $u = x_1/x_0$  and  $v = x_2/x_0$  in (18), we obtain

$$ds^2 = 4\gamma^2 \frac{(4\gamma^2 - v^2)du^2 + 2uvdu dv + (4\gamma^2 - u^2)dv^2}{(4\gamma^2 - u^2 - v^2)^2}.$$

This is the form of linear element used by Beltrami in the paper cited above. This form is such that geodesics are given by linear equations in  $u$  and  $v$ . Hence geodesics of the surface correspond to lines of the hyperbolic plane.

It is to be noted that the curvature of a surface, while often defined in terms of a Euclidean space in which the surface is supposed to be situated, is a function of  $E$ ,  $F$ , and  $G$  and therefore an internal property of the surface, i.e. a property stated in terms of curves ( $u = c$  and  $v = c$ ) in the surface and entirely independent of its being situated in a space.

Another remark which may save misunderstanding by a beginner is that the geometries corresponding to real values of  $\gamma$  are identical. The choice of  $\gamma$  amounts to a determination of the unit of length, as was shown in § 137.

### EXERCISES

1. Express the differential of angle in terms of  $(x_0, x_1, x_2)$  and their derivatives (cf. Clebsch-Lindemann, loc. cit., Vol. II, p. 477).

\*2. Develop the theory of areas in the hyperbolic plane. For a treatment by differential geometry cf. Clebsch-Lindemann, loc. cit., p. 489. For a development by elementary geometry of a theory of areas of polygons which is equally available in hyperbolic, parabolic, and elliptic geometry, see A. Finzel, *Mathematische Annalen*, Vol. LXXII (1912), p. 262.

**140. Hyperbolic geometry of three dimensions.** A *hyperbolic space* of three dimensions is the interior (cf. Ex. 2, § 129) of a nonruled quadric surface, called the *absolute quadric*, and the hyperbolic geometry of three dimensions is the set of theorems stating properties of this space which are not disturbed by the projective collineations leaving the quadric invariant. The definitions of the terms "displacement," "congruent," "perpendicular," etc. are obtained by direct generalization of the definition in § 132 and the corresponding definitions in the chapters on Euclidean geometry. They will be taken for granted in what follows, without being formally written down.

The fundamental theorems on congruence may be obtained from the observations (1) that any displacement of space leaving a plane invariant effects in this plane a displacement or a symmetry in the sense of § 132, and (2) that no two displacements of space leaving a plane invariant effect the same displacement or symmetry in this plane. From this we infer, by reference to § 135,

**THEOREM 11.** *In the real three-dimensional hyperbolic geometry Assumptions I–XVI of §§ 29 and 66 are all true except Assumption IX, which is false.*

By § 100 there is a simple isomorphism between the displacements of a hyperbolic space and the direct circular transformations of the inversion plane. Hence the theorems of inversion geometry or of the theory of projectivities of complex one-dimensional forms can all be translated into theorems of hyperbolic geometry. The reader who carries this out in detail will find that many of the theorems of Chap. VI assume very interesting forms when carried over into the hyperbolic geometry.

In particular, if an *orthogonal line reflection*, or *half turn*, is defined as a line reflection (§ 101) whose directrices are polar with respect to the absolute, it follows at once that every displacement is a product of two orthogonal line reflections. With this basis the theory of displacements is very similar to the corresponding theory in Euclidean geometry, but many of the proofs are simpler.

The formulas for distance and angle are identical with those of § 138, and the differential formulas with those of § 139 if  $f(X, X')$  be understood to be a bilinear form in  $(x_0, x_1, x_2, x_3)$  and  $(x'_0, x'_1, x'_2, x'_3)$ .

### EXERCISES

1. The product of three half turns is a half turn if and only if their three ordinary directrices have a common intersecting perpendicular line.

2. If a simple hexagon be inscribed in the absolute, the common intersecting perpendicular lines of pairs of opposite edges are met by a common intersecting perpendicular line (cf. § 108).

3. Determine the projectively distinct types of displacements.

\*4. Defining a *horosphere* as a real quadric interior to the absolute and transformable into the absolute by means of an elation whose center is on the absolute and whose plane of fixed points is tangent to the absolute, prove that the hyperbolic geometry of a horosphere is equivalent to the Euclidean plane geometry.

\*5. Classify the quadric surfaces from the point of view of hyperbolic geometry.

\*6. Given the existence of a hyperbolic space, define a set of ideal points such that the extended space is projective. Cf. R. Bonola, *Giornale di Matematiche*, Vol. XXXVIII (1900), p. 105, and F. W. Owens, *Transactions of the American Mathematical Society*, Vol. XI (1910), p. 140.

\*7. Obtain theorems analogous to those in the exercises of §§ 122, 123 with regard to the hyperbolic displacements.

\*8. Study the theory of volumes in hyperbolic geometry by methods of differential geometry.

**141. Elliptic plane geometry. Definition.** The geometry corresponding to the group of projective collineations in a real\* projective plane  $\pi$  which leave an imaginary ellipse  $E^2$  invariant is called the *two-dimensional elliptic geometry* or *elliptic plane geometry*. The imaginary conic  $E^2$  is called the *absolute conic* or the *absolute*. The projective plane  $\pi$  is sometimes referred to as the *elliptic plane*.

The order relations in this geometry are of course identical with those of the projective plane (Chap. II). The congruence relations are defined as in § 132, with suitable modifications corresponding to the fact that  $E^2$  is imaginary. Some of the theorems which run parallel to the corresponding theorems of hyperbolic geometry are put down in the following list of exercises.

The formula for the measure of angle used in hyperbolic geometry may be taken over without change, i.e.

$$\theta = m(l_1 l_2) = -\frac{i}{2} \log R(l_1 l_2, i_1 i_2),$$

where  $l_1$  and  $l_2$  are intersecting lines and  $i_1$  and  $i_2$  are tangents to the absolute in the same flat pencil with  $l_1$  and  $l_2$ . The formula for distance may also be taken from hyperbolic geometry:

$$d = \text{Dist}(PQ) = \gamma \log R(PQ, P_\infty Q_\infty).$$

In order that this shall give a real value for the distance between two real points,  $\gamma$  must be a pure imaginary. So we write

$$\gamma = -\frac{ki}{2};$$

\* This geometry can in large part be developed on the basis of Assumptions A, E, S, P alone, the imaginary conic being replaced by the corresponding elliptic polar system, the existence and properties of which are studied in § 89. As a matter of fact there is considerable interest attached to the elliptic geometry in a modular plane, but the point of view which we are taking in this chapter puts order relations in the foreground.

and in order to have formulas in the simplest possible form, we may choose  $k=1$ , so that

$$d = -\frac{i}{2} \log R(PQ, P_\infty Q_\infty).$$

The discussion in § 138 is applicable at once to elliptic geometry if  $f(X, X')$  be taken to be a bilinear form in three variables such that  $f(X, X) = 0$  is the equation of the absolute of elliptic geometry. Thus we have

$$(20) \quad d = \text{Dist}(YZ) = \frac{-ik}{2} \log \frac{(f(Y, Z) + \sqrt{f^2(Y, Z) - f(Y, Y)f(Z, Z)})^2}{f(Y, Y)f(Z, Z)},$$

$$(21) \quad \theta = m(uv) = \frac{-i}{2} \log \frac{(F(u, v) + \sqrt{F^2(u, v) - F(u, u)F(v, v)})^2}{F(u, u)F(v, v)},$$

$$(22) \quad \cos \frac{d}{k} = \sqrt{\frac{f^2(Y, Z)}{f(Y, Y)f(Z, Z)}},$$

$$(23) \quad \cos \theta = \sqrt{\frac{F^2(u, v)}{F(u, u)F(v, v)}}.$$

### EXERCISES

1. The principle of duality holds good in the elliptic geometry.
2. The elliptic geometry is identical with the set of theorems about the geometry of the plane at infinity in three-dimensional Euclidean geometry.
3. The pairs of perpendicular lines at any point are pairs of an elliptic involution.
4. The lines perpendicular to a line  $l$  all meet in the pole of  $l$  with respect to the absolute. Through any point except the pole of  $l$  there is one and but one line perpendicular to  $l$ .
5. Defining a *ray* as a segment whose ends are conjugate with respect to the absolute, prove that Assumption X, § 66, holds in the single elliptic geometry if the restrictions be added that  $A$  and  $B$  are on the same ray.
6. Assumptions XI and XIII of § 66 hold for single elliptic geometry.
7. How may Assumptions XII, XIV, and XV be modified so as to be valid for single elliptic geometry?
8. A circle is a conic touching the absolute in two conjugate imaginary points.
9. A circle is the locus of a point at a fixed distance from a fixed line.
10. If  $A, B, C$  are three collinear points,

$$\text{Dist}(A, B) + \text{Dist}(BC) + \text{Dist}(CA) = \pi.$$

In other words, the total length of a line is  $\pi$ .

11. The sum of the angles of a triangle is less than  $\pi$ .

12. Let  $K^2$  be a circle in a Euclidean plane, and let  $[C^2]$  be the set of circles which meet  $K^2$  in pairs of points on its diameters. An elliptic plane is determined by defining as "elliptic points" all the Euclidean points interior to  $K^2$  and all the pairs of Euclidean points in which  $K^2$  is met by its diameters, and defining as collinear any set of elliptic points on a circle  $C^2$ .

142. **Elliptic geometry of three dimensions.** The three-dimensional elliptic geometry is the set of theorems about a three-dimensional projective space which state properties undisturbed by the projective collineations leaving invariant an arbitrary but fixed projective polar system, called the *absolute polar system*, in which no point is on its polar plane. It is a direct generalization of the elliptic geometry of the plane and may be based on a similar set of assumptions.

In a real space this polar system is that of an imaginary quadric (called the *absolute quadric*) with respect to which each real point has a real polar plane, and the equation of the absolute quadric may be taken to be

$$x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0.$$

A *displacement* is defined as a direct\* projective collineation (cf. § 32) which leaves the absolute polar system invariant; a *symmetry* is defined as a nondirect projective collineation leaving the absolute polar system invariant. The definitions of congruence, perpendicularity, distance, etc. follow the pattern of the hyperbolic and parabolic geometries, and the same method may be used, as in those geometries, to extend the theorems on congruence from the plane to space.

It can easily be proved by means of the theorems on the quadric in Chap. VI that any displacement is a product of two line reflections whose axes are polar with regard to the absolute. From this proposition a series of theorems on displacements can be derived, just as in the parabolic and hyperbolic geometries.

Through a given point not on a given line  $l$  there is no line parallel to  $l$  in the sense in which the term is used in parabolic or hyperbolic geometry. There is, however, a generalization of the Euclidean notion of parallelism to elliptic three-dimensional space which preserves many of the properties of Euclidean parallelism and is, if possible, more interesting.

\* Without appealing to order relations, the direct collineations may be characterized as those which do not interchange the reguli on the absolute quadric.

Any real line  $l$  meets the absolute in two conjugate imaginary points, and through these points there are two lines  $p_1, p_2$  of one regulus and two lines  $q_1, q_2$  of the other regulus. The lines  $p_1, p_2$  are conjugate imaginary lines of the second kind (§ 109), and  $l$  is one line of an elliptic congruence of which  $p_1, p_2$  are directrices. A similar remark applies to the conjugate imaginary lines  $q_1, q_2$ . Any line of the elliptic congruences having  $p_1, p_2$  or  $q_1, q_2$  as directrices is called a *Clifford parallel*\* of  $l$  or a *paratactic*† of  $l$ . Thus there are two Clifford parallels to  $l$  through any point not on  $l$ , and  $l$  is a Clifford parallel to itself.

The two Clifford parallels to any line through any point not on it may be distinguished as follows: Let  $R_1^2$  and  $R_2^2$  be the two reguli on the absolute. Two real lines  $l, m$  meeting two conjugate imaginary lines  $p_1, p_2$  of  $R_1^2$  are *right-handed* Clifford parallels, or paratactics; and two real lines  $l', m'$  meeting two conjugate imaginary lines  $q_1, q_2$  of  $R_2^2$  are *left-handed* Clifford parallels, or paratactics.

The distinction between right-handed and left-handed Clifford parallels may be drawn entirely in terms of real elements by means of the notion of sense-class (§ 32), and thus connected with the intuitive distinction between right and left. This matter will be taken up again in the next chapter. In the meantime it may be remarked that the definition in terms of the two reguli on the absolute is independent of all question of order relations and is based on Assumptions A, E, P alone.

From the definition it follows immediately that if  $l$  is a right-handed Clifford parallel to  $m$ ,  $m$  is a right-handed Clifford parallel to  $l$ ; that if  $m$  is also a right-handed Clifford parallel to  $n$ ,  $l$  is a right-handed Clifford parallel to  $n$ . In general, two lines have one and only one common intersecting perpendicular; but if they are right-handed Clifford parallels, there is a regulus of common intersecting perpendiculars, and the latter are all left-handed Clifford parallels.

The product of two orthogonal line reflections whose axes are Clifford parallels leaves each line of the congruence of Clifford parallels perpendicular to the axes invariant, and is called a *translation*. A

\* Cf. Clifford, A Preliminary Sketch of Biquaternions, Mathematical Papers (London, 1882), p. 181, and Klein, Autographierte Vorlesungen über nicht-euklidische Geometrie, Vol. II (Göttingen, 1892), p. 245.

† E. Study, Jahresbericht der Deutschen Mathematiker-Vereinigung, Vol. XI (1903), p. 319.

translation is right-handed or left-handed according as the congruence of its invariant lines is right-handed or left-handed. Any displacement can be expressed as a product of two translations.

For a discussion of Clifford parallels and related questions see Appendix II of the book by Bonola referred to above, F. Klein, *Mathematische Annalen*, Vol. XXXVII (1890), p. 544, and the other references given above in this section.

**143. Double elliptic geometry.** The geometry corresponding to the group of projective collineations transforming a sphere  $S^2$  in a Euclidean three-space into itself is called *spherical or double elliptic plane geometry*. The sphere  $S$  is called the *double elliptic plane*. The circles in which  $S^2$  is met by planes through its center are called *lines*, and two figures are said to be *congruent* if conjugate under the group of direct projective collineations transforming the sphere into itself.

The plane which is called elliptic in § 141 is sometimes called *single elliptic* to distinguish it from the double elliptic plane here described. Since the plane at infinity  $\pi_\infty$  of a Euclidean space is a single elliptic plane, and since each line through the center of  $S^2$  meets  $S^2$  in two points and  $\pi_\infty$  in one point, there is a correspondence between a single elliptic plane and a double elliptic plane, in which each point of the first corresponds to a pair of points of the latter. By means of this correspondence any result of either geometry can be carried over into the other geometry.

These remarks can all be generalized to  $n$ -dimensions. For a set of assumptions for double elliptic geometry as a separate science, see J. R. Kline, *Annals of Mathematics*, 2d Ser., Vol. XIX (1916), p. 31.

**144. Euclidean geometry as a limiting case of non-Euclidean.** In the two-dimensional case we have seen that the equation of the absolute may be taken as

$$(24) \quad x_1^2 + x_2^2 - 4\gamma^2 x_0^2 = 0,$$

or in line coördinates, as

$$(25) \quad \frac{u_0^2}{4\gamma^2} - (u_1^2 + u_2^2) = 0.$$

The formulas of hyperbolic geometry arise if  $\gamma$  is real and not zero, and of elliptic geometry if  $\gamma$  is imaginary. If we set  $c = \frac{1}{4\gamma^2} = 0$ , (25) may be regarded as the equation of the circle at infinity of the

Euclidean geometry in the form used in § 72. Moreover, if we set  $c = 0$  in the formulas of §§ 138 and 141, we obtain

$$\cos \theta = \frac{u_1 v_1 + u_2 v_2}{\sqrt{(u_1^2 + u_2^2)(v_1^2 + v_2^2)}}$$

and 
$$\theta = -\frac{i}{2} \log \frac{u_1 v_1 + u_2 v_2 + i(u_1 v_2 - u_2 v_1)}{u_1 v_1 + u_2 v_2 - i(u_1 v_2 - u_2 v_1)},$$

which agree with the formulas of Euclidean geometry given in § 72. In like manner, if we set  $c = 0$  in the formula for the differential of distance in § 139, we obtain  $ds^2 = du^2 + dv^2$ . The generalization of these remarks to three or  $n$  dimensions is of course obvious.

If  $c$  changes by continuous variation from a positive to a negative value, it must pass through zero. Since the corresponding geometry is elliptic while  $c$  is positive, parabolic when  $c$  is zero, and hyperbolic while  $c$  is negative, the parabolic geometry is often spoken of as a limiting case both of elliptic and of hyperbolic geometry.

This point of view is reënforced by observing that the formula (10) makes the measure of a fixed angle a continuous function of  $c$ , so that for a small variation of  $c$  the value given by (10) for  $\theta$  suffers a correspondingly small variation. A like remark can be made about the distance between a fixed pair of points.

This has the consequence that for a given figure  $F$  consisting of a finite number of points and lines, and for a given number  $\epsilon$ , a number  $\delta$  can be found such that if  $c$  varies between  $-\delta$  and  $\delta$ , the distance of point-pairs and the angular measure of line-pairs of  $F$  do not vary more than  $\epsilon$ . Nevertheless, in this interval of variation of  $c$  the geometry according to which the distances and angles are measured changes from elliptic through parabolic to hyperbolic.

For example, if  $F$  were a triangle, and the sum of the angles were found by physical measurement to be between  $\pi + \epsilon$  and  $\pi - \epsilon$ , the geometry according to which the measurements were made might be either parabolic, hyperbolic, or elliptic. Further refinements of experimental methods might decrease  $\epsilon$ , but according to current physical doctrine could not reduce it to zero. Hence, while experiment might conceivably prove that the geometry at the bottom of the system of measurements was elliptic or hyperbolic, it could not prove it to be parabolic.

For the details of showing that the Euclidean formula for distance is a limiting case of the non-Euclidean formula, see Clebsch-Lindemann, loc. cit., Vol II, p. 530.

**145. Parameter representation of elliptic displacements.** Suppose the coördinate system so chosen that the equation of the absolute is

$$x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0.$$

The projective collineations which leave the lines of a regulus on the absolute invariant have been proved to have matrices of the form (50) or (51) in § 126. Let  $R_1^2$  be the regulus on the absolute left invariant by the transformations of type (50), and  $R_2^2$  that left invariant by those of type (51). The transformations of type (50) are the translations leaving systems of right-handed Clifford parallels invariant, and those of type (51) the translations leaving systems of left-handed Clifford parallels invariant.

Since any transformation leaving the quadric invariant is a product of one leaving the lines of  $R_1^2$  invariant by one leaving the lines of  $R_2^2$  invariant, any displacement is a product of a transformation of type (50) by one of type (51). Denoting (50) by  $A$  and (51) by  $B$ , the matrix  $\Delta$  of any displacement can be written

$$(26) \quad \Delta = B \cdot A = \begin{pmatrix} \beta_0 & \beta_1 & \beta_2 & \beta_3 \\ -\beta_1 & \beta_0 & \beta_3 & -\beta_2 \\ -\beta_2 & -\beta_3 & \beta_0 & \beta_1 \\ -\beta_3 & \beta_2 & -\beta_1 & \beta_0 \end{pmatrix} \cdot \begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 \\ -\alpha_1 & \alpha_0 & -\alpha_3 & \alpha_2 \\ -\alpha_2 & \alpha_3 & \alpha_0 & -\alpha_1 \\ -\alpha_3 & -\alpha_2 & \alpha_1 & \alpha_0 \end{pmatrix}$$

$$= \begin{pmatrix} \beta_0\alpha_0 - \beta_1\alpha_1 - \beta_2\alpha_2 - \beta_3\alpha_3 & \beta_0\alpha_1 + \beta_1\alpha_0 + \beta_2\alpha_3 - \beta_3\alpha_2 \\ -\beta_1\alpha_0 - \beta_0\alpha_1 - \beta_2\alpha_2 + \beta_3\alpha_3 & -\beta_1\alpha_1 + \beta_0\alpha_0 + \beta_2\alpha_3 + \beta_3\alpha_2 \\ -\beta_2\alpha_0 + \beta_3\alpha_1 - \beta_0\alpha_2 - \beta_1\alpha_3 & -\beta_2\alpha_1 - \beta_3\alpha_0 + \beta_0\alpha_3 - \beta_1\alpha_2 \\ -\beta_3\alpha_0 - \beta_2\alpha_1 + \beta_1\alpha_2 - \beta_0\alpha_3 & -\beta_3\alpha_1 + \beta_2\alpha_0 - \beta_1\alpha_3 - \beta_0\alpha_2 \end{pmatrix}$$

$$\begin{pmatrix} \beta_0\alpha_2 - \beta_1\alpha_3 + \beta_2\alpha_0 + \beta_3\alpha_1 & \beta_0\alpha_3 + \beta_1\alpha_2 - \beta_2\alpha_1 + \beta_3\alpha_0 \\ -\beta_1\alpha_2 - \beta_0\alpha_3 + \beta_2\alpha_0 - \beta_3\alpha_1 & -\beta_1\alpha_3 + \beta_0\alpha_2 - \beta_2\alpha_1 - \beta_3\alpha_0 \\ -\beta_2\alpha_3 + \beta_3\alpha_2 + \beta_0\alpha_0 + \beta_1\alpha_1 & -\beta_2\alpha_2 - \beta_3\alpha_1 - \beta_0\alpha_1 + \beta_1\alpha_0 \\ -\beta_3\alpha_2 - \beta_2\alpha_3 - \beta_1\alpha_0 + \beta_0\alpha_1 & -\beta_3\alpha_3 + \beta_2\alpha_2 + \beta_1\alpha_1 + \beta_0\alpha_0 \end{pmatrix}.$$

If  $\Delta' = B'A'$  is the matrix of a second displacement, and  $B'$  and  $A'$  are of the types (50) and (51) respectively,

$$(27) \quad \Delta' \cdot \Delta = B'A'BA = B'B \cdot A'A,$$

because any displacement leaving all lines of  $R_1^2$  invariant is commutative with any displacement leaving all lines of  $R_2^2$  invariant.

Thus any displacement

$$\begin{aligned}
 (28) \quad x'_0 &= \alpha_{00}x_0 + \alpha_{01}x_1 + \alpha_{02}x_2 + \alpha_{03}x_3, \\
 x'_1 &= \alpha_{10}x_0 + \alpha_{11}x_1 + \alpha_{12}x_2 + \alpha_{13}x_3, \\
 x'_2 &= \alpha_{20}x_0 + \alpha_{21}x_1 + \alpha_{22}x_2 + \alpha_{23}x_3, \\
 x'_3 &= \alpha_{30}x_0 + \alpha_{31}x_1 + \alpha_{32}x_2 + \alpha_{33}x_3
 \end{aligned}$$

is given parametrically in terms of two sets of homogeneous parameters  $\alpha_0, \alpha_1, \alpha_2, \alpha_3$  and  $\beta_0, \beta_1, \beta_2, \beta_3$  by means of the formulas obtained by equating  $a_{ij}$  to the corresponding element of the last matrix in Equation (26).

The formulas for the parameters of the product of two displacements are determined by (27), for if  $\Delta'' = B''A'' = \Delta'\Delta$ , then  $B'' = B'B$  and  $A'' = A'A$ , and hence

$$\begin{aligned}
 (29) \quad \alpha''_0 &= \alpha'_0\alpha_0 - \alpha'_1\alpha_1 - \alpha'_2\alpha_2 - \alpha'_3\alpha_3, \\
 \alpha''_1 &= \alpha'_0\alpha_1 + \alpha'_1\alpha_0 + \alpha'_2\alpha_3 - \alpha'_3\alpha_2, \\
 \alpha''_2 &= \alpha'_0\alpha_2 - \alpha'_1\alpha_3 + \alpha'_2\alpha_0 + \alpha'_3\alpha_1, \\
 \alpha''_3 &= \alpha'_0\alpha_3 + \alpha'_1\alpha_2 - \alpha'_2\alpha_1 + \alpha'_3\alpha_0, \\
 (30) \quad \beta''_0 &= \beta'_0\beta_0 - \beta'_1\beta_1 - \beta'_2\beta_2 - \beta'_3\beta_3, \\
 \beta''_1 &= \beta'_0\beta_1 + \beta'_1\beta_0 - \beta'_2\beta_3 + \beta'_3\beta_2, \\
 \beta''_2 &= \beta'_0\beta_2 + \beta'_1\beta_3 + \beta'_2\beta_0 - \beta'_3\beta_1, \\
 \beta''_3 &= \beta'_0\beta_3 - \beta'_1\beta_2 + \beta'_2\beta_1 + \beta'_3\beta_0.
 \end{aligned}$$

The formulas for the  $\alpha$ 's are, by § 127, the same as for the multiplication of quaternions, and the formulas for the  $\beta$ 's are given by the following quaternion formula:

$$(\beta'_0 - \beta'_1i - \beta'_2j - \beta'_3k)(\beta_0 - \beta_1i - \beta_2j - \beta_3k) = \beta''_0 - \beta''_1i - \beta''_2j - \beta''_3k.$$

Now let  $\lambda_1$  and  $\lambda_2$  be two symbols defined by the multiplication table

$$(31) \quad \begin{array}{c|cc} & \lambda_1 & \lambda_2 \\ \hline \lambda_1 & \lambda_1 & 0 \\ \lambda_2 & 0 & \lambda_2 \end{array}$$

and the conditions  $\lambda_1q = q\lambda_1$ ,  $\lambda_2q = q\lambda_2$ , where  $q$  is any quaternion. If we write

$$\begin{aligned}
 (32) \quad & [\lambda_1(\alpha'_0 + \alpha'_1i + \alpha'_2j + \alpha'_3k) + \lambda_2(\beta'_0 - \beta'_1i - \beta'_2j - \beta'_3k)] \\
 & \cdot [\lambda_1(\alpha_0 + \alpha_1i + \alpha_2j + \alpha_3k) + \lambda_2(\beta_0 - \beta_1i - \beta_2j - \beta_3k)] \\
 & = \lambda_1(\alpha''_0 + \alpha''_1i + \alpha''_2j + \alpha''_3k) + \lambda_2(\beta''_0 - \beta''_1i - \beta''_2j - \beta''_3k),
 \end{aligned}$$

the  $\alpha''$ 's and  $\beta''$ 's are given in terms of the  $\alpha$ 's,  $\beta$ 's,  $\alpha'$ 's, and  $\beta'$ 's by the equations (29) and (30).

The number system whose elements are  $\lambda_1 q_1 + \lambda_2 q_2$ , where  $q_1$  and  $q_2$  are quaternions, is one of the systems of biquaternions referred to in the footnote of § 130. It is often given a form which may be derived as follows:

$$\begin{aligned} \text{Let} \quad e_1 &= \lambda_1 + \lambda_2, & \lambda_1 &= \frac{e_1 + e_2}{2}, \\ (33) \quad e_2 &= \lambda_1 - \lambda_2, & \lambda_2 &= \frac{e_1 - e_2}{2}. \end{aligned}$$

Then  $e_1$  and  $e_2$  obey the multiplication table

$$(34) \quad \begin{array}{c|cc} & e_1 & e_2 \\ \hline e_1 & e_1 & e_2 \\ e_2 & e_2 & e_1 \end{array}$$

$$\begin{aligned} \text{and we have} \quad 2(\lambda_1 q_1 + \lambda_2 q_2) &= (e_1 + e_2)q_1 + (e_1 - e_2)q_2 \\ &= e_1(q_1 + q_2) + e_2(q_1 - q_2), \end{aligned}$$

$$\begin{aligned} \text{or} \quad \lambda_1(\alpha_0 + \alpha_1 i + \alpha_2 j + \alpha_3 k) + \lambda_2(\beta_0 - \beta_1 i - \beta_2 j - \beta_3 k) \\ = e_1 \left( \frac{\alpha_0 + \beta_0}{2} + \left( \frac{\alpha_1 - \beta_1}{2} \right) i + \left( \frac{\alpha_2 - \beta_2}{2} \right) j + \left( \frac{\alpha_3 - \beta_3}{2} \right) k \right) \\ + e_2 \left( \frac{\alpha_0 - \beta_0}{2} + \left( \frac{\alpha_1 + \beta_1}{2} \right) i + \left( \frac{\alpha_2 + \beta_2}{2} \right) j + \left( \frac{\alpha_3 + \beta_3}{2} \right) k \right). \end{aligned}$$

Let us write

$$\begin{aligned} (35) \quad \gamma_0 &= \frac{\alpha_0 + \beta_0}{2}, & \gamma_1 &= \frac{\alpha_1 - \beta_1}{2}, & \gamma_2 &= \frac{\alpha_2 - \beta_2}{2}, & \gamma_3 &= \frac{\alpha_3 - \beta_3}{2}, \\ \delta_0 &= \frac{\alpha_0 - \beta_0}{2}, & \delta_1 &= \frac{\alpha_1 + \beta_1}{2}, & \delta_2 &= \frac{\alpha_2 + \beta_2}{2}, & \delta_3 &= \frac{\alpha_3 + \beta_3}{2}. \end{aligned}$$

The rule for multiplying biquaternions,

$$\begin{aligned} [e_1(\gamma'_0 + \gamma'_1 i + \gamma'_2 j + \gamma'_3 k) + e_2(\delta'_0 + \delta'_1 i + \delta'_2 j + \delta'_3 k)] \\ \cdot [e_1(\gamma''_0 + \gamma''_1 i + \gamma''_2 j + \gamma''_3 k) + e_2(\delta''_0 + \delta''_1 i + \delta''_2 j + \delta''_3 k)] \\ = e_1(\gamma'_0 \gamma''_0 + \gamma'_1 \gamma''_1 + \gamma'_2 \gamma''_2 + \gamma'_3 \gamma''_3) + e_2(\delta'_0 \delta''_0 + \delta'_1 \delta''_1 + \delta'_2 \delta''_2 + \delta'_3 \delta''_3), \end{aligned}$$

gives the following equations:

$$\begin{aligned} (36) \quad \gamma''_0 &= \gamma'_0 \gamma_0 - \gamma'_1 \gamma_1 - \gamma'_2 \gamma_2 - \gamma'_3 \gamma_3 + \delta'_0 \delta_0 - \delta'_1 \delta_1 - \delta'_2 \delta_2 - \delta'_3 \delta_3, \\ \gamma''_1 &= \gamma'_0 \gamma_1 + \gamma'_1 \gamma_0 + \gamma'_2 \gamma_3 - \gamma'_3 \gamma_2 + \delta'_0 \delta_1 + \delta'_1 \delta_0 + \delta'_2 \delta_3 - \delta'_3 \delta_2, \\ \gamma''_2 &= \gamma'_0 \gamma_2 - \gamma'_1 \gamma_3 + \gamma'_2 \gamma_0 + \gamma'_3 \gamma_1 + \delta'_0 \delta_2 - \delta'_1 \delta_3 + \delta'_2 \delta_0 + \delta'_3 \delta_1, \\ \gamma''_3 &= \gamma'_0 \gamma_3 + \gamma'_1 \gamma_2 - \gamma'_2 \gamma_1 + \gamma'_3 \gamma_0 + \delta'_0 \delta_3 + \delta'_1 \delta_2 - \delta'_2 \delta_1 + \delta'_3 \delta_0, \\ \delta''_0 &= \gamma'_0 \delta_0 - \gamma'_1 \delta_1 - \gamma'_2 \delta_2 - \gamma'_3 \delta_3 + \delta'_0 \gamma_0 - \delta'_1 \gamma_1 - \delta'_2 \gamma_2 - \delta'_3 \gamma_3, \\ \delta''_1 &= \gamma'_0 \delta_1 + \gamma'_1 \delta_0 + \gamma'_2 \delta_3 - \gamma'_3 \delta_2 + \delta'_0 \gamma_1 + \delta'_1 \gamma_0 + \delta'_2 \gamma_3 - \delta'_3 \gamma_2, \\ \delta''_2 &= \gamma'_0 \delta_2 - \gamma'_1 \delta_3 + \gamma'_2 \delta_0 + \gamma'_3 \delta_1 + \delta'_0 \gamma_2 - \delta'_1 \gamma_3 + \delta'_2 \gamma_0 + \delta'_3 \gamma_1, \\ \delta''_3 &= \gamma'_0 \delta_3 + \gamma'_1 \delta_2 - \gamma'_2 \delta_1 + \gamma'_3 \delta_0 + \delta'_0 \gamma_3 + \delta'_1 \gamma_2 - \delta'_2 \gamma_1 + \delta'_3 \gamma_0. \end{aligned}$$

The  $\gamma$ 's and  $\delta$ 's given by (36) may be regarded as a new set of parameters for the elliptic displacements. Since the  $\alpha$ 's and  $\beta$ 's are separate sets of homogeneous variables, they may be subjected to the relation

$$(37) \quad \alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = \beta_0^2 + \beta_1^2 + \beta_2^2 + \beta_3^2.$$

By means of (35) the relation (37) becomes

$$(38) \quad \gamma_0\delta_0 + \gamma_1\delta_1 + \gamma_2\delta_2 + \gamma_3\delta_3 = 0.$$

The formulas for the coefficients of a displacement (28) in terms of the new parameters are found by substituting

$$\begin{aligned} \alpha_0 &= \gamma_0 + \delta_0, & \alpha_1 &= \gamma_1 + \delta_1, & \alpha_2 &= \gamma_2 + \delta_2, & \alpha_3 &= \gamma_3 + \delta_3, \\ \beta_0 &= \gamma_0 - \delta_0, & \beta_1 &= -\gamma_1 + \delta_1, & \beta_2 &= -\gamma_2 + \delta_2, & \beta_3 &= -\gamma_3 + \delta_3 \end{aligned}$$

in the formulas for  $\alpha_\gamma$  in terms of the  $\alpha$ 's and  $\beta$ 's. In other words, the matrix of the displacement corresponding to  $(\gamma_0, \gamma_1, \gamma_2, \gamma_3; \delta_0, \delta_1, \delta_2, \delta_3)$  is

$$\begin{aligned} & \begin{bmatrix} \gamma_0 - \gamma_1 - \gamma_2 - \gamma_3 \\ \gamma_1 & \gamma_0 - \gamma_3 & \gamma_2 \\ \gamma_2 & \gamma_3 & \gamma_0 - \gamma_1 \\ \gamma_3 - \gamma_2 & \gamma_1 & \gamma_0 \end{bmatrix} - \begin{bmatrix} \delta_0 - \delta_1 - \delta_2 - \delta_3 \\ \delta_1 & \delta_0 - \delta_3 & \delta_2 \\ \delta_2 & \delta_3 & \delta_0 - \delta_1 \\ \delta_3 - \delta_2 & \delta_1 & \delta_0 \end{bmatrix} \\ & \cdot \left[ \begin{bmatrix} \gamma_0 & \gamma_1 & \gamma_2 & \gamma_3 \\ -\gamma_1 & \gamma_0 - \gamma_3 & \gamma_2 \\ -\gamma_2 & \gamma_3 & \gamma_0 - \gamma_1 \\ -\gamma_3 - \gamma_2 & \gamma_1 & \gamma_0 \end{bmatrix} + \begin{bmatrix} \delta_0 & \delta_1 & \delta_2 & \delta_3 \\ -\delta_1 & \delta_0 - \delta_3 & \delta_2 \\ -\delta_2 & \delta_3 & \delta_0 - \delta_1 \\ -\delta_3 - \delta_2 & \delta_1 & \delta_0 \end{bmatrix} \right], \end{aligned}$$

and the formulas for the composition of two displacements are (36).

### EXERCISE

The elliptic displacements are orthogonal transformations in four homogeneous variables. Work out the parameter representation determined by the formula

$$R = (1 - S)(1 + S)^{-1}$$

of § 125.

**146. Parameter representation of hyperbolic displacements.** Let the equation of the absolute be taken in the form

$$(39) \quad x_0^2 + \mu^2(x_1^2 + x_2^2 + x_3^2) = 0.$$

If  $\mu$  is real, the corresponding geometry is elliptic; and if  $\mu$  is a pure imaginary, the corresponding geometry is hyperbolic. No generality is lost by taking  $\mu = 1$  (as in the section above) for the elliptic case and  $\mu = \sqrt{-1}$  in the hyperbolic case. For the sake of the limiting process referred to at the end of the section, we shall, however, carry out the discussion for an arbitrary  $\mu$ .

By precisely the reasoning used in § 126 it is seen that any collineation leaving one regulus on the absolute invariant has the matrix

$$A = \begin{pmatrix} \alpha_0 & \mu\alpha_1 & \mu\alpha_2 & \mu\alpha_3 \\ -\frac{1}{\mu}\alpha_1 & \alpha_0 & -\alpha_3 & \alpha_2 \\ -\frac{1}{\mu}\alpha_2 & \alpha_3 & \alpha_0 & -\alpha_1 \\ -\frac{1}{\mu}\alpha_3 & -\alpha_2 & \alpha_1 & \alpha_0 \end{pmatrix},$$

and any collineation leaving the other regulus invariant has the matrix

$$B = \begin{pmatrix} \beta_0 & \mu\beta_1 & \mu\beta_2 & \mu\beta_3 \\ -\frac{1}{\mu}\beta_1 & \beta_0 & \beta_3 & -\beta_2 \\ -\frac{1}{\mu}\beta_2 & -\beta_3 & \beta_0 & \beta_1 \\ -\frac{1}{\mu}\beta_3 & \beta_2 & -\beta_1 & \beta_0 \end{pmatrix}.$$

Hence any displacement has a matrix  $BA$ . In other words, if

$$\begin{aligned} a_{00} &= \beta_0\alpha_0 - \beta_1\alpha_1 - \beta_2\alpha_2 - \beta_3\alpha_3, \\ a_{01} &= \mu(\beta_0\alpha_1 + \beta_1\alpha_0 + \beta_2\alpha_3 - \beta_3\alpha_2), \\ a_{02} &= \mu(\beta_0\alpha_2 - \beta_1\alpha_3 + \beta_2\alpha_0 + \beta_3\alpha_1), \\ a_{03} &= \mu(\beta_0\alpha_3 + \beta_1\alpha_2 - \beta_2\alpha_1 + \beta_3\alpha_0), \\ a_{10} &= -\frac{1}{\mu}(\beta_1\alpha_0 + \beta_0\alpha_1 + \beta_3\alpha_2 - \beta_2\alpha_3), \\ a_{11} &= -\beta_1\alpha_1 + \beta_0\alpha_0 + \beta_3\alpha_3 + \beta_2\alpha_2, \\ a_{12} &= -\beta_1\alpha_2 - \beta_0\alpha_3 + \beta_3\alpha_0 - \beta_2\alpha_1, \\ a_{13} &= -\beta_1\alpha_3 + \beta_0\alpha_2 - \beta_3\alpha_1 - \beta_2\alpha_0, \\ a_{20} &= -\frac{1}{\mu}(\beta_2\alpha_0 - \beta_3\alpha_1 + \beta_0\alpha_3 + \beta_1\alpha_2), \\ a_{21} &= -\beta_2\alpha_1 - \beta_3\alpha_0 + \beta_0\alpha_3 - \beta_1\alpha_2, \\ a_{22} &= -\beta_2\alpha_2 + \beta_3\alpha_3 + \beta_0\alpha_0 + \beta_1\alpha_1, \\ a_{23} &= -\beta_2\alpha_3 - \beta_3\alpha_2 - \beta_0\alpha_1 + \beta_1\alpha_0, \\ a_{30} &= -\frac{1}{\mu}(\beta_3\alpha_0 + \beta_2\alpha_1 - \beta_1\alpha_2 + \beta_0\alpha_3), \\ a_{31} &= -\beta_3\alpha_1 + \beta_2\alpha_0 - \beta_1\alpha_3 - \beta_0\alpha_2, \\ a_{32} &= -\beta_3\alpha_2 - \beta_2\alpha_3 - \beta_1\alpha_0 + \beta_0\alpha_1, \\ a_{33} &= -\beta_3\alpha_3 + \beta_2\alpha_2 + \beta_1\alpha_1 + \beta_0\alpha_0, \end{aligned} \tag{40}$$

the transformation (28) is a displacement.

As we have already seen in the elliptic case, if  $A'$  and  $B'$  are matrices analogous to  $A$  and  $B$ ,

$$B'A' \cdot BA = B'B \cdot A'A.$$

Hence the product of two displacements  $BA$  and  $B'A'$  is a displacement  $B''A''$  such that

$$A'' = A'A$$

and

$$B'' = B'B.$$

On multiplying out the two matrix products  $A'A$  and  $B'B$ , it is evident that the elements of  $A''$  and  $B''$  are given by the formulas (29) and (30) found above for the elliptic case. These formulas are associated with the biquaternions determined by the table (31).

The remark must now be made that if  $\mu = \sqrt{-1}$ , the parameter representation above does not give real values of  $\alpha_v$  for real values of the  $\alpha$ 's and  $\beta$ 's. Suppose, however, that we transform the biquaternions  $\lambda_1 q_1 + \lambda_2 q_2$  as follows:

$$\begin{aligned} \epsilon_1 &= \lambda_1 + \lambda_2, & \lambda_1 &= \frac{\mu\epsilon_1 + \epsilon_2}{2\mu}, \\ \epsilon_2 &= \mu(\lambda_1 - \lambda_2), & \lambda_2 &= \frac{\mu\epsilon_1 - \epsilon_2}{2\mu}. \end{aligned}$$

Then  $\epsilon_1$  and  $\epsilon_2$  obey the multiplication table

$$(41) \quad \begin{array}{c|cc} & \epsilon_1 & \epsilon_2 \\ \hline \epsilon_1 & \epsilon_1 & \epsilon_2 \\ \epsilon_2 & \epsilon_2 & \mu^2 \epsilon_1 \end{array},$$

$$\begin{aligned} \text{and we have } 2\mu(\lambda_1 q_1 + \lambda_2 q_2) &= (\mu\epsilon_1 + \epsilon_2)q_1 + (\mu\epsilon_1 - \epsilon_2)q_2 \\ &= \mu(q_1 + q_2)\epsilon_1 + (q_1 - q_2)\epsilon_2, \end{aligned}$$

$$\begin{aligned} \text{or } \lambda_1(\alpha_0 + \alpha_1 i + \alpha_2 j + \alpha_3 k) + \lambda_2(\beta_0 - \beta_1 i - \beta_2 j - \beta_3 k) \\ = \epsilon_1(\gamma_0 + \gamma_1 i + \gamma_2 j + \gamma_3 k) + \epsilon_2(\delta_0 + \delta_1 i + \delta_2 j + \delta_3 k), \end{aligned}$$

where

$$(42) \quad \begin{aligned} \gamma_0 &= \frac{(\alpha_0 + \beta_0)}{2}, & \gamma_1 &= \frac{(\alpha_1 - \beta_1)}{2}, & \gamma_2 &= \frac{(\alpha_2 - \beta_2)}{2}, & \gamma_3 &= \frac{(\alpha_3 - \beta_3)}{2}, \\ \delta_0 &= \frac{\alpha_0 - \beta_0}{2\mu}, & \delta_1 &= \frac{\alpha_1 + \beta_1}{2\mu}, & \delta_2 &= \frac{\alpha_2 + \beta_2}{2\mu}, & \delta_3 &= \frac{\alpha_3 + \beta_3}{2\mu}. \end{aligned}$$

The rule for multiplying biquaternions

$$\begin{aligned} \epsilon_1(\gamma_0'' + \gamma_1''i + \gamma_2''j + \gamma_3''k) + \epsilon_2(\delta_0'' + \delta_1''i + \delta_2''j + \delta_3''k) \\ = [\epsilon_1(\gamma_0' + \gamma_1'i + \gamma_2'j + \gamma_3'k) + \epsilon_2(\delta_0' + \delta_1'i + \delta_2'j + \delta_3'k)] \\ \cdot [\epsilon_1(\gamma_0 + \gamma_1i + \gamma_2j + \gamma_3k) + \epsilon_2(\delta_0 + \delta_1i + \delta_2j + \delta_3k)], \end{aligned}$$

according to (41), gives the following equations:

$$\begin{aligned} (43) \quad \gamma_0'' &= \gamma_0'\gamma_0 - \gamma_1'\gamma_1 - \gamma_2'\gamma_2 - \gamma_3'\gamma_3 + \mu^2(\delta_0'\delta_0 - \delta_1'\delta_1 - \delta_2'\delta_2 - \delta_3'\delta_3), \\ \gamma_1'' &= \gamma_0'\gamma_1 + \gamma_1'\gamma_0 + \gamma_2'\gamma_3 - \gamma_3'\gamma_2 + \mu^2(\delta_0'\delta_1 + \delta_1'\delta_0 + \delta_2'\delta_3 - \delta_3'\delta_2), \\ \gamma_2'' &= \gamma_0'\gamma_2 - \gamma_1'\gamma_3 + \gamma_2'\gamma_0 + \gamma_3'\gamma_1 + \mu^2(\delta_0'\delta_2 - \delta_1'\delta_3 + \delta_2'\delta_0 + \delta_3'\delta_1), \\ \gamma_3'' &= \gamma_0'\gamma_3 + \gamma_1'\gamma_2 - \gamma_2'\gamma_1 + \gamma_3'\gamma_0 + \mu^2(\delta_0'\delta_3 + \delta_1'\delta_2 - \delta_2'\delta_1 + \delta_3'\delta_0), \\ \delta_0'' &= \gamma_0'\delta_0 - \gamma_1'\delta_1 - \gamma_2'\delta_2 - \gamma_3'\delta_3 + \delta_0'\gamma_0 - \delta_1'\gamma_1 - \delta_2'\gamma_2 - \delta_3'\gamma_3, \\ \delta_1'' &= \gamma_0'\delta_1 + \gamma_1'\delta_0 + \gamma_2'\delta_3 - \gamma_3'\delta_2 + \delta_0'\gamma_1 + \delta_1'\gamma_0 + \delta_2'\gamma_3 - \delta_3'\gamma_2, \\ \delta_2'' &= \gamma_0'\delta_2 - \gamma_1'\delta_3 + \gamma_2'\delta_0 + \gamma_3'\delta_1 + \delta_0'\gamma_2 - \delta_1'\gamma_3 + \delta_2'\gamma_0 + \delta_3'\gamma_1, \\ \delta_3'' &= \gamma_0'\delta_3 + \gamma_1'\delta_2 - \gamma_2'\delta_1 + \gamma_3'\delta_0 + \delta_0'\gamma_3 + \delta_1'\gamma_2 - \delta_2'\gamma_1 + \delta_3'\gamma_0. \end{aligned}$$

For  $\mu = 0$  these equations reduce to (64) and (65) of § 130, and for  $\mu^2 = 1$  they reduce to (36). For  $\mu^2 = -1$  they give the standard formulas for combining hyperbolic displacements. Thus there are three essentially distinct systems of biquaternions, determined respectively by the conditions  $\mu^2 = 1$ ,  $\mu^2 = -1$ ,  $\mu = 0$ . The first corresponds to the elliptic, the second to the hyperbolic, and the third to the parabolic geometry. The geometry in each case is determined by an absolute whose equation in point coördinates is (39), and in plane coördinates,

$$(44) \quad \mu^2 u_0^2 + u_1^2 + u_2^2 + u_3^2 = 0.$$

Since the same geometry corresponds to any two real values of  $\mu$ , there must be a simple isomorphism between any two systems of biquaternions corresponding to positive values of  $\mu^2$ ; and a like statement holds with regard to the systems of biquaternions corresponding to negative values of  $\mu^2$ . The biquaternions for which  $\mu = 0$  may be regarded as a limiting case between those for which  $\mu^2$  is positive and those for which  $\mu^2$  is negative, just as the parabolic geometry is regarded as a limiting case between the hyperbolic and elliptic (§ 144).

In these remarks it is understood that the coefficients  $\gamma_0, \gamma_1, \gamma_2, \gamma_3, \delta_0, \delta_1, \delta_2, \delta_3$  are always real. From the geometrical discussion above it is clear that if these coefficients were taken as complex, the

biquaternions for which  $\mu^2=1$  would be isomorphic with those for which  $\mu^2=-1$ .

The multiplication table (41), in case  $\mu^2=-1$ , is satisfied if we take  $\epsilon_1=1$  and  $\epsilon_2=\sqrt{-1}$ . Hence the biquaternions with real coefficients,

$$\epsilon_1(\gamma_0 + \gamma_1 i + \gamma_2 j + \gamma_3 k) + \epsilon_2(\delta_0 + \delta_1 i + \delta_2 j + \delta_3 k),$$

are equivalent, in case  $\mu^2=-1$ , to the quaternions with ordinary complex coefficients,

$$\alpha_0 + \beta_0 \sqrt{-1} + (\alpha_1 + \beta_1 \sqrt{-1})i + (\alpha_2 + \beta_2 \sqrt{-1})j + (\alpha_3 + \beta_3 \sqrt{-1})k.$$

The biquaternions for which  $\mu=0$ , when taken with complex coefficients, may be regarded as a number system of sixteen units with real coefficients. This is the number system (§ 130) which is needed to study the displacements in the complex Euclidean geometry, and it may be regarded as containing the other systems of real biquaternions.

## CHAPTER IX

### THEOREMS ON SENSE AND SEPARATION

**147. Plan of the chapter.** The theorems and definitions of Chapter II are for the most part special cases of more general concepts of Analysis Situs. The present chapter develops these ideas further, so that the two chapters together lay the foundation for the class of theorems which are particularly of use in the application of geometry to analysis, and vice versa.

In most of the chapter attention is confined to theorems which can be proved without the use of the continuity assumptions (C, R). Many of the theorems are proved on the basis of A, E, S alone and others on the basis of A, E, S, P.

In the first sections (§§ 148–153) of this chapter we prove some of the general theorems about convex regions. These are followed (§§ 154–157) by the definitions of some very general concepts, such as curve, region, continuous group, etc. It will not be necessary (or possible in the remaining pages) to develop the corresponding general theory to any considerable extent. Nevertheless, these general notions underlie and give unity to the rest of the chapter, which may in fact be regarded as a study of certain continuous families of figures by special methods.

In §§ 158–181 the theory of sense-classes is developed in considerable detail for the various cases considered in earlier chapters and for other cases, the principal idea involved being that of an elementary transformation. Finally (§§ 182–199), we prove the fundamental theorems on the regions determined in a plane by polygons and in space by polyhedra.

**148. Convex regions.** **THEOREM 1.** *If  $l$  is a line coplanar with a triangular region  $R$  and containing a point of  $R$ , the points of  $R$  on  $l$  constitute a segment.*

*Proof.* A line coplanar with a triangle and not containing more than one vertex meets the sides of the triangle in at least two and at most three points. These points, by § 22, are the ends of two or

three segments. By Theorem 20, Chap. II, the points of any one of these segments are in the same one of the triangular regions determined by the triangle, and two points in different segments are in different triangular regions.

**COROLLARY.** *The points common to a tetrahedral region and a line containing one of its points constitute a linear segment.*

*Proof.* A line not on one of the planes of a tetrahedron meets these planes in at least two and at most four points. The rest of the argument is the same as for the theorem above, replacing Theorem 20, Chap. II, by Theorem 21 of the same chapter.

Convex regions on a line have been defined and studied in § 23.

**DEFINITION.** A set of points in a plane is said to be a *two-dimensional* (or *planar*) *convex region* if and only if it satisfies the following conditions: (1) Any two points of the set are joined by an interval consisting entirely of points of the set, (2) every point of the set is interior to a triangular region containing no point not in the set, and (3) there is at least one line coplanar with and not containing any point of the set.

A triangular region, a Euclidean plane, and the interior of a conic are examples of planar convex regions.

**THEOREM 2.** *If  $l$  is a line coplanar with a two-dimensional convex region  $R$  and containing a point of  $R$ , the points of  $R$  on  $l$  constitute a linear convex region.*

*Proof.* The definition of a linear convex region is given in § 23. That the points of  $R$  on  $l$  satisfy (1) of that definition follows directly from (1) of the definition of a planar convex region. To prove (2) that any point  $P$  of  $R$  on  $l$  is interior to a segment of points of  $R$  on  $l$ , we observe that by (2) of the definition of a planar convex region  $P$  is interior to a triangular region consisting entirely of points of  $R$  and that by Theorem 1 the points common to  $l$  and this triangular region are a linear segment. Condition (3) of the definition of a linear convex region is satisfied by the points of  $R$  on  $l$  because  $l$  contains one point of the line coplanar with  $R$  and not containing any point of  $R$ .

**DEFINITION.** A set of points in space is said to be a *three-dimensional* (or *spatial*) *convex region* if and only if it satisfies the following conditions: (1) Any two points of the set are joined by an interval

consisting entirely of points of the set, (2) every point of the set is interior to a tetrahedral region containing no points not in the set, and (3) there is at least one plane containing no point of the set.

A tetrahedral region, a Euclidean space, and a hyperbolic space are examples of three-dimensional convex regions.

**THEOREM 3.** *If a line  $l$  contains a point of a three-dimensional convex region  $R$ , the points of  $R$  on  $l$  constitute a linear convex region.*

The proof of this theorem follows the same lines as that of Theorem 2, the corollary of Theorem 1 being used instead of Theorem 1 in showing that the points of  $l$  in  $R$  satisfy Condition (2) of the definition of linear convex region.

In consequence of Theorems 2 and 3 the definitions (*between, precede, ray, sense*; etc.) and theorems of § 23 are applicable to collinear sets of points in two- and three-dimensional convex regions. In the rest of this chapter the segment  $AB$  where  $A$  and  $B$  are in a given convex region  $R$  always means the segment  $AB$  of points of  $R$ .

**THEOREM 4.** *If  $ABC$  are three noncollinear points of a convex region  $R$ ,  $D$  a point of  $R$  in the order  $\{BCD\}$ , and  $E$  a point of  $R$  in the order  $\{CEA\}$ , there exists a point  $F$  of  $R$  in the orders  $\{AFB\}$  and  $\{DEF\}$ .*

*Proof.* Let  $F$  be defined as the point of intersection of the lines  $DE$  and  $AB$  (fig. 77, p. 351). By (3) of the definition of a two- or three-dimensional convex region there is a line  $l_\infty$  coplanar with  $A, B$ , and  $C$  and containing no point of  $R$ . Hence  $l_\infty$  does not meet any of the segments  $AB, BC, CA$ . Hence (Theorem 19, Chap. II) the line  $DE$  which meets the segment  $CA$  and does not meet  $BC$  must meet  $AB$ . Hence  $\{AFB\}$ .

The line  $l_\infty$  does not meet any of the segments  $FB, BD, DF$ , and the line  $AC$  meets the segment  $BD$  and does not meet the segment  $BF$ . Hence  $AC$  meets the segment  $DF$ . Hence  $\{DEF\}$ .

**THEOREM 5.** *A three-dimensional convex region  $R$  satisfies Assumptions I-VIII of the set given for a Euclidean space in § 29.*

*Proof.* Assumptions I, II, III, V, VIII are direct consequences of Theorem 3 and the theorems of § 23. Assumptions VI and VII are consequences of Condition (2) of the definition of a three-dimensional convex region. Assumption IV is a consequence of Theorem 4.

The theory of order relations in convex regions can be based entirely on Theorem 5. This amounts to developing the consequences of Assumptions I-VIII of § 29. Since both the Euclidean and the hyperbolic spaces satisfy these assumptions, this method of treating convex regions is of considerable interest from the point of view of foundations of geometry (cf. references in § 29). The methods required to prove the theorems on this basis are but little different from those used in the next section.

**COROLLARY.** *In a real projective space a convex region also satisfies Assumption XVII of § 29.*

### EXERCISES

1. The set of all points common to a set of convex regions which are all contained in a single convex region is, if existent, a convex region. (In other words, the logical product of a set of convex regions contained in a convex region is a convex region.)

2. Prove on the basis of Assumptions I-VIII of § 29 that for any set of points  $P_1, P_2, \dots, P_n$ , finite in number, there is a line  $l$  such that  $P_1, P_2, \dots, P_n$  are all on the same side of  $l$ .

\*3. A set of points in a projective space such that any two points of the set are joined by *one and only one* segment consisting entirely of points of the set and such that every point of the set is interior to at least one tetrahedral region consisting entirely of points of the set, is a convex region.

\*4. Study the set of assumptions for projective geometry consisting of  $A, E$  and the assumption that in the projective space there is a set of points satisfying the Assumptions I-VIII, XVII for a convex region.

**149. Further theorems on convex regions. THEOREM 6.** *If  $A, B, C$  are three noncollinear points of a convex region  $R$ , they are the vertices of one and only one triangular region consisting entirely of points of  $R$ . This triangular region consists of all points on the segments joining  $A$  to the points of the segment  $BC$ .*

*Proof.* By Theorem 4 a line joining  $B$  to a point of the segment  $CA$  meets a segment joining  $A$  to any point  $A_1$  of the segment  $BC$ ; and by the same theorem any point of the segment  $AA_1$  is joined to  $B$  by a line meeting the segment  $CA$ . Hence the set of points  $[P]$  on the segments joining  $A$  to the points of the segment  $BC$  is identical with the set of points of intersection of lines joining  $A$  to points of the segment  $BC$  with lines joining  $B$  to points of the segment  $CA$ . By similar reasoning  $[P]$  is the set of points of intersection of lines joining  $A$  to points of the segment  $BC$  with lines

joining  $C$  to points of the segment  $AB$ . The points  $[P]$  form a triangular region because they are all the points not separated from a particular  $P$  by any pair of the three lines  $AB$ ,  $BC$ ,  $CA$ .

The other three triangular regions having  $A$ ,  $B$ ,  $C$  as vertices contain points of the line which by (3) of the definition of a convex region is coplanar with  $ABC$  and contains no point of  $R$ . Hence  $[P]$  is the only triangular region satisfying the conditions of the theorem.

In the rest of this section the triangular region determined by three noncollinear points  $A$ ,  $B$ ,  $C$  of a convex region  $R$  according to Theorem 6 shall be called *the triangular region  $ABC$* . It is also called the *interior* of the triangle  $ABC$ .

**COROLLARY.** *If  $ABCD$  are four noncoplanar points of a convex region  $R$ , they are the vertices of one and only one tetrahedral region consisting entirely of points of  $R$ . This tetrahedral region consists of the segments of points of  $R$  joining  $A$  to points of the triangular region  $BCD$ .*

*Proof.* Let  $[\alpha]$  be the set of segments joining  $A$  to points of the triangular region  $BCD$  and  $[P]$  the set of all points on the segments  $[\alpha]$ . Any  $P$  is also on a segment joining  $B$  to a point of the triangular region  $ACD$ , as is seen by applying the theorem above to the figure obtained by taking a section of the tetrahedron  $ABCD$  by the plane  $ABP$ . In like manner any  $P$  is on a segment joining  $C$  to a point of the triangular region  $DAB$ , and on a segment joining  $D$  to a point of the triangular region  $ABC$ .

The same argument shows that any point of intersection of a line joining  $A$  to a point of the triangular region  $BCD$  with a line joining  $B$  to a point of the triangular region  $CAD$  is in the set  $[P]$  and that every  $P$  is a point of this description. From this it follows that  $[P]$  contains all points not separated from a particular  $P$  by the faces of the tetrahedron  $ABCD$ . Hence by Theorem 21, Chap. II,  $[P]$  is a tetrahedral region.

Any tetrahedral region having  $ABCD$  as vertices and distinct from  $[P]$  contains points not in  $R$ , because it either contains points on the segments complementary to  $[\alpha]$  or on the lines joining  $A$  to the points of the triangular regions different from  $BCD$  in the plane  $BCD$ .

**THEOREM 7.** *If a plane  $\pi$  contains a point of a three-dimensional convex region  $R$ , the points of  $R$  on  $\pi$  constitute a planar convex region.*

*Proof.* The points of  $R$  on  $\pi$  satisfy Conditions (1) and (3) of the definition of a planar convex region because  $R$  satisfies Conditions (1) and (3) of the definition of a three-dimensional convex region. To prove that the points of  $R$  on  $\pi$  satisfy (2) of the definition of a planar convex region, let  $P$  be a point of  $R$  on  $\pi$  and  $l$  a line on  $P$  and  $\pi$ . By Theorem 3 there are two points  $A, A_1$  of  $R$  on  $l$  such that the segment  $APA_1$  is composed entirely of points of  $R$ . Let  $a$  be a line on  $A_1$  and  $\pi$  but distinct from  $l$ . By the same reasoning as before there are two points  $B, C$  of  $R$  on  $a$  such that the segment  $BA_1C$  is composed entirely of points of  $R$ . By Theorem 6 the triangular region having  $A, B, C$  as vertices and containing  $P$  contains no points not in  $R$ . Hence the points of  $R$  on  $\pi$  satisfy Condition (2) of the definition of a planar convex region.

**THEOREM 8.** *If  $l$  is any line coplanar with and containing a point of a planar convex region  $R$ , the points of  $R$  not on  $l$  constitute two convex regions such that the segment joining any point of one to any point of the other meets the linear convex region which  $l$  has in common with  $R$ .*

*Proof.* By definition there is a line  $m$  coplanar with  $R$  and containing no point of  $R$ . By Theorem 18, Cor. 1, Chap. II, all points of the plane not on  $l$  or  $m$  fall into two classes  $[O]$  and  $[P]$  such that (1) two points  $O, P$  of different classes are separated by  $l$  and  $m$  and (2) two points of the same class are not separated by  $l$  and  $m$ . The region  $R$  contains points of both of these classes. For let  $I$  be any point of  $R$  on  $l$ . By Theorem 2 any line through  $I$  coplanar with  $R$  and distinct from  $l$  contains a segment of points of  $R$  of which  $I$  is one point. If  $A$  and  $B$  are two points of this segment in the order  $\{AIB\}$ ,  $A$  and  $B$  are separated by  $l$  and  $m$  and also are points of  $R$ . Hence there exist two mutually exclusive classes  $[O']$  and  $[P']$ , subsets of  $[O]$  and  $[P]$  respectively, which contain all points of  $R$  not on  $l$ .

Since any  $O'$  and any  $P'$  are separated by  $l$  and  $m$  and no segment  $O'P'$  contains a point of  $m$ , every segment  $O'P'$  contains a point of  $l$ .

Since two points of the same class ( $[O']$  or  $[P']$ ) are not separated by  $l$  and  $m$ , and since the segment joining them does not contain a point of  $m$ , it does not contain a point of  $l$ .

It remains to show that any point of either of the classes, say  $[O']$ , is interior to a triangular region consisting entirely of points of this class. Let  $p$  be any line on a point  $O'$  and coplanar with  $R$ . Let  $O'_1$  and  $O'_2$  be two points of  $R$  on  $p$  in the order  $\{O'_1 O' O'_2\}$  and such that the segment  $O'_1 O'_2$  does not contain a point of  $l$ . Let  $q$  be any line distinct from  $p$ , coplanar with  $R$  and on  $O'_2$ , and let  $O'_3, O'_4$  be two points of  $R$  on  $q$  in the order  $\{O'_3 O'_2 O'_4\}$  and such that the segment  $O'_3 O'_4$  does not contain a point of  $l$ . By Theorem 6 there is a unique triangular region with  $O'_1, O'_3, O'_4$  as vertices consisting only of points of  $R$  and containing all points of the segment  $O'_1 O'_2$ . Since  $l$  does not meet any of the segments  $O'_1 O'_2, O'_2 O'_3, O'_2 O'_4$ , it cannot meet any segments joining  $O'_1$  to a point of the segment  $O'_3 O'_4$  (Theorem 4). Hence the triangular region  $O'_1 O'_3 O'_4$  consists entirely of points of  $[O']$ .

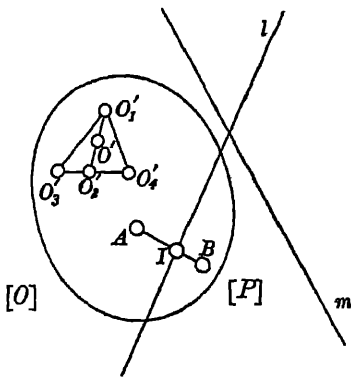


FIG. 80

**COROLLARY 1.** *If  $\pi$  is any plane containing a point of a three-dimensional convex region  $R$ , the points of  $R$  not on  $\pi$  constitute two three-dimensional convex regions such that the segment joining any point of one to any point of the other meets the planar convex region which  $\pi$  has in common with  $R$ .*

*Proof.* The proof is a strict generalization of that of the theorem above to space, using the corollary of Theorem 6 instead of Theorem 6.

**COROLLARY 2.** *For a given line  $l$  (or plane  $\pi$ ) and a given convex region  $R$ , there is only one pair of regions of the sort described in Theorem 8 (or Cor. 1).*

*Proof.* If  $O$  is any point of  $R$  not on  $l$ , the class containing  $O$  must include all points joined to  $O$  by segments not meeting  $l$ . Hence it must be identical with one of the classes given by the theorem.

DEFINITION. The two convex regions determined according to Theorem 8 by a line in a planar convex region are called the *sides* of the line relative to the convex region. The two convex regions determined according to Cor. 1 by a plane in a convex region are called the two *sides* of the plane relative to the convex region.

DEFINITION. Two sets of points  $[P]$ ,  $[Q]$  in a convex region or in a projective plane or space are said to be *separated* by a set  $[S]$  if every segment of the convex region or of the projective plane or space which joins a  $P$  to a  $Q$  contains an  $S$ .

### EXERCISE

Given two lines containing points of a convex region but intersecting in a point  $P$  outside the region. Construct the line joining  $P$  to a point  $Q$  in the region by means of linear constructions involving only points and lines in the region. Cf. Ex. 4, § 20, Vol. I.

150. **Boundary of a convex region.** DEFINITION. A point  $B$  is a *boundary point* of a set of points  $[P]$  if every tetrahedral region containing  $B$  contains a point  $P$  and a point not in  $[P]$ . The set of all boundary points of  $[P]$  is called the *boundary* of  $[P]$ .

THEOREM 9. *All boundary points of a set of points on a line  $l$  are on  $l$ . All boundary points of a set of points on a plane  $\pi$  are on  $\pi$ .*

*Proof.* If  $Q$  is a point not on a line  $l$ , any tetrahedron one of whose faces contains  $l$  and none of whose faces contains  $Q$  will determine a tetrahedral region (§ 26) which contains  $Q$  and does not contain any point of  $l$ . Hence  $Q$  is not a boundary point of any set of points on  $l$ . A like argument proves the second statement in the theorem.

COROLLARY 1. *A boundary point  $B$  of a set of points  $[P]$  on a line  $l$  is any point such that any segment of  $l$  containing  $B$  contains a  $P$  and a point not in  $[P]$ .*

COROLLARY 2. *A boundary point  $B$  of a set of points  $[P]$  on a plane  $\pi$  is any point such that any triangular region of  $\pi$  containing  $B$  contains a  $P$  and a point not in  $[P]$ .*

THEOREM 10. *Let  $\sigma$  be the convex region common to a line  $l$  and a planar convex region  $R$  and let  $R_1$  and  $R_2$  be the convex regions formed by the points of  $R$  which are not on  $\sigma$ . The boundaries of  $R_1$  and of  $R_2$  contain  $\sigma$  and all boundary points of  $\sigma$ . Each boundary*

*point of  $R$  is a boundary point of  $R_1$  or of  $R_2$ , and each boundary point of  $R_1$  or of  $R_2$  which is not on  $l$  is a boundary point of  $R$ .*

*Proof.* If  $Q$  is any point of  $\sigma$ , and  $m$  a line on  $Q$  coplanar with  $R_1$  and distinct from  $l$ , any segment of  $m$  containing  $Q$  contains points both of  $R_1$  and of  $R_2$ . Since any triangular region containing  $Q$  contains a segment of  $m$  containing  $Q$ , it contains points both of  $R_1$  and of  $R_2$ . Hence  $Q$  is a boundary point both of  $R_1$  and of  $R_2$ . If  $B$  is a boundary point of  $\sigma$ , any triangular region containing  $B$  contains a point  $Q$  of  $\sigma$ , and hence, by the argument just given, contains points both of  $R_1$  and of  $R_2$ . Hence  $B$  is a boundary point both of  $R_1$  and of  $R_2$ .

Let  $A$  be a boundary point of  $R$ . Any triangular region  $T$  containing  $A$  contains at least one point not in  $R_1$  or  $R_2$ , namely,  $A$  itself. Since  $A$  is a boundary point of  $R$ ,  $T$  contains at least one point of  $R$ , which may be in  $R_1$  or in  $R_2$  or in  $\sigma$ . In the latter case  $T$  contains points of  $R_1$  and  $R_2$  both, by the paragraph above. Hence in every case  $T$  contains points of  $R_1$  or  $R_2$ . If every triangular region containing  $A$  contains points of  $R_1$  and of  $R_2$ ,  $A$  is a boundary point of both  $R_1$  and  $R_2$ . If this does not happen, some triangular region  $T_0$  containing  $A$  contains points of one of  $R_1$  and  $R_2$  (say  $R_1$ ) and not of the other. Any triangular region  $T$  containing  $A$  then contains points of  $R_1$  because by an easy construction we obtain a triangular region  $T'$  containing  $A$  and contained in both  $T$  and  $T_0$ ; and since  $T'$  contains  $A$ , it contains points of  $R$ , which because they are in  $T_0$  must be points of  $R_1$ . Hence  $A$  is a boundary point of  $R_1$ .

Let  $C$  be a boundary point of  $R_1$  which is not on  $l$ . Any triangular region  $T$  containing  $C$  contains points of  $R$ , because it contains points of  $R_1$ . It also contains points not in  $R_1$ . One of these points is not in  $R$  unless  $T$  consists entirely of points of  $R_1$ ,  $R_2$ , and  $l$ . If the latter case should arise, since  $C$  is not on  $l$  a triangular region  $T'$  could be constructed containing  $C$ , interior to  $T$ , and not containing any point of  $l$ .  $T'$  then would contain points of both  $R_1$  and  $R_2$  and hence would contain a segment joining a point of  $R_1$  to a point of  $R_2$ ; which segment, by Theorem 8, would contain a point of  $l$ , contrary to hypothesis. Hence  $T$  contains points not in  $R$ , and  $C$  is a boundary point of  $R$ .

**COROLLARY.** *Let  $\sigma$  be the convex region common to a plane  $\pi$  and a three-dimensional convex region  $R$ , and let  $R_1$  and  $R_2$  be the convex regions formed by the points of  $R$  which are not on  $\pi$ . The boundaries of  $R_1$  and  $R_2$  contain  $\sigma$  and all boundary points of  $\sigma$ . Each boundary point of  $R$  is a boundary point of  $R_1$  or of  $R_2$ , and each boundary point of  $R_1$  or of  $R_2$  which is not on  $\pi$  is a boundary point of  $R$ .*

It is to be noted that we have not proved that a convex region always has a boundary. Cf. Ex. 7, below.

### EXERCISES

1. If  $A$  and  $B$  are two points of the boundary of a convex region  $R$ , one of the segments joining them consists entirely of points of  $R$  or entirely of points of the boundary of  $R$ .

2. A line has no points, one point, two points, or one interval in common with the boundary of a convex region.

3. If a segment consists of boundary points of a given set, its ends are also boundary points.

4. Using the notation of Theorem 10, no point of  $l$  not in  $\sigma$  or its boundary can be a boundary point of  $R$ . Hence if  $P$  is a point of a two- or three-dimensional convex region  $R$ , and  $B$  a boundary point of  $R$ , the points  $P$  and  $B$  are joined by a segment consisting entirely of points of  $R$ .

5. Using the notation of the corollary of Theorem 10, no point of  $\pi$  not in  $\sigma$  or its boundary can be a boundary point of  $R$ .

6. Using the notation of Theorem 10, if  $R$  and its boundary are contained in another convex region  $R'$ , then no point of the boundary of  $R_1$  not on  $\sigma$  or its boundary can be on the boundary of  $R_2$ .

7. Give an example of a space containing a convex region which has no boundary.

8. A ray whose origin is in the interior of a triangle meets the boundary of this triangular region in one and only one point.

\*9. Let  $O$  be an arbitrary point of a Euclidean plane, and  $R_0$  an arbitrary convex region containing  $O$  and having a boundary which is met in two points by every line which contains a point of  $R_0$ . Let any set of points into which the boundary of  $R_0$  can be transformed by a homothetic transformation (§ 47) be called a *circle*. Let the point to which  $O$  is transformed by the homothetic transformation which carries the boundary of  $R_0$  into any circle be called the *center* of this circle. Let two point-pairs  $AB$  and  $A'B'$  be said to be *congruent* if and only if there is a circle with  $A$  as center and passing through  $B$  which can be carried by a translation into one with  $A'$  as center and passing through  $B'$ . The geometry based on these definitions is analogous to the Euclidean plane geometry. Develop its main theorems. Cf. the memoir of H. Minkowski by D. Hilbert, *Mathematische Annalen*, Vol. LXVIII (1910), p. 445.

**151. Triangular regions.** The theorems of the last sections can be used to complete the discussion of the regions determined by a triangle. We shall continue to use the notation of § 26 and shall denote the sides  $AB, BC, CA$  by  $c, a$ , and  $b$  respectively. The points of the plane which are not on  $a$  form a convex region, of which  $a$  is the boundary. By Theorem 8 the points not on  $a$  or  $b$  fall into two convex regions, of each of which  $a$  and  $b$  together (by Theorem 10) constitute the boundary. The line  $c$  meets  $a$  and  $b$  in the points  $B$  and  $A$  respectively and hence has the segment  $\gamma$  in common with one of the regions and  $\bar{\gamma}$  in common with the other. By Theorem 8 the region containing  $\gamma$  is separated into two convex regions, each having  $\gamma$  on its boundary, and the other into two, each having  $\bar{\gamma}$  on its boundary. Thus the three lines  $a, b, c$  determine four planar convex regions which are identical with the four triangular regions of Theorem 20, Chap. II. Since the lines enter symmetrically, each of the segments  $\alpha, \beta, \gamma, \bar{\alpha}, \bar{\beta}, \bar{\gamma}$  is on the boundary of two and only two of the triangular regions.

The three vertices  $A, B, C$  are on the boundaries of all four triangular regions, because every point of the plane can be joined to these three points by segments not meeting the lines  $a, b, c$ . No point not on  $a, b$ , or  $c$  can be a boundary point of any of the triangular regions, because such a point is an interior point of one of them.

Since any line  $m$  which meets one of the four planar convex regions meets it in a segment the ends of which are the only points of  $m$  on the boundary, the three segments which bound one of the four triangular regions cannot be met by the same line. The boundaries of the four regions therefore consist respectively (cf. fig. 16) of the vertices of the triangle, together with

- $\bar{\alpha}, \bar{\beta}, \bar{\gamma}$  for Region I,
- $\alpha, \beta, \bar{\gamma}$  for Region II,
- $\bar{\alpha}, \beta, \gamma$  for Region III,
- $\alpha, \bar{\beta}, \gamma$  for Region IV.

In addition to what has already been stated in Theorem 2, the discussion above gives us the following information:

**THEOREM 11.** *A triangular region is bounded by the three vertices of the triangle, together with three segments joining them which cannot all be met by a line.*

If  $\overline{AOB}$  and  $\overline{APC}$  are two noncollinear segments they may be denoted by  $\alpha$  and  $\beta$ . The two segments whose ends are  $B$  and  $C$  may be denoted by  $\gamma$  and  $\bar{\gamma}$ ,  $\gamma$  being the one met by the line  $OP$ . As we have just seen,  $\alpha$ ,  $\beta$ , and  $\bar{\gamma}$ , together with the vertices of the triangle, are the boundary of a convex region, and there is one and only one of the four convex regions of whose boundary  $\alpha$  and  $\beta$  form part. Hence

**THEOREM 12.** *For any two noncollinear segments  $\alpha$ ,  $\beta$  having a common end there is a unique triangular region and a unique segment  $\bar{\gamma}$  such that  $\alpha$ ,  $\beta$ , and  $\bar{\gamma}$ , together with the ends of  $\alpha$  and  $\beta$ , form the boundary of the triangular region.*

**COROLLARY 1.** *On any point coplanar with but not in a given triangular region  $T$ , there is at least one line composed entirely of points not in  $T$ .*

**COROLLARY 2.** *The triangular region determined according to Theorem 12 by two noncollinear segments  $\overline{CB'A}$  and  $\overline{CA'B}$  consists of the points of intersection of the lines joining  $B$  to the points of the first segment with the lines joining  $A$  to the points of the second segment.*

The complete set of relations among the points, segments, and triangular regions determined by three noncollinear parts  $A$ ,  $B$ ,  $C$  may be indicated by the following tables,

$H_1:$

	$\alpha$	$\bar{\alpha}$	$\beta$	$\bar{\beta}$	$\gamma$	$\bar{\gamma}$
$A$	0	0	1	1	1	1
$B$	1	1	0	0	1	1
$C$	1	1	1	1	0	0

$H_2:$

	I	II	III	IV
$\alpha$	0	1	0	1
$\bar{\alpha}$	1	0	1	0
$\beta$	0	1	1	0
$\bar{\beta}$	1	0	0	1
$\gamma$	0	0	1	1
$\bar{\gamma}$	1	1	0	0

where in the first table a "1" or a "0" is placed in the  $i$ th row and  $j$ th column according as the point whose name appears at the beginning of the  $i$ th row is or is not an end of the segment whose name appears at the top of the  $j$ th column; and where in the second table a "1" or a "0" is placed in the  $i$ th row and  $j$ th column according as the segment whose name appears at the beginning of the  $i$ th row is or is not a part of the boundary of the triangular region whose name appears at the top of the  $j$ th column.

## EXERCISES

1. The lines polar (§ 18, Vol. I) with respect to a triangle  $ABC$  to the points of one of the four triangular regions determined by  $ABC$  constitute one of the four sets of lines determined by  $ABC$ , according to the dual of Theorem 20, Chap. II. The points on these lines constitute the set of all points coplanar with but not on the given triangular region or its boundary.

2. Divide the lines of the plane of a complete quadrangle into classes according as the point pairs in which they meet the pairs of opposite sides separate one another or not. Apply the results to the problem: When can a real conic be drawn through four given points and tangent to a given line? Dualize.

152. **The tetrahedron.** The discussion in § 151 generalizes at once to space. Let us use the notation of § 26. The points not on  $\alpha_1$  constitute a convex region of which  $\alpha_1$  is the boundary. By Theorem 8, Cor. 1, the points not on  $\alpha_1$  and  $\alpha_2$  constitute two convex regions, of each of which, by Theorem 10,  $\alpha_1$  and  $\alpha_2$  form the boundary.

The plane  $\alpha_3$  has points in each of the three-dimensional convex regions bounded by  $\alpha_1$  and  $\alpha_2$  and hence by Theorem 7 has a planar convex region in common with each of them. By Theorem 8, Cor. 1, each of these planar convex regions separates the spatial convex region in which it lies into two spatial convex regions, of each of which (Theorem 10, Cor.) it forms part of the boundary. Thus the points not on  $\alpha_1, \alpha_2, \alpha_3$  form four spatial convex regions. Since any plane not on  $A_4$  meets  $\alpha_1, \alpha_2$ , and  $\alpha_3$  in a triangle, it meets each of these four spatial convex regions in a triangular region. Thus, since the planes  $\alpha_1, \alpha_2, \alpha_3$  enter symmetrically, we have

**THEOREM 13. DEFINITION.** *Three planes  $\alpha_1, \alpha_2, \alpha_3$  meet by pairs in three lines, and each pair of these lines bounds two planar convex regions. The points not on  $\alpha_1, \alpha_2$ , and  $\alpha_3$  form four spatial convex regions (called trihedral regions) each bounded by the three lines and three of the planar convex regions. The relations among these regions are fully represented by the matrices of § 151 if the three lines are denoted by  $A, B, C$ , the planar convex regions by  $\alpha, \beta, \gamma, \bar{\alpha}, \bar{\beta}, \bar{\gamma}$ , and the three-dimensional regions by I, II, III, IV.*

Each of the four spatial convex regions determined by  $\alpha_1, \alpha_2, \alpha_3$  is met by  $\alpha_4$  in a triangular region and separated by it into two convex regions each of which is partially bounded by the triangular region. Hence the points not on  $\alpha_1, \alpha_2, \alpha_3, \alpha_4$  form eight convex

spatial regions which must be identical with the tetrahedral regions of Theorem 21, Chap. II. Since the planes  $\alpha_1, \alpha_2, \alpha_3, \alpha_4$  enter symmetrically, there are sixteen triangular regions each of which is on the boundary of two and only two three-dimensional regions; and, moreover, each tetrahedral region has one and only one triangular region from each of the four planes on its boundary.

Since any point not on  $\alpha_1, \alpha_2, \alpha_3, \alpha_4$  can be joined to any of the points  $A_1, A_2, A_3, A_4$  by a segment not containing any point of  $\alpha_1, \alpha_2, \alpha_3$ , or  $\alpha_4$ , the points  $A_1, A_2, A_3, A_4$  are on the boundary of all eight tetrahedral regions; and by similar reasoning each segment which bounds a triangular region also bounds each of the tetrahedral regions bounded by the triangular region.

**THEOREM 14.** *The boundary of a tetrahedral region consists of its four vertices, together with four triangular regions and the six segments bounding the four triangular regions and bounded by the four vertices.*

**COROLLARY.** *Three noncoplanar segments having a common end are on the boundary of one and only one of the tetrahedral regions having their ends as vertices.*

The complete set of relations among the points, segments, triangular regions, and tetrahedral regions determined by  $A_1, A_2, A_3, A_4$  may be indicated by three matrices analogous to those employed in § 151. That the points  $A_i$  and  $A_j$  are ends of the segments  $\sigma_{ij}$  and  $\bar{\sigma}_{ij}$  is indicated in the first matrix, a "1" in the  $i$ th row and  $j$ th column signifying that the point whose name appears at the beginning of the  $i$ th row is an end of the segment whose name appears at the top of the  $j$ th column, and a "0" signifying that it is not.

	$\sigma_{12}$	$\bar{\sigma}_{12}$	$\sigma_{13}$	$\bar{\sigma}_{13}$	$\sigma_{14}$	$\bar{\sigma}_{14}$	$\sigma_{23}$	$\bar{\sigma}_{23}$	$\sigma_{24}$	$\bar{\sigma}_{24}$	$\sigma_{34}$	$\bar{\sigma}_{34}$
$A_1$	1	1	1	1	1	1	0	0	0	0	0	0
$A_2$	1	1	0	0	0	0	1	1	1	1	0	0
$A_3$	0	0	1	1	0	0	1	1	0	0	1	1
$A_4$	0	0	0	0	1	1	0	0	1	1	1	1

The four triangular regions in the plane  $\alpha_i (i=1, 2, 3, 4)$  determined by the lines in which the other three planes meet  $\alpha_i$  may be denoted by  $\tau_{i1}, \tau_{i2}, \tau_{i3}, \tau_{i4}$ . Applying the results of

§ 151 to each plane we have the following matrix, in which a "1" or a "0" appears in the  $i$ th row and  $j$ th column according as the segment whose name is at the beginning of the  $i$ th row is or is not on the boundary of the triangular region whose name is at the top of the  $j$ th column.

$H_2$ :

	$\tau_{11}$	$\tau_{12}$	$\tau_{13}$	$\tau_{14}$	$\tau_{21}$	$\tau_{22}$	$\tau_{23}$	$\tau_{24}$	$\tau_{31}$	$\tau_{32}$	$\tau_{33}$	$\tau_{34}$	$\tau_{41}$	$\tau_{42}$	$\tau_{43}$	$\tau_{44}$
$\sigma_{12}$	0	0	0	0	0	0	0	0	0	1	0	1	0	1	0	1
$\bar{\sigma}_{12}$	0	0	0	0	0	0	0	0	1	0	1	0	1	0	1	0
$\sigma_{13}$	0	0	0	0	0	1	0	1	0	0	0	0	0	1	1	0
$\bar{\sigma}_{13}$	0	0	0	0	1	0	1	0	0	0	0	0	1	0	0	1
$\sigma_{14}$	0	0	0	0	0	1	1	0	0	1	1	0	0	0	0	0
$\bar{\sigma}_{14}$	0	0	0	0	1	0	0	1	1	0	0	1	0	0	0	0
$\sigma_{23}$	0	1	0	1	0	0	0	0	0	0	0	0	0	0	1	1
$\bar{\sigma}_{23}$	1	0	1	0	0	0	0	0	0	0	0	0	1	1	0	0
$\sigma_{24}$	0	1	1	0	0	0	0	0	0	0	1	1	0	0	0	0
$\bar{\sigma}_{24}$	1	0	0	1	0	0	0	0	1	1	0	0	0	0	0	0
$\sigma_{34}$	0	0	1	1	0	0	1	1	0	0	0	0	0	0	0	0
$\bar{\sigma}_{34}$	1	1	0	0	1	1	0	0	0	0	0	0	0	0	0	0

Let us denote the eight tetrahedral regions by  $T_1, \dots, T_8$  and construct a matrix analogous to the preceding ones, in which a "1" or a "0" appears in the  $i$ th row and  $j$ th column according as the triangular region whose name is at the beginning of the  $i$ th row is or is not on the boundary of the tetrahedral region whose name is at the top of the  $j$ th column. By definition there is a plane  $\pi$  which meets all the six segments  $\sigma_v$  and none of the segments  $\bar{\sigma}_v$ . There is one and only one tetrahedral region not met by  $\pi$ . Let us assign the notation so that this region is called  $T_1$ . As  $\pi$  cannot meet the segments and triangular regions on the boundaries of  $T_1$ , these segments must be the six segments  $\bar{\sigma}_v$  and these triangular regions must be those bounded by  $\bar{\sigma}_v$ . The latter can be found by means of the matrix  $H_2$ . This determines the first column of the matrix to be constructed. The other columns are found by considering successively the planes of the seven other classes of planes described in

§ 26. Thus, for example,  $T_2$  is the region on whose boundary are the segments  $\bar{\sigma}_{12}$ ,  $\bar{\sigma}_{13}$ ,  $\sigma_{14}$ ,  $\bar{\sigma}_{23}$ ,  $\sigma_{24}$ ,  $\sigma_{34}$ .

$H_2$  :

	$T_1$	$T_2$	$T_3$	$T_4$	$T_5$	$T_6$	$T_7$	$T_8$
$\tau_{11}$	1	0	0	0	0	0	0	1
$\tau_{12}$	0	0	0	1	1	0	0	0
$\tau_{13}$	0	1	0	0	0	0	1	0
$\tau_{14}$	0	0	1	0	0	1	0	0
$\tau_{21}$	1	0	0	0	1	0	0	0
$\tau_{22}$	0	0	0	1	0	0	0	1
$\tau_{23}$	0	1	0	0	0	1	0	0
$\tau_{24}$	0	0	1	0	0	0	1	0
$\tau_{31}$	1	0	1	0	0	0	0	0
$\tau_{32}$	0	0	0	0	0	1	0	1
$\tau_{33}$	0	1	0	1	0	0	0	0
$\tau_{34}$	0	0	0	0	1	0	1	0
$\tau_{41}$	1	1	0	0	0	0	0	0
$\tau_{42}$	0	0	0	0	0	0	1	1
$\tau_{43}$	0	0	1	1	0	0	0	0
$\tau_{44}$	0	0	0	0	1	1	0	0

### EXERCISE

The planes polar (§ 18, Vol. I) with respect to a tetrahedron  $ABCD$  to the points of one of the tetrahedral regions determined by  $ABCD$  constitute one of the four sets of planes determined by  $ABCD$  according to § 26. The points on these planes constitute the set of all points not on the given tetrahedral region or its boundary.

\*153. **Generalization to  $n$  dimensions.** The generalization to  $n$  dimensions of the point pair, triangle, and tetrahedron is the  $(n+1)$ -point in  $n$ -space. This is any set of  $n+1$  points no  $n$  of which are in the same  $(n-1)$ -space, together with the lines, planes, 3-spaces, etc. which they determine by pairs, triads, tetrads, etc. By a direct generalization of § 26 one proves that the points not on the  $n-1$  spaces of an  $(n+1)$ -point fall into  $2^n$  mutually

exclusive sets  $R_1, \dots, R_n$  such that any two points of the same set are joined by a segment of points of the set and that any segment joining two points not in the same set contains at least one point on an  $(n-1)$ -space of the  $(n+1)$ -point. Any one of the sets  $R_1, \dots, R_n$  is called a *simplex* or *n-dimensional segment*.

Thus the simplex is a generalization of the linear segment, triangular region, and tetrahedral region. By replacing triangular and tetrahedral regions by simplexes throughout §§ 148–152 we obtain immediately the theory of  $n$ -dimensional convex regions. A like process applied to §§ 154–157, below, gives the theory of  $n$ -dimensional connected sets, regions, continuous families of sets of points, continuous families of transformations, continuous groups, etc. We leave both series of generalizations to the reader.

**154. Curves.** DEFINITION. Let  $[T]$  be the set of all points on an interval  $T_0T_1$  of a line  $l$ . A set of points  $[P]$  is called a *continuous curve* or, more simply, a *curve*, if it is in such a correspondence  $\Gamma$  with  $[T]$  that

- (1) for every  $T$  there is one and only one  $P$  such that  $P = \Gamma(T)$ ;
- (2) for every  $P$  there is at least one  $T$  such that  $P = \Gamma(T)$ ;
- (3) for every  $T$ , say  $T'$ , and for every tetrahedral region  $R$  containing  $\Gamma(T')$ , there is a segment  $\sigma$  of  $l$  containing  $T'$  and such that for every  $T$  in  $\sigma$ ,  $\Gamma(T)$  is in  $R$ .

A curve is said to be *closed* if  $\Gamma(T_0) = \Gamma(T_1)$ . It is said to be *simple* if  $\Gamma$  can be chosen so as to satisfy (1), (2), (3) and so that if  $T' \neq T''$ ,  $\Gamma(T') \neq \Gamma(T'')$  unless the pair  $T'T''$  is identical with the pair  $T_0T_1$ .

The point  $\Gamma(T)$  is said to *describe* the curve as  $T$  varies. The curve is said to *join* the points  $\Gamma(T_0)$  and  $\Gamma(T_1)$ .

In view of the definition of the geometric number system in Chap. VI, Vol. I, and the theorems in Chap. I, Vol. II, this definition could also be stated in the following form: Let  $(t)$  be the set of numbers such that  $0 \leq t \leq 1$ . A set of points  $[P]$  is called a *curve* if it is in such a correspondence  $\Gamma$  with  $(t)$  that (1) for every  $t$  there is one and only one  $P = \Gamma(t)$ , (2) for every  $P$  there is at least one  $t$  such that  $P = \Gamma(t)$ , and (3) for every  $t$ , say  $t'$ , and for every tetrahedral region  $R$  containing  $\Gamma(t')$  there is a number  $\delta > 0$  such that if  $t' - \delta < t < t' + \delta$ ,  $\Gamma(t)$  is in  $R$ .

In the Euclidean or non-Euclidean spaces (3) may be replaced by the condition: For every  $t'$  and every positive number  $\epsilon$  there is a positive number  $\delta$  such that if  $t' - \delta < t < t' + \delta$ , the distance between  $\Gamma(t)$  and  $\Gamma(t')$  is less than  $\epsilon$ .

The most obvious examples of simple closed curves are the projective line and the point conic. The proof that these are simple closed curves will be given for the planar case, and may be extended at once to the three-dimensional case by substituting tetrahedral regions for triangular ones.

**THEOREM 15.** *A projective line is a simple closed curve.*

*Proof.* Let  $[P]$  be the set of points on a projective line and let  $P_0, P_1, P_2, P_3$  be four particular values of  $[P]$  in the order  $\{P_0 P_1 P_2 P_3\}$ . Let  $T_0, T_1, T_2, T_3, T_4$  be five collinear points in the order  $\{T_0 T_1 T_2 T_3 T_4\}$ , and let  $[T]$  be the set of all points of the interval  $T_0 T_1 T_4$ . If  $T$  is on the interval  $T_0 T_1 T_2$ , let  $\Gamma(T)$  be the point to which  $T$  is carried by a projective correspondence\* which takes the points  $T_0, T_1, T_2$  into  $P_0, P_1, P$  respectively; and if  $T$  is on the interval  $T_2 T_3 T_4$ , let  $\Gamma(T)$  be the point to which  $T$  is carried by a projectivity which carries the points  $T_2, T_3, T_4$  into  $P_2, P_3, P$  respectively.

The correspondence  $\Gamma$  is defined so that there is one and only one point  $P = \Gamma(T)$  for each  $T$ ; and also so that  $\Gamma(T') \neq \Gamma(T'')$ , unless  $T' = T''$ , or  $T' = T_0$  and  $T'' = T_4$ , or  $T' = T_4$  and  $T'' = T_0$ . Thus  $[P]$  satisfies conditions (1) and (2) of the definition of a curve and the condition that a curve be simple.

Let  $R$  be any triangular region containing a point  $P' = \Gamma(T')$ . By Theorem 1 there is a segment of the projective line  $[P]$  containing  $P'$  and contained in  $R$ ; let  $P'' = \Gamma(T'')$  and  $P''' = \Gamma(T''')$  be the ends of this segment. This segment is the image either of the points  $T$  between  $T''$  and  $T'''$  or of the points  $T$  not between  $T''$  and  $T'''$ . Hence if  $\sigma$  be any segment of the line  $T_0 T_1$  containing  $T'$  and not containing  $T''$  or  $T'''$ , every point  $T$  on  $\sigma$  is such that  $\Gamma(T)$  is in  $R$ . Hence  $[P]$  satisfies Condition (3) of the definition of a curve.

**THEOREM 16.** *A point conic is a simple closed curve.*

*Proof.* The proof is precisely the same as that of Theorem 15 except that  $[P]$  is the set of points on a conic, and the following lemma is used instead of Theorem 1.

\*This does not use Assumption P, because it requires only the *existence* of a projectivity, and this may be set up as a series of perspectivities (cf. Chap. III, Vol. I).

LEMMA. If a point  $P$  of a conic  $C^2$  is in a triangular region  $R$  coplanar with  $C^2$ , there is a segment  $\sigma$  of  $C^2$  which contains  $P$  and is contained in  $R$ .

*Proof.* If  $C^2$  is entirely in  $R$  the conclusion of the theorem is obvious. If not, let  $Q$  be a point of  $C^2$  not in  $R$ . By § 75 the points of the line  $PQ$  interior to  $C^2$  constitute a segment having  $P$  and  $Q$  as ends. Let  $R$  be a point of this segment which is also on the segment containing  $P$  (Theorem 1), which the line  $PQ$  has in common with  $R$  (fig. 81).

Let  $T$  be the common point of the tangents at  $P$  and  $Q$  and let  $T'$  and  $T''$  be points of  $R$  in the order  $\{TT'PT''\}$ . Let  $S'$  and  $S''$  be the points in which  $QT'$  and  $QT''$  meet  $TR$ ; so that  $\{TS'RS''\}$ . Let  $S_1$  and  $S_2$  be points interior to  $R$ , interior to  $C^2$ , and in the order  $\{TS'S_1RS_2S''\}$  (Theorem 1). The lines  $QS_1$  and  $QS_2$  meet  $TP$  in two points  $T_1$  and  $T_2$  respectively in the order  $\{TT'T_1PT_2T''\}$ . Since these points are on the segment  $T'PT''$  they are in  $R$ . Since  $Q$  is on the conic  $C^2$  the lines  $QT_1$  and  $QT_2$  meet  $C^2$  in two points  $P_1$  and  $P_2$  respectively.

Since  $S_1$  is interior and  $T_1$  (a point of a tangent) exterior to  $C^2$ , we have the order  $\{QS_1P_1T_1\}$ . But  $S_1$  and  $T_1$  are in  $R$  and  $Q$  is not in  $R$ . Hence by Theorem 1,  $P_1$  is in  $R$ . In like manner  $P_2$  is in  $R$ .

The segment  $\overline{P_1P_2}$  of the conic  $C^2$  is now easily seen to consist entirely of points of  $R$ . For if  $\bar{P}$  is any point of this segment, and  $\bar{T}$  and  $\bar{S}$  the points in which  $Q\bar{P}$  meets  $PT$  and  $RT$  respectively,

$$P\bar{P}R_1P_2 \wedge P\bar{T}T_1T_2 \frac{Q}{\wedge} R\bar{S}S_1S_2.$$

Hence  $\bar{T}$  is on the segment  $\overline{T_1PT_2}$ , and  $\bar{S}$  is on the segment  $\overline{S_1RS_2}$ . Hence  $\bar{T}$  and  $\bar{S}$  are interior to  $R$ , and  $\bar{S}$  interior to  $C^2$ . Since  $\bar{T}$  is exterior to  $C^2$ , it follows that  $\bar{S}$  and  $\bar{T}$  separate  $\bar{P}$  and  $Q$ . Therefore, as  $Q$  is not in  $R$ ,  $\bar{P}$  is in  $R$ .

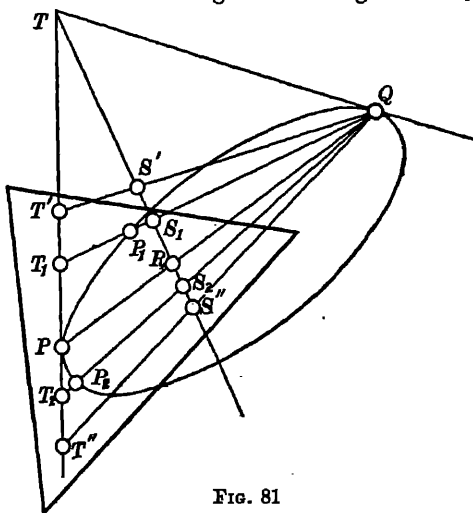


FIG. 81

## EXERCISE

The boundary of a triangular region is a simple closed curve.

155. **Connected sets, regions, etc.** A set of points is said to be *connected* if and only if any two points of the set are joined by a curve consisting entirely of points of the set. A connected set is sometimes called a *continuous family of points*. In a space satisfying Assumptions A, E, H, C (or A, E, K or A, E, J) a connected set is also called a *continuum*. A connected set in a plane such that every point of the set is in a triangular region containing no points not in the set is called a *planar region*. A connected set of points in space such that every point of the set is in a tetrahedral region containing no points not in the set is called a *three-dimensional region*.

A one-to-one transformation  $\Gamma$  carrying a set of points  $[X]$  into a set of points  $[Y]$  is said to be *continuous* if and only if for every  $X$ , say  $X'$ , and every tetrahedral region  $T$  containing  $\Gamma(X')$ , there is a tetrahedral region  $R$  containing  $X'$  and such that for every  $X$  in  $R$ ,  $\Gamma(X)$  is in  $T$ .

If a linear interval joining two points  $A, B$  is subjected to a continuous one-to-one reciprocal transformation, it goes into a curve joining the transforms of  $A$  and  $B$  (§ 154). The set of points on the curve, excluding the transforms of  $A$  and  $B$ , is called a *1-cell*.

If a triangular region and its boundary are subjected to a continuous one-to-one reciprocal transformation, the set of points into which the triangular region goes is called a *simply connected element of surface*, or a *2-cell*.

If a tetrahedral region and its boundary are subjected to a continuous one-to-one reciprocal transformation, the set of points into which the boundary goes is called a *simply connected surface*, or *simple surface*, and the set of points into which the tetrahedral region goes is called a *simply connected three-dimensional region*, or a *3-cell*.

## EXERCISES

1. A region contains no point of its boundary.
2. If  $A$  and  $B$  are any two points of a planar region  $R$ , there exists a finite number of triangular regions  $t_1, t_2, \dots, t_n$  such that  $t_i$  has a point in common with  $t_{i+1}$  ( $i = 1, \dots, n-1$ ) and  $t_1$  contains  $A$  and  $t_n$  contains  $B$ . This property could be taken as the definition of a region in a plane.

3. Given any set of regions all contained in a convex region. The set of all points in triangular regions whose vertices are in the given regions is a convex region. This region is contained in every convex region containing the given set of regions (J. W. Alexander).

4. The set of all points on segments joining pairs of points of an arbitrary region  $R$  contained in a convex region constitutes a convex region  $R'$ . The region  $R'$  is contained in every convex region containing  $R$ .

5. The boundary (§ 150) of a region in a plane (space) separates (§ 149) the set of all points in the region from the set of all points of the plane (space) not in the region.

6. A continuous one-to-one reciprocal transformation of space transforms any region into a region.

**156. Continuous families of sets of points.** The notion of continuous curve has the following direct generalization:

DEFINITION. Let  $[T]$  be the set of all points on an interval  $T_0T_1$  of a line  $l$ . A set of sets of coplanar points  $[S]$  is called a *continuous one-parameter family of sets of points* if it is in such a correspondence  $\Gamma$  with  $T$  that

- (1) for every  $T$  there is one and only one set  $S$  such that  $S = \Gamma(T)$ ;
- (2) for every set  $S$  there is at least one  $T$  such that  $S = \Gamma(T)$ ;
- (3) for every  $T$ , say  $T'$ , and for every triangular region  $R$  including a point of the set  $\Gamma(T')$ , there is a segment  $\sigma$  of  $l$  containing  $T'$  and such that if  $T$  is in  $\sigma$  at least one point of the set  $\Gamma(T)$  is in  $R$ .

The definition of a continuous one-parameter family of sets of points in space is obtained by replacing the triangular region  $R$  in the statements above by a tetrahedral region.

If the sets  $S$  are taken to be lines, planes, conics, quadrics, etc., this gives the definition of one-parameter continuous families of lines, planes, conics, quadrics, etc., respectively. Cf. Exs. 1-5, below.

DEFINITION. A *connected set of sets of points* or a *continuous family of sets of points* is a set of sets of points  $[S]$  such that any two sets  $S_1, S_2$  are members of a continuous one-parameter family of sets of  $[S]$ .

For example, the discussions given below in terms of elementary transformations establish in each case that a sense-class is a connected set of sets of points. Cf. also Exs. 6-7, below.

The definition of a continuous family may be extended in an obvious way so as to include sets whose elements are points, sets of points, sets of sets of points, etc.

## EXERCISES

1. Defining an *envelope* of lines as the plane dual of a curve, prove that an envelope is a continuous one-parameter family of lines.
2. The space dual of a curve is a continuous one-parameter family of planes.
3. Pencils of lines and planes are continuous one-parameter families.
4. A line conic or a regulus is a continuous one-parameter family of lines.
5. A pencil of point conics is a continuous one-parameter family of curves.
6. The set of all lines in a plane or space or in a linear congruence or a linear complex is a connected set of sets of points.
7. The set of all planes in space or of all planes tangent to a quadric is a connected set of sets of points.

157. **Continuous families of transformations.** Let  $[T]$  be the set of all points on an interval  $T_0T_1$  of a line  $l$ . Let  $[\Pi_T]$  be a set of transformations of a set of points  $[P]$ . If (1) to every  $T$  there corresponds one and only one transformation  $\Pi_T$ , and (2) for every point  $P$  the set of points  $[\Pi_T(P)]$  is a curve for which the defining correspondence  $\Gamma$  (in the notation of § 154) may be taken to be the correspondence between  $T$  and  $\Pi_T(P)$ , then  $[\Pi_T]$  is said to be a *continuous one-parameter family of transformations*. The curves  $[\Pi_T(P)]$  are called the *path curves* of  $[\Pi_T]$ .

The term "continuous one-parameter family of transformations" may also be applied to a set of transformations  $[\Pi_T]$  of a set  $S$  of points  $P$  and of sets of points  $S$  (e.g.  $S$  may be a set of figures as defined in § 13, Vol. I). In this case (1) and (2) must be satisfied, and also the following condition: (3) For every set of points  $S$ ,  $[\Pi_T(S)]$  is a one-parameter continuous family of sets of points for which the defining correspondence  $\Gamma$  (in the notation of § 156) may be taken to be the correspondence between  $T$  and  $\Pi_T(S)$ .

If the set of correspondences  $[\Pi_T]$  is both a group and a continuous one-parameter family of transformations, it is called a *one-parameter continuous group*.

A set of transformations  $[\Pi]$  of a set of points and of sets of points, such that any two transformations of  $[\Pi]$  are members of a continuous one-parameter family of transformations of  $[\Pi]$ , is called a *continuous family of transformations*. If  $[\Pi]$  is also a group, it is called a *continuous group*.

If  $[\Pi_T]$  is a continuous one-parameter family of one-to-one reciprocal transformations of a figure  $F$ , and if  $\Pi_{\tau_0}$  is the identity, then  $F$  is said to be *moved*, or *deformed*, to the figure  $\Pi_{\tau_1}(F)$  through the set

of *intermediate positions*  $[\Pi_T(F)]$ . Any one of the transformations  $\Pi_T$  is called a *deformation*; if  $F$  is a set of points and all the transformations of the family  $[\Pi_T]$  are continuous, the deformation is said to be a *continuous deformation*.

**158. Affine theorems on sense.** Let us recapitulate some of the main propositions about sense-classes in Euclidean spaces by enumerating the one-dimensional propositions of which they are generalizations.

The group of all projectivities  $x' = ax + b$  on a Euclidean line has a subgroup of *direct projectivities* for which  $a > 0$ . This subgroup is self-conjugate, because if a transformation of the group be denoted by  $\Sigma$ , and any other transformation  $x' = ax + \beta$  by  $T$ , then  $T\Sigma T^{-1}$  is

$$x' = a \left( a \left( \frac{1}{a} x - \frac{\beta}{a} \right) + b \right) + \beta,$$

a transformation in which the coefficient of  $x$  is positive. From the fact that the subgroup is self-conjugate, it follows as in § 18 that the same subgroup is defined by the condition  $a > 0$ , no matter how the scale is chosen, so long as  $P_\infty$  is the point at infinity. These statements are generalized to the plane in § 30 and to spaces of any dimensionality in § 31. The generalization consists in replacing  $a$  by the determinant

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

for the two-dimensional case, and by the corresponding  $n$ -rowed determinant in the  $n$ -dimensional case.

A sense-class  $S(AB)$  is the set of all ordered pairs of points into which a pair of distinct points can be carried by direct projectivities (§ 23). This proposition is generalized to the plane in § 30 and to  $n$ -space in § 31.

A particular arbitrarily chosen sense-class shall be called *positive* and the other sense-class shall be called *negative*. This statement reads the same for any number of dimensions. In the three-dimensional case the positive sense-class is also called *right-handed* and the negative sense-class *left-handed* (see the fine print in § 162).

In the one-dimensional case a nonhomogeneous coordinate system is called *positive* if  $S(P_0P_1)$  is positive. In the two-dimensional

case a nonhomogeneous coördinate system is called *positive* if  $S(OXY)$  is positive when  $O = (0, 0)$ ,  $X = (1, 0)$ , and  $Y = (0, 1)$ . In the three-dimensional case a nonhomogeneous coördinate system is called *positive* or *right-handed* if  $S(OXYZ)$  is positive when  $O = (0, 0, 0)$ ,  $X = (1, 0, 0)$ ,  $Y = (0, 1, 0)$ , and  $Z = (0, 0, 1)$ .

On the Euclidean line two ordered pairs of points  $AB$  and  $A'B'$  are in the same sense-class if and only if

$$\begin{vmatrix} 1 & a \\ 1 & b \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} 1 & a' \\ 1 & b' \end{vmatrix}$$

have the same sign,  $a, b, a', b'$  being the nonhomogeneous coördinates of  $A, B, A', B'$  respectively. Hence, if the coördinate system is positive,  $S(AB)$  is positive or negative according as  $(b - a)$  is positive or negative. Similar criteria for the plane and space are given in §§ 30, 31. It follows immediately that if the coördinate system in the plane is positive,  $S(ABC)$  is positive or negative according as the determinant

$$\begin{vmatrix} 1 & a_1 & a_2 \\ 1 & b_1 & b_2 \\ 1 & c_1 & c_2 \end{vmatrix}$$

is positive or negative, where  $A = (a_1, a_2)$ ,  $B = (b_1, b_2)$ ,  $C = (c_1, c_2)$ . If the coördinate system in space is positive,  $S(ABCD)$  is positive or negative according as the determinant

$$\begin{vmatrix} 1 & a_1 & a_2 & a_3 \\ 1 & b_1 & b_2 & b_3 \\ 1 & c_1 & c_2 & c_3 \\ 1 & d_1 & d_2 & d_3 \end{vmatrix}$$

is positive or negative, where  $A = (a_1, a_2, a_3)$ ,  $B = (b_1, b_2, b_3)$ ,  $C = (c_1, c_2, c_3)$ ,  $D = (d_1, d_2, d_3)$ .

In the one-dimensional case  $B$  is on one or the other of the rays having  $A$  as origin according as  $S(AB)$  is positive or negative. In a Euclidean plane  $C$  is on one side of the line  $AB$  or the other according as  $S(ABC)$  is positive or negative (§ 30). In a Euclidean space  $D$  is on one side or the other of the plane  $ABC$  according as  $S(ABCD)$  is positive or negative.

The projectivities  $x' = ax + b$  of the Euclidean line are in one-to-one reciprocal correspondence with the points  $(a, b)$  of the Euclidean plane. The direct projectivities correspond to the points on one side of the line  $a = 0$  and the opposite ones to those on the

other side. From this it readily follows that the set of all direct projectivities forms a continuous group, whereas the set of all projectivities is a group which is not continuous.

In like manner the transformations

$$x' = a_{11}x + a_{12}y + a_{10},$$

$$y' = a_{21}x + a_{22}y + a_{20}$$

can be set in correspondence with the points of a six-dimensional Euclidean space, the direct and opposite collineations respectively corresponding to points of two regions separated by the locus

$$a_{11}a_{22} - a_{12}a_{21} = 0.$$

Similarly, the direct and opposite collineations in a Euclidean space of three dimensions may be represented by points of two regions in a space of twelve dimensions. In all three cases the set of all direct collineations forms a continuous group, but the set of all collineations does not.

Another way of coming at the same result is this: Let the ordered pairs of points  $AB$  of a Euclidean line be represented by the points  $(a, b)$  of a Euclidean plane,  $a$  being the nonhomogeneous coordinate of  $A$ , and  $b$  that of  $B$ . Under this convention the points representing pairs of the positive sense-class are on one side of the line  $b - a = 0$  and those representing pairs of the negative sense-class on the other side of this line. The one-dimensional affine projectivities are in one-to-one reciprocal correspondence with the ordered point pairs to which they carry a fixed ordered point pair  $PQ$ . The direct projectivities thus correspond to point pairs represented by points on one side of the line  $b - a = 0$  and the opposite projectivities to point pairs represented by points on the other side.

### 159. Elementary transformations on a Euclidean line. DEFINITION.

Given an ordered pair of points  $AB$  of a Euclidean line, the operation of replacing one of the points by a second point not separated from it by the other point is called an *elementary transformation* of the pair  $AB$ .\*

Thus  $AB$  may be transformed into  $AB'$  if  $\{ABB'\}$  or  $\{AB'B\}$ . In other words (cf. § 23)  $B$  can be transformed to any point  $B'$  such that  $S(AB) = S(AB')$ , and into no other. Hence it follows that if  $AB$  is transformable to  $A'B'$  by any sequence of elementary transformations,  $S(AB) = S(A'B')$ .

Conversely, if  $S(AB) = S(A'B')$ , it is easy to see, as follows, that by a sequence of elementary transformations  $AB$  can be transformed

\* The transformations which we have considered heretofore have usually been transformations of the line, plane, or space as a whole. Here we are considering a transformation of a single pair of points.

to  $A'B'$ . From the theorems on linear order in Chap. II it follows that there are two points  $A''$  and  $B''$  satisfying the order relations

$$\{AB, A''B''\} \text{ and } \{A'B', A''B''\}.$$

By elementary transformations  $AB$  goes to  $AB''$ ;  $AB''$  to  $A''B''$ ;  $A''B''$  to  $A'B''$ ; and  $A'B''$  to  $A'B'$ . Hence we have

**THEOREM 17.** *On a Euclidean line the set of all ordered pairs of points into which an ordered pair of distinct points  $AB$  can be transformed by elementary transformations is the sense-class  $S(AB)$ .*

An elementary transformation may be regarded as a special type of *continuous deformation* (§ 157). If  $AB$  is carried by an elementary transformation to  $A'B'$ , the point  $B$  may be thought of as moved (§ 157) along the segment  $BB'$  from  $B$  to  $B'$ , and since this segment does not contain  $A$ , the motion is such that the pair of distinct points never degenerates into a coincident pair. Thus we may say that a sense-class consists of all pairs obtainable from a fixed pair by deformations in which no pair ever degenerates.

When the ordered point pairs are represented by points in a Euclidean plane, as explained at the end of the last section, an elementary transformation corresponds to moving a point  $(a, b)$  parallel to the  $a$ -axis or the  $b$ -axis in such a way as not to intersect the line  $a = b$ .

**DEFINITION.** An elementary transformation of a pair of points  $AB$  is said to be *restricted with respect to a set of points  $[P]$*  if and only if it carries one of the pair, say  $B$ , into a point  $B'$  such that the segment  $BB'$  does not contain any one of the points  $P$ . (Any one of these points may, however, be an end of the segment  $BB'$ .)

It is evident that any elementary transformation can be effected as a resultant of a sequence of elementary transformations which are restricted with respect to an arbitrary finite set of points. Hence, Theorem 17 has the following corollary:

**COROLLARY.** *Let  $P_1, P_2, \dots, P_n$  be any finite set of points on a line  $l$ . Two ordered pairs of points are in the same sense-class if and only if one can be carried into the other by a sequence of elementary transformations restricted with respect to  $P_1, P_2, \dots, P_n$ .*

The concept of a restricted elementary transformation is intimately connected with the idea of a "small motion." In the metric geometry the points  $P_1, P_2, \dots, P_n$  can be chosen so as to be in the order  $\{P_1, P_2, \dots, P_n\}$  and so that the segments  $P_iP_{i+1}$  are arbitrarily small. Any elementary transformation of a pair of points on the interval  $P_iP_{i+1}$  will be effected by a small motion of one of the points in the pair.

### 160. Elementary transformations in the Euclidean plane and space.

**DEFINITION.** Given an ordered set of three noncollinear points in a Euclidean plane, an *elementary transformation* is the operation of replacing one of them by a point which is joined to it by a segment not meeting the line on the other two.

As in the one-dimensional case, an elementary transformation may be regarded as effected by a continuous deformation of a point triad. A path is specified along which a point may be moved without allowing the triad to degenerate into a collinear one.

Let  $A, B, C$  be three noncollinear points and let  $C'$  and  $B'$  be points of the segments  $AB$  and  $CA$  respectively. Then by elementary transformations (cf. fig. 84, p. 423)  $ABC$  goes to  $C'BC$ ; and this to  $C'BB'$ ; and this to  $C'CB'$ ; and this to  $BCB'$ ; and this to  $BCA$ . In like manner it can be shown that  $ABC$  can be carried to  $CAB$  by a sequence of elementary transformations. Hence any even permutation of three noncollinear points can be effected by elementary transformations.

By Theorem 27, § 30, an elementary transformation leaves the sense of an ordered triad invariant. Hence, by Theorem 26, § 30, no odd permutation can be effected by elementary transformations.

If  $A', B', C'$  are any three noncollinear points,  $ABC$  can be carried into some permutation of  $A'B'C'$  by elementary transformations. For since at most one side of the triangle  $A'B'C'$  is parallel to the line  $AB$ , this line meets two of the sides in points which we may denote by  $A''$  and  $B''$ . By one-dimensional elementary transformations on the line  $AB$ , the ordered pair  $AB$  can be carried either to  $A''B''$  or to  $B''A''$ . These one-dimensional elementary transformations determine a sequence of two-dimensional elementary transformations leaving  $C$  invariant and carrying  $ABC$  to  $A''B''C$  or to  $B''A''C$ . The point  $C$  can be carried by an obvious elementary transformation to a point  $C''$  such that  $A''C''$  is not parallel to any side of  $A'B'C'$ , and then  $A''C''$  can be carried to two of the points, say  $A'''B'''C'''$ , in which the line  $A''C''$  meets the sides of the triangle  $A'B'C'$ . The points  $A'''B'''C'''$  are on the sides of the triangle  $A'B'C'$ , and the one-dimensional elementary transformations on the sides which carry them into the vertices determine two-dimensional elementary transformations which carry  $A'''B'''C'''$  to some permutation of  $A'B'C'$ .

Since  $ABC$  cannot be carried into  $A'B'C'$  if  $S(ABC) \neq S(A'B'C')$ , and since all even permutations of  $A'B'C'$  can be effected by elementary transformations, it follows that  $ABC$  can be carried into  $A'B'C'$  by a sequence of elementary transformations if  $S(ABC) = S(A'B'C')$ . Hence we have

**THEOREM 18.** *In a Euclidean plane  $S(ABC) = S(A'B'C')$  if and only if there exists a finite set of elementary transformations carrying the noncollinear points  $A, B, C$  into the points  $A', B', C'$  respectively.*

**DEFINITION.** Given an ordered set of four noncoplanar points, an *elementary transformation* is the operation of replacing one of them by another point which is joined to it by a segment containing no point of the plane on the other three.

Let  $ABCD$  be four noncoplanar points. Holding  $D$  fixed,  $ABC$  may be subjected to precisely the sequence of elementary transformations given above in the planar case for carrying  $ABC$  into  $BCA$ . This effects the permutation

$$\begin{pmatrix} A & B & C & D \\ B & C & A & D \end{pmatrix},$$

the symbol for each point being written above that for the point into which it is transformed. In like manner we obtain the permutations

$$\begin{pmatrix} A & B & C & D \\ B & D & C & A \end{pmatrix}, \quad \begin{pmatrix} A & B & C & D \\ C & B & D & A \end{pmatrix}, \quad \begin{pmatrix} A & B & C & D \\ A & C & D & B \end{pmatrix},$$

and it is easily verifiable that any even permutation of  $ABCD$  is a product of these permutations. Hence any even permutation of a set of four points may be effected by elementary transformations.

By Theorem 23, § 27, an elementary transformation of four points  $(a_1, a_2, a_3), (b_1, b_2, b_3), (c_1, c_2, c_3), (d_1, d_2, d_3)$  leaves the sign of

$$\begin{vmatrix} a_1 & a_2 & a_3 & 1 \\ b_1 & b_2 & b_3 & 1 \\ c_1 & c_2 & c_3 & 1 \\ d_1 & d_2 & d_3 & 1 \end{vmatrix}$$

invariant, and hence leaves their sense-class invariant. Hence (§ 31) no odd permutations of four noncoplanar points can be effected by elementary transformations.

An ordered tetrad  $ABCD$  of noncoplanar points can be carried into some permutation of an ordered tetrad  $A'B'C'D'$  of noncoplanar points. For the line  $AB$  is not parallel to more than two planes of the tetrahedron, and hence by the one-dimensional case  $AB$  can be

carried into two points  $A''B''$  of the planes of the tetrahedron  $A'B'C'D'$ . By repeating this argument it is easily proved that  $C$  and  $D$  can also be carried to points  $C''D''$  on these planes. By the two-dimensional case it follows that the ordered tetrad  $A''B''C''D''$  of points on the planes of the tetrahedron  $A'B'C'D'$  can be carried into some permutation of its vertices. Since  $ABCD$  cannot be carried into  $A'B'C'D'$ , if  $S(ABCD) \neq S(A'B'C'D')$  it follows by the last paragraph but one that it can be carried into  $A'B'C'D'$  if  $S(ABCD) = S(A'B'C'D')$ . Thus we have

**THEOREM 19.** *In a Euclidean space  $S(ABCD) = S(A'B'C'D')$  if and only if there exists a finite set of elementary transformations carrying the noncoplanar points  $A, B, C, D$  into the points  $A', B', C', D'$  respectively.*

The theorems and definitions of the last two sections can be regarded as based on any one of the sets of assumptions  $A, E, H, C, R$  or  $A, E, K$  or  $A, E, P, S$ . Assumption  $P$  is used wherever coördinates are employed, but it is possible to make the argument without the aid of coördinates and thus to base it on  $A, E, S$  alone (cf. Ex. 2, § 161).

**161. Sense in a convex region.** **DEFINITION.** Given a set of three noncollinear points of a planar convex region  $R$ , the operation of replacing any one of them by any other point of  $R$  on the same side of the line joining the other two is called an *elementary transformation*. The set of all ordered triads obtainable by finite sequences of elementary transformations from one noncollinear ordered triad of points  $ABC$  is called a *sense-class* and is denoted by  $S(ABC)$ .

This definition is in agreement with the propositions about sense given for the special case of a Euclidean plane. Moreover, if  $R$  is any convex region, and  $l_\infty$  is any line coplanar with  $R$  but containing no point of  $R$ , two triads of points of  $R$  are in the same sense with respect to  $R$  if and only if they are in the same sense with respect to the Euclidean plane containing  $R$  and having  $l_\infty$  as singular line. Hence the theorems of § 160 may be taken over at once to convex regions in general. This result may be stated as follows:

**THEOREM 20.** *In a planar convex region there are two and only two senses. Sense is preserved by even and altered by odd permutations of three noncollinear points. Two points  $C$  and  $D$  are on opposite sides of a line  $AB$  if and only if  $S(ABC) \neq S(ABD)$ .*

DEFINITION. Given a set of four noncoplanar points of a three-dimensional convex region  $R$ , the operation of replacing any one of them by any point of  $R$  on the same side of the plane of the other three is called an *elementary transformation*. The set of all ordered tetrads obtainable by finite sequences of elementary transformations from one noncoplanar ordered tetrad of points  $ABCD$  is called a *sense-class* and is denoted by  $S(ABCD)$ .

The theories of sense in a three-dimensional convex region and in a three-dimensional Euclidean space are related in just the same way as the corresponding planar theories. Hence we have

THEOREM 21. *In a three-dimensional convex region there are two and only two senses. Sense is preserved by even and altered by odd permutations of four points. Two points  $D$  and  $E$  are on opposite sides of a plane  $ABC$  if and only if  $S(ABCD) \neq S(ABCE)$ .*

### EXERCISES

1. The whole theory of order relations can be developed by defining sense-class on a line by means of elementary transformations instead of as in Chap. II.

\*2. Develop the theory of order in two- and three-dimensional convex regions, defining sense-class in terms of elementary transformations and using Assumptions A, E, S or Assumptions I-VIII of § 29 (cf. Theorem 5, § 148) as basis.

3. An elementary transformation of a triad of points  $ABC$  is said to be *restricted with respect to a set of points*  $P_1, P_2, \dots, P_n$  if it carries a point of the triad, say  $C$ , into a point  $C'$  such that the segment  $CC'$  does not contain any point collinear with two of the points  $P_1, P_2, \dots, P_n$ . Two ordered triads of points are in the same sense-class if and only if there is a sequence of restricted elementary transformations carrying the one triad into the other.

4. Generalize the notion of restricted elementary transformation to space.

162. **Euclidean theorems on sense.** The involutions which leave the point at infinity of a Euclidean line invariant may be called *point reflections*. The product of two point reflections is a parabolic projectivity leaving the point at infinity invariant, and may be called a *translation*. A point reflection has an equation of the form

$$(1) \quad x' = -x + b,$$

and a translation has one of the form

$$(2) \quad x' = x + b.$$

The point reflections interchange the two sense-classes of the Euclidean line, and the translations leave them invariant.

In generalizing these propositions to the plane, the point reflections may be replaced by the orthogonal line reflections (Chap. IV) or, indeed, by the set of all symmetries, and the one-dimensional translations by the set of all displacements in the plane. Since an orthogonal line reflection in the plane interchanges the two sense-classes, any symmetry interchanges them, but any displacement leaves each of them invariant. The generalization to three-dimensions is similar.

The equations of a displacement in two or three (or any number of) dimensions are a direct generalization of the one-dimensional equations, namely,

$$(3) \quad x'_i = \sum_{j=1}^n a_{ij} x_j + b_i, \quad (i = 1, 2, \dots, n)$$

where the matrix  $(a_{ij})$  is orthogonal and the determinant  $|a_{ij}|$  is  $+1$ . The equations of a symmetry satisfy the same condition except that the determinant  $|a_{ij}|$  is  $-1$  instead of  $+1$ .

It is worthy of comment that the distinction between displacements and symmetries holds in the complex space just as well as in the real, whereas the distinction between direct and opposite collineations holds only in the real space. Algebraically, this is because the distinction of sense depends merely on the sign of the determinant  $|a_{ij}|$ , whereas the distinction between displacements and symmetries is between collineations satisfying the condition  $|a_{ij}| = +1$  and  $|a_{ij}| = -1$ . In the representative spaces of six and twelve dimensions referred to in § 158,  $|a_{ij}| = 1$  and  $|a_{ij}| = -1$  are the equations of nonintersecting loci.

From the point of view of Euclidean geometry, as has been said above, the two sense-classes are indistinguishable.\* In the applications of geometry, however, a number of extra-geometrical elements enter which make the two

\* This does not contradict the existence of a geometry in which one sense-class is specified absolutely in the assumptions. The group of such a geometry is unlike the Euclidean group in that it does not include symmetries though it does include displacements. Its relation to the Euclidean geometry is similar to that of the geometries mentioned in the fine print in § 116. Those geometries, however, correspond to groups which are not self-conjugate under the Euclidean group, whereas this one corresponds to a self-conjugate subgroup. On the foundations of geometry in terms of sense-relations taken either absolutely or relatively, see the article by Schweitzer referred to in § 15.

sense-classes play essentially different rôles. Thus any normal human being who identifies the abstract Euclidean space with the space in which he views himself and other material objects may single out one of the sense-classes as follows: Let him hold his right hand in such a way that the index finger is in line with his arm, his middle finger at right angles to his index finger, and his thumb at right angles to the two fingers (fig. 82). Let a point in his palm be denoted by  $O$ , and the tips of his thumb, index finger, and middle finger by  $X$ ,  $Y$ ,  $Z$  respectively. The sense-class  $S(OXYZ)$  shall be called *right-handed* or *positive*, and the other *left-handed* or *negative*. This designation is unique because of the mechanical structure of the body.

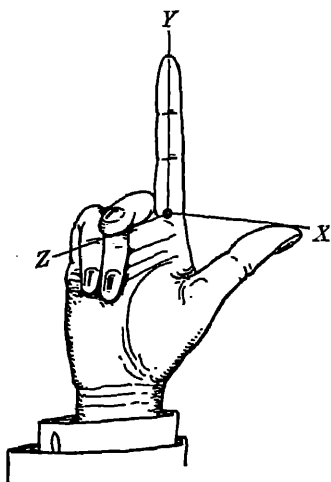


FIG. 82

A nonhomogeneous coordinate system is called *right-handed* or *positive* if and only if  $S(OXYZ)$  is positive when  $O = (0, 0, 0)$ ,  $X = (1, 0, 0)$ ,  $Y = (0, 1, 0)$ , and  $Z = (0, 0, 1)$ . The reader will find it convenient whenever an arbitrary sense-class is called positive to identify it with the intuitively right-handed sense-class.\*

**163. Positive and negative displacements.** On a Euclidean line, if a translation carries one point  $A$  to a point  $B$  such that  $S(AB)$  is positive, it carries any point  $A$  to a point  $B$  such that  $S(AB)$  is positive. Such a translation is called *positive*. Any other translation is called *negative* and has the property that if it carries  $C$  to  $D$ ,  $S(CD)$  is negative. Any translation carries positive translations into positive translations; i.e. if  $T'$  is a positive translation and  $T$  any translation,  $TT'T^{-1}$  is a positive translation. A translation  $x' = x + b$  is positive or negative according as  $b$  is positive or negative, provided that the scale is such that  $S(P_0P_1)$  is positive. The inverse of a positive translation is negative.

The distinction between positive and negative translations is quite distinct from that between direct and opposite projectivities, for all translations are direct.

\* An interesting account of the way in which this choice is made in various branches of mathematics and other sciences is to be found in an article by E. Study, *Archiv der Mathematik und Physik*, 3d series, Vol. XXI (1918), p. 193.

A like subdivision of the Euclidean displacements of a plane which are neither translations nor point reflections nor the identity may be made as follows: A rotation leaving a point  $O$  fixed and carrying a point  $A$  to a point  $B$  not collinear with  $O$  and  $A$  is said to be *positive* if  $S(OAB)$  is positive and to be *negative* if  $S(OAB)$  is negative. It is easily proved that if  $S(OAB)$  is positive for one value of  $A$  it is positive for all values of  $A$ . The inverse of a positive rotation is negative. Any displacement transforms a positive rotation into a positive rotation.

A rotation is a product of two orthogonal line reflections  $\{lL_\infty\}$  and  $\{mM_\infty\}$  such that the lines  $l$  and  $m$  intersect in  $O$ . Hence the ordered pairs of lines which intersect and are not perpendicular fall into two classes, which we shall call *positive* and *negative* respectively, according as the rotations which they determine are positive or negative.

In a three-dimensional Euclidean space let  $A$  be a point not on the axis of a given twist which is not a half-twist, let  $O$  be the foot of a perpendicular from  $A$  on the axis of the twist, and let  $A'$  and  $O'$  be the points to which  $A$  and  $O$  respectively are carried by the twist. The twist is said to be *positive* or *right-handed* if  $S(OAO'A')$  is positive or right-handed and to be *negative* or *left-handed* if  $S(OAO'A')$  is negative.

It is easily seen that  $S(OAO'A')$  is the same for all choices of  $A$ , so that the definition just made is independent of the choice of  $A$ . The inverse of the twist carrying  $O$  and  $A$  to  $O'$  and  $A'$  carries  $O'$  and  $A'$  to  $O$  and  $A$ , and thus is positive if and only if  $S(O'A'OA)$  is positive. Since  $S(O'A'OA) = S(OAO'A')$ , the inverse of a positive twist is positive. Any direct similarity transformation carries a positive twist into a positive twist.

With the choice of the right-handed sense-class described in the fine print in § 162, the definition here given is such that a right-handed twist is the displacement suffered by a commercial right-handed screw driven a short distance into a piece of wood.

Since a twist is a product of two orthogonal line reflections,  $\{lL_\infty\} \cdot \{mm_\infty\}$ , it follows that the pairs of ordinary lines  $lm$  which are not parallel, intersecting, or perpendicular fall into two classes, according as the twist  $\{mm_\infty\} \cdot \{lL_\infty\}$  is positive or negative. We shall call the line pairs of these two classes *positive* and *right-handed*

or *negative* and *left-handed* respectively. Since the inverse of a positive twist is positive, the ordered pair  $ml$  is positive if  $lm$  is positive. Hence a pair of lines is right-handed or left-handed without regard to the order of its members. Any direct similarity transformation carries a right-handed pair of lines into a right-handed pair and a left-handed pair into a left-handed pair.

### EXERCISES

1. The collineations which are commutative with a positive displacement (or with a negative displacement) are all direct.

2. By the definition in § 69,  $0 < \angle AOB < \pi$  or  $\pi < \angle AOB < 2\pi$  according as  $S(OAB)$  is positive or negative, provided that the points  $O, P_0, P_1$  are so chosen that  $S(OP_0P_1)$  is positive.

3. By the definition in § 72,  $0 < m(l_1l_2) < \frac{\pi}{2}$  or  $\frac{\pi}{2} < m(l_1l_2) < \pi$  according as the ordered line pair  $l_1l_2$  is positive or negative.

4. Let us define an *elementary transformation* of an ordered line pair  $l_1l_2$  in a plane as being either the operation of replacing  $l_1$  or  $l_2$  by a line parallel to itself, or the operation of replacing  $l_1$  or  $l_2$ , say  $l_1$ , by a line through the point  $l_1l_2$  which is not separated from  $l_1$  by  $l_2$  and the line through  $l_1l_2$  perpendicular to  $l_2$ . Two ordered pairs of nonparallel and nonperpendicular lines are equivalent under elementary transformations if and only if they are both in the positive or both in the negative class.

5. Let us define an *elementary transformation* of a pair of nonparallel and nonperpendicular lines  $l_1l_2$  in space as the operation of replacing one of the lines, say  $l_1$ , by a line intersecting  $l_1$  and not separated from  $l_1$  by the plane through the point of intersection perpendicular to  $l_1$  and the plane through this point and  $l_2$ . The pair  $l_1l_2$  can be transformed into a pair of lines  $m_1m_2$  by a sequence of elementary transformations if and only if both pairs are right-handed or both pairs are left-handed.

164. *Sense-classes in projective spaces.* It has been seen in Chap. II (cf. §§ 18 and 32) that the distinction between direct and opposite collineations can be drawn in any projective space of an odd number of dimensions which is real or, more generally, which satisfies  $A, E, S$ . This depends (§ 32) on the fact that the sign of a determinant  $|a_{ij}|$  ( $i, j = 0, 1, \dots, n$ ) cannot be changed by multiplying every element by the same factor if  $n$  is odd, and can be changed by multiplying every element by  $-1$  if  $n$  is even.

In a real projective space of odd dimensionality the direct collineations form a self-conjugate subgroup of the projective group and thus give rise to the definitions of sense-class in §§ 19 and 32. The same remarks are made about the independence of this definition

of the frame of reference as in the Euclidean cases, and the criteria for sense in terms of products of determinants are given in §§ 24 and 32. If one forms the analogous determinant products for the projective spaces of even dimensionality, it is found that the sign of the product may be changed by multiplying the coördinates of one point by  $-1$ , which verifies in a second way that there is only one sense-class in a projective space of an even number of dimensions.

The projectivity

$$x' = \frac{a_{11}x + a_{12}}{a_{21}x + a_{22}}$$

may be represented by means of a point  $(a_{11}, a_{12}, a_{21}, a_{22})$  in a projective space of three dimensions. The points representing direct projectivities are on one side of the ruled quadric

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = 0,$$

and those representing opposite projectivities on the other side. This representation of projectivities by points is in fact identical with that considered in § 129. It can be generalized to any number of dimensions just as are the analogous representations in § 158.

It readily follows that the group of all projective collineations in a real space of  $n$  dimensions is continuous if  $n$  is even, and not continuous if  $n$  is odd. If  $n$  is odd the group of direct collineations is continuous.

In the following sections (§§ 165–167) we shall discuss the sense-classes of projective spaces by means of elementary transformations, the latter term being used as before to designate a particular type of continuous deformation. After this (§§ 169–181) similar considerations will be applied to other figures.

**165. Elementary transformations on a projective line.** DEFINITION. Given a set of three collinear points  $A, B, C$ , an *elementary transformation* is the operation of replacing any one of them, say  $A$ , by another point  $A'$  such that there is a segment  $AA'$  not containing  $B$  or  $C$ .

**THEOREM 22.** *Two ordered triads of points on a real projective line have the same sense if and only if one is transformable into the other by a finite number of elementary transformations.*

*Proof.* Comparing the definitions of elementary transformation and of segment (§ 22), it is clear that a single elementary transformation cannot change the sense of a triad of points. Hence two

triads of points have the same sense if one can be transformed into the other by a finite number of elementary transformations. The converse statement, namely, that a triad  $A, B, C$  can be transformed by elementary transformations into any other triad  $A'B'C'$  in the same sense-class, follows at once if we establish (1) that  $ABC$  can be transformed by elementary transformations into  $BCA$  and  $CAB$  and (2) that any ordered triad of points  $A, B, C$  can be transformed by elementary transformations into one of the six ordered triads formed by any three points  $A', B', C'$ .

(1) Let  $D$  be a point in the order  $\{ABCD\}$ . Then by elementary transformations we can change  $ABC$  into  $ABD$ , then into  $ACD$ , then into  $BCD$ , and then into  $BCA$ . By repeating these steps once more  $ABC$  can be transformed into  $CAB$ .

(2) If  $A'$  does not coincide with one of the points  $A, B, C$ , it is on one of the three mutually exclusive segments (§ 22) of which they are the ends; and by (1) the points  $ABC$  may be transformed so that the ends of this segment are  $B$  and  $C$ . Hence we have  $\{ABA'C\}$ , and by elementary transformations  $ABC$  goes successively into  $AA'C, BA'C, BA'A, BA'C$ . If  $A'$  does coincide with one of the points  $A, B, C$ , the triad  $ABC$  may be transformed according to (1) so that  $A' = A$ . In like manner the three points  $A'BC$  can be transformed into  $A', B', C$  in some order, and then  $A'B'C$  into  $A'B'C'$  in some order.

The proof given for this theorem holds good without change on the basis of Assumptions A, E, S. Cf. § 15.

**DEFINITION.** An elementary transformation of a triad of points  $ABC$  of a line  $l$  is said to be *restricted with respect to a set of points*  $P_1, P_2, \dots, P_n$  if it carries one point of the triad, say  $C$ , into a point  $C'$  such that  $C$  and  $C'$  are not separated by any pair of the points  $P_1, P_2, \dots, P_n$ . ( $C$  or  $C'$  may coincide with any of the points  $P_1, \dots, P_n$ .)

It is obvious that any elementary transformation whatever is the resultant of a finite number of restricted elementary transformations. Hence Theorem 22 has the following immediate corollary:

**COROLLARY.** Let  $P_1, P_2, \dots, P_n$  be any finite set of points on a line  $l$ . Two ordered triads of points of  $l$  have the same sense if and only if one is transformable into the other by a finite number of elementary transformations restricted with respect to  $P_1, P_2, \dots, P_n$ .

The concept of "restricted elementary transformation" connects with the intuitive idea of "small motions." Let a line be set into projective correspondence with a conic, say a circle. For any  $n$  there is a set of points  $P_1, P_2, \dots, P_n$  on the circle such that the intervals  $P_1P_2$ , etc. are equal. By increasing  $n$  these intervals can be made arbitrarily small, and thus the elementary transformations restricted with respect to  $P_1, P_2, \dots, P_n$  can be made arbitrarily small.

**166. Elementary transformations in a projective plane.** DEFINITION. Given a set of four points in a projective plane, no three being collinear, an *elementary transformation* is the operation of replacing one of them by a point of the same plane joined to the point replaced by a segment not meeting any side of the triangle of the other three points.

**THEOREM 23.** *If  $ABCD$  and  $A'B'C'D'$  are any two complete quadrangles in the same projective plane, there exists a finite set of elementary transformations changing the points  $A, B, C, D$  into  $A', B', C', D'$  respectively.*

*Proof.* It can be shown by means of the result for the one-dimensional case, just as in the proof of Theorem 18, first that the ordered tetrad  $ABCD$  can be carried by elementary transformations into an

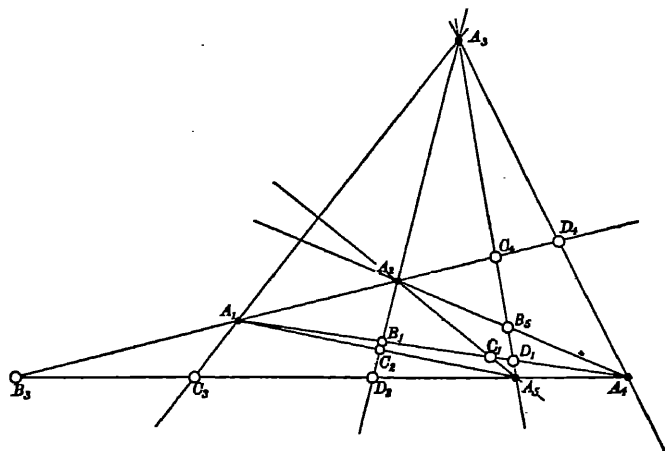


FIG. 88

ordered tetrad  $A''B''C''D''$  of points on the sides of the quadrangle  $A'B'C'D'$  and then that  $A''B''C''D''$  can be carried by elementary transformations into some permutation of  $A'B'C'D'$ .

To complete the proof it is necessary to show that any permutation of the vertices of a complete quadrangle can be effected by elementary transformations.

Given a complete quadrangle  $A_1A_2A_3A_4$ , let  $B_1$  be the point of intersection of the lines  $A_2A_3$  and  $A_1A_4$ , and let  $C_1$  and  $D_1$  be two points in the order  $\{A_1B_1C_1D_1A_4\}$ . Let  $A_5$  be the point of intersection of  $A_2C_1$  with  $A_3D_1$  and let  $C_2, D_2, B_2, C_3, C_4, D_4, B_5$  be the points defined by the following perspectivities (fig. 83):

$$\begin{aligned} A_1B_1C_1D_1A_4 &\stackrel{A_5}{\wedge} C_2B_1A_2A_3D_2 \stackrel{A_1}{\wedge} A_5A_4B_3C_3D_3 \\ &\stackrel{A_2}{\wedge} C_4D_4B_3A_1A_3 \stackrel{A_4}{\wedge} C_4A_3A_5D_1B_5. \end{aligned}$$

By Theorem 7, Chap. II, it follows that no two of the pairs of points  $A_1A_4$ ,  $A_2A_3$ ,  $A_4A_5$ ,  $A_1A_3$ , and  $A_5A_5$  are separated by the lines joining the other three of the points  $A_1, A_2, A_3, A_4, A_5$ . Hence there exist elementary transformations changing each of the following sets of four points into the one written below it:

$$\begin{aligned} &A_1 A_2 A_3 A_4 \\ &A_1 A_2 A_3 A_5 \\ &A_4 A_2 A_3 A_5 \\ &A_4 A_1 A_3 A_5 \\ &A_4 A_1 A_3 A_5 \\ &A_4 A_1 A_2 A_3 \end{aligned}$$

Hence the permutation

$$\Pi_1 = \begin{pmatrix} A_1A_2A_3A_4 \\ A_4A_1A_2A_3 \end{pmatrix}$$

can be effected by elementary transformations. By changing the notation in  $\Pi_1$  it is clear that

$$\Pi_2 = \begin{pmatrix} A_1A_4A_3A_5 \\ A_3A_1A_4A_2 \end{pmatrix}$$

can be effected by elementary transformations. Hence the product  $\Pi_2\Pi_1^2$  (i.e. the resultant of  $\Pi_1$  applied twice and followed by  $\Pi_2$ ), which is

$$\begin{pmatrix} A_1A_2A_3A_4 \\ A_2A_1A_3A_4 \end{pmatrix},$$

can also be effected by elementary transformations. Hence any two vertices of the quadrangle can be interchanged by a sequence

of elementary transformations, and hence any permutation of the vertices can be effected by means of elementary transformations.

**167. Elementary transformations in a projective space.** DEFINITION. Given a set of five points in a projective space, no four of the points being coplanar, an *elementary transformation* is the operation of replacing any one of them by a point joined to it by a segment not meeting any plane on three of the other four.

It follows from § 27 that the determinant product (25) of § 32 is unaltered in sign by any sequence of elementary transformations of the points whose coördinates are the columns of (21) in § 32. Hence a sequence of elementary transformations cannot carry an ordered pentad of points from one sense-class into the other.

Hence the odd permutations of the vertices of a complete five-point cannot be effected by elementary transformations. That the even permutations can be thus effected

may be seen as follows: Let the vertices be denoted by  $A, B, C, D, E$  and let the line  $DE$  meet the plane  $ABC$  in a point  $F$ . This point is not on a side of the triangle  $ABC$ . Let  $A'$  be the point of intersection of the lines  $FA$  and  $BC$ ,  $B'$  that of  $FB$  and  $CA$ , and  $C'$  that of  $FC$  and  $AB$ . Let  $A_1$  be a point in the order  $\{BA_1A'C\}$

(fig. 84) and  $B_1$  the point in which the line  $FA_1$  meets  $AC$ , so that  $\{AB_1B'C\}$ . Let  $B_2$  be a point in the order  $\{AB_1B_2B'C\}$ .

We now can transform  $ABCDE$  by elementary transformations successively into  $AA_1CDE$ ,  $AA_1B_2DE$ ,  $BA_1B_2DE$ ,  $BCB_2DE$ ,  $BCADE$ . Thus the even permutation

$$\begin{pmatrix} ABCDE \\ BCADE \end{pmatrix}$$

can be effected by elementary transformations. It is easily verifiable that any even permutation is a product of even permutations of this type.

It can be proved by the same methods as in Theorems 18 and 19 that any five points no four of which are coplanar can be carried into some permutation of any other such set of five points. The

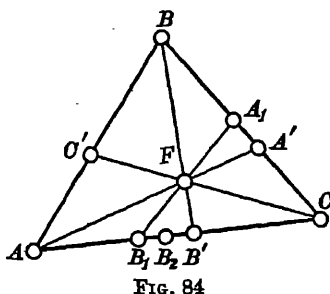


FIG. 84

details of this proof are left as an exercise to the reader. When this is combined with the paragraph above, we obtain

**THEOREM 24.** *In a real projective space,  $S(ABCDE) = S(A'B'C'D'E')$  if and only if there exists a sequence of elementary transformations carrying the points  $A, B, C, D, E$  into  $A', B', C', D', E'$  respectively.*

The proof just outlined for this theorem holds good on the basis of Assumptions A, E, S, P. Assumption P comes in because of the use of a coördinate system. This, however, can be avoided; and the construction of a proof on the basis of A, E, S alone is recommended to the reader as an interesting exercise.

**\*168. Sense in overlapping convex regions.** The discussion of sense in convex regions by means of elementary transformations (as made in §§159-161) is essentially the same for any number of dimensions. Now if two regions of the same dimensionality have a point in common, they have at least one convex region of that dimensionality in common. Assigning a positive sense in this region determines a positive sense in each of the given regions. Thus if we have a set of convex regions including all points of a space, we should have, on assigning a positive sense to a tetrad of points in one region, a positive sense determined for any tetrad of points in any of the regions. Since, however, it is in general possible to pass from one region to another by means of different sets of intermediate regions, the possibility arises that this determination of sense may not be unique. In other words, it is logically possible that a given tetrad in a given region might, according to this definition, have both positive and negative senses.

The determination of sense by this method is unique in projective spaces of odd dimensionality and is not unique in projective spaces of even dimensionality. We shall prove this for the two- and three-dimensional cases, but since it reduces merely to a question of even and odd permutations the generalization is obvious.

**THEOREM 25.** *There exists a unique determination of sense for all three-dimensional convex regions in a real projective three-space, but not for all two-dimensional convex regions in a real projective plane.*

*Proof.* Consider first the plane and in it a triangle  $ABC$  decomposing it into four triangular regions, which we shall denote by the

notation of § 26, Chap. II. Any one of these regions, say Region I, is contained in a convex region, say  $I'$  (e.g., a Euclidean plane with line at infinity not meeting Region I), which contains the boundary of the triangular region. So the determination of sense for Region I extends to all the points of its boundary and also to a portion of Region II.

Let the sense of  $ABC$  with respect to Region I be positive. The segment  $\overline{y}$ , one of the segments  $AB$  (fig. 16), is common to the boundaries of I and II and hence is contained in Region  $I'$ . If  $C'$  is any point common to  $I'$  and II,  $C$  and  $C'$  are on opposite sides of the line  $AB$  in Region  $I'$ . Hence, according to § 29,  $S(BAC')$  is positive in Region II. Hence  $S(BAC)$  is positive with respect to Region II.

Regions II and IV have in common a segment  $BC$ , and thus by a repetition of this argument  $S(CAB)$  is positive with respect to Region IV. The latter region has a segment  $AC$  in common with Region I, and hence  $S(ACB)$  is positive with respect to Region I. But by hypothesis  $S(ABC)$  is positive with respect to Region I. Hence there is not a unique determination of sense in a real projective plane.

To show that there is a unique determination of sense for a real projective three-space, let a given sense-class  $S(ABCDE)$  (cf. § 164) be designated as right-handed, and in any convex region let a sense-class  $S(A'B'C'D')$  be right-handed if  $S(OA'B'C'D')$  is positive, where  $O$  is interior to the tetrahedron  $A'B'C'D'$ . This convention satisfies the requirements laid down above for overlapping convex regions and, by § 167, is unique for the projective three-space.

Any two-dimensional region whatever is, by definition (§ 155), the set of all points in an infinite set of triangular regions, i.e. in an infinite set of convex regions. In like manner, any three-dimensional region is the set of all points in a set of three-dimensional regions. The method given above may be applied to determine the positive sense-class in all convex regions in a given region  $R$ , and  $R$  may be said to be two-sided or one-sided according as this determination is or is not unique. Another, slightly different, method of treating this question is given in § 173.

**\*169. Oriented points in a plane.** By the principles of duality the lines of a flat pencil or the planes of an axial pencil satisfy the same theorems on order as the points of a projective line.

This proposition is valid whether the pencils are considered in a projective or in a Euclidean space.\*

DEFINITION. In a plane any point associated with one of the sense-classes among the lines on this point is called an *oriented point*, and a line associated with one of the sense-classes among its points is called an *oriented line*. Two oriented points are said to be *similarly oriented with respect to a line  $l$*  if their sense-classes are perspective with the same sense-class in the points of  $l$ . By Ex. 1, § 26, if two oriented points are similarly oriented with respect to a line  $l$ , they are similarly oriented with respect to a line  $m$  if and only if  $l$  and  $m$  do not separate the two points.

By § 30 a direct collineation of a Euclidean plane transforms any oriented point into one which is similarly oriented with respect to the line at infinity. Hence the oriented points fall into two classes such that any two oriented points of the same class are equivalent under direct collineations and that the two classes are interchanged by any nondirect collineation.

No such statement as this can be made about the oriented lines in a Euclidean plane, because any oriented line can be carried by a direct collineation to any other oriented line. This is obvious because (1) an affine collineation exists carrying an arbitrary line to any other line and (2) the two sense-classes on any line are interchanged by a harmonic homology whose center is the point at infinity of the line.

It is a corollary of the last paragraph that any oriented line of a projective plane can be carried into any other oriented line of the projective plane by a direct collineation. By duality the same proposition holds for oriented points in a projective plane.

The oriented points determined by associating the points of a segment  $\gamma$  with sense-classes in the flat pencils of which they are centers fall into two sets, all points of either set being similarly oriented with respect to any line not meeting  $\gamma$ . These two sets shall be called *segments* of oriented points and may be denoted by  $\gamma^{(+)}$  and  $\gamma^{(-)}$ . If  $A$  and  $B$  are the ends of  $\gamma$ , the two oriented points determined by  $A$  and  $B$  and oriented similarly to  $\gamma^{(+)}$  with respect to a

\*In general, the geometry of a Euclidean space or, indeed, of any space of  $n$  dimensions involves the study of the projective geometry of  $n - 1$  dimensions, in order to describe the relations among the lines, planes, etc. on a fixed point.

line  $l$  not meeting  $\gamma$  or either of its ends are called the *ends* of  $\gamma^{(+)}$  and may be denoted by  $A^{(+)}$  and  $B^{(+)}$ . The other two oriented points determined by  $A$  and  $B$  are the ends of  $\gamma^{(-)}$  and may be denoted by  $A^{(-)}$  and  $B^{(-)}$ .

In terms of these definitions it is clear that each of the two classes of similarly oriented points determined by a Euclidean plane satisfies a set of order relations such that it may be regarded as a Euclidean plane.

The situation in the projective plane is entirely different. Let us first consider a projective line, and let  $\gamma$  and  $\delta$  be two complementary segments whose ends are  $A$  and  $B$ . Let  $A^{(+)}$ ,  $B^{(+)}$ ,  $A^{(-)}$ ,  $B^{(-)}$ ,  $\gamma^{(+)}$ ,  $\gamma^{(-)}$  be defined as above, and let  $\delta^{(+)}$  and  $\delta^{(-)}$  be the two segments of oriented points determined by  $\delta$  and oriented similarly to  $A^{(+)}$  and  $A^{(-)}$  respectively with respect to a line  $m$  not meeting  $\delta$  or either of its ends. Since  $A^{(+)}$  and  $B^{(+)}$  are similarly oriented with respect to  $l$ , and  $A$  and  $B$  are separated by  $l$  and  $m$ ,  $A^{(+)}$  and  $B^{(-)}$  are similarly oriented with respect to  $m$  (cf. Ex. 1, § 26, and fig. 85). Hence the ends of  $\delta^{(+)}$  are  $A^{(+)}$  and  $B^{(-)}$ , and the ends of  $\delta^{(-)}$  are  $A^{(-)}$  and  $B^{(+)}$ . Hence the oriented points and segments are arranged as follows:

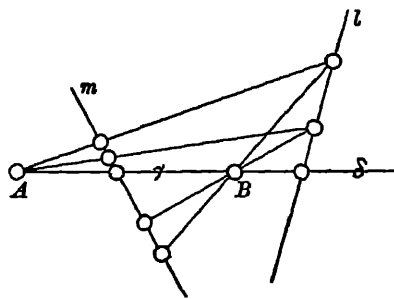


FIG. 85

$$A^{(+)}, \gamma^{(+)}, B^{(+)}, \delta^{(-)}, A^{(-)}, \gamma^{(-)}, B^{(-)}, \delta^{(+)}, A^{(+)},$$

the symbols for segments and their ends being written adjacent.

Let  $A_1, B_1, A_2, B_2$  be four points in the order  $\{A_1 B_1 A_2 B_2\}$  on a projective line or on a conic. They separate the line (§ 21) into four mutually exclusive segments  $\gamma_1, \delta_1, \gamma_2, \delta_2$  arranged as follows:

$$A_1, \gamma_1, B_1, \delta_2, A_2, \gamma_2, B_2, \delta_1, A_1,$$

the symbols for segments and their ends being written adjacent. Letting  $A_1$  correspond to  $A^{(+)}$ ,  $\gamma_1$  to  $\gamma^{(+)}$ , etc., it is obvious that there is a one-to-one reciprocal correspondence preserving order between the points of a real projective line or conic and the oriented points of a real projective line.

Thus, if an oriented point be moved along a projective line in such a way that all oriented points of any segment described are similarly oriented with respect to a line not meeting the segment, the oriented point must describe the line twice before returning to its first position. A motion of this sort will obviously carry any oriented point of the projective plane into any other oriented point. Thus the oriented points either of a projective line or of a projective plane constitute a continuous family in the sense of § 156.

Let  $\pi$  denote the projective plane under consideration here and let us suppose it contained in a projective space  $S$ , and let  $S'$  be a Euclidean space obtained by removing from  $S$  a plane different from  $\pi$  which contains the line  $AB$ . Let  $S^2$  be a sphere of  $S'$  tangent to  $\pi$  at a point  $P_1$ , let  $O$  be the center of  $S^2$ , and let  $P_2$  be the other point in which the line  $OP_1$  meets the sphere. Let  $P^{(+)}$  and  $P^{(-)}$  be the two oriented points of  $\pi$  determined by  $P_1$ .

A correspondence  $\Gamma$  between the points of the sphere  $S^2$  and the oriented points of the projective plane  $\pi$  may now be set up by the following rule: Let  $P_1$  correspond to  $P^{(+)}$ , and  $P_2$  to  $P^{(-)}$ ; if  $X$  is any point of  $\pi$  not on the line at infinity, denote by  $X_1$  and  $X_2$  the points in which the line  $OX$  meets the sphere, assigning the notation so that each of the angles  $\angle P_1 O X_1$  and  $\angle P_2 O X_2$  is less than a right angle (i.e. so that the points  $X_1$  are all on the same side as  $P_1$  of the plane through  $O$  parallel to  $\pi$ , and the points  $X_2$  are on the other side of this plane); and denote by  $X^{(+)}$  the oriented point of  $\pi$  determined by  $X$  and joined to  $P^{(+)}$  by a segment of oriented points containing no point of the line at infinity  $AB$ , and by  $X^{(-)}$  the other oriented point determined by  $X$ . Let  $X_1$  correspond to  $X^{(+)}$ , and  $X_2$  to  $X^{(-)}$ . If  $Y$  is any point of the line at infinity  $AB$ , and  $Y^{(=)}$  one of the oriented points determined by it,  $Y^{(=)}$  is an end of a segment  $\sigma^{(+)}$  of points  $X^{(+)}$  whose other end is  $P^{(+)}$  and of a segment  $\sigma^{(-)}$  of points  $X^{(-)}$  whose other end is  $P^{(-)}$ . The line  $OY$  meets the sphere in two points one of which,  $Y_1$ , is an end both of a segment of points  $X_1$  corresponding to  $\sigma^{(+)}$  and of a segment of points  $X_2$  corresponding to  $\sigma^{(-)}$ . Let  $Y_1$  correspond to  $Y^{(=)}$ . This construction evidently makes the oriented point other than  $Y^{(=)}$  which is determined by  $Y$  correspond to the point other than  $Y_1$  in which  $OY$  meets the sphere.

The correspondence  $\Gamma$  is one-to-one and reciprocal and makes each segment of oriented points of  $\pi$  correspond to a segment of points

on  $S^2$ . In view of the correspondence between the sphere and the inversion plane, this result may be stated in the following form:

**THEOREM 26.** *There is a one-to-one reciprocal correspondence preserving order-relations between the oriented points of a real projective plane and the points of a real inversion plane.*

The treatment of oriented points in this section does not generalize directly to three dimensions, because there is only one sense-class in a projective plane and, therefore, also only one in a bundle of lines. The discussion of sense in terms of the set of all lines through a point is therefore possible along these lines only in spaces of an even number of dimensions.

A discussion which is uniform for spaces of any number of dimensions can, however, be made in terms of rays. An outline of the theory of pencils and bundles of rays which may be used for this purpose is given in the next three sections, and an outline of one way of generalizing the contents of the present section is given in § 173.

Another type of generalization of the theory of oriented points in the plane is the theory of doubly oriented lines in three dimensions which is given in § 180, below.

**\*170. Pencils of rays.** The term "ray"\* is defined in § 23 for a linear convex region and extended to any convex region in § 148. The definition of *angle* in § 23 will be carried over to any convex region.

**DEFINITION.** The set of all rays with a common origin in a planar convex region is called a *pencil* of rays. The common origin is called the *center* of the pencil.

The order relations in a pencil of rays are essentially the same as those among the points of a projective line. This can be shown by setting up a correspondence between the rays through the center of a circle and the points in which they meet the circle, as in § 69. It can also be done on the basis of Assumptions A, E, P alone by proving Theorems 27-33, below. The proofs of the theorems are not given, because they are not very different from those of other theorems in this chapter. A third way of deriving these relations is indicated in Theorems 34, 35, and a fourth in Theorems 37-41.

**THEOREM 27.** *If  $a, b, c$  are three rays of a pencil, and if any segment joining a point of  $a$  to a point of  $c$  contains a point of  $b$ , then every segment joining a point of  $a$  to a point of  $c$  contains a point of  $b$ .*

\* In some books the term "ray" is used as synonymous with "projective line," and "pencil of rays" with "pencil of lines."

DEFINITION. If  $a, b, c$  are three rays of a pencil,  $b$  is said to be *between*  $a$  and  $c$  if and only if (1)  $a$  and  $c$  are not collinear and (2) any segment joining a point of  $a$  to a point of  $c$  contains a point of  $b$ .

THEOREM 28. *If  $b$  is any ray between two rays  $a$  and  $c$ , any other ray between  $a$  and  $c$  is either between  $a$  and  $b$  or between  $b$  and  $c$ . No ray is both between  $a$  and  $b$  and between  $b$  and  $c$ . Any ray between  $a$  and  $b$  is between  $a$  and  $c$ .*

THEOREM 29. *There is a one-to-one reciprocal correspondence preserving all order relations between the points of a segment of a line and the rays between two rays of a pencil.*

THEOREM 30. *If three rays  $a, b, c$  of a pencil are such that no two of them are collinear and no one of them is between the other two, then any other ray of this pencil is between  $a$  and  $b$  or between  $b$  and  $c$  or between  $c$  and  $a$ .*

DEFINITION. Given a set of three distinct rays  $a, b, c$  of a pencil, by an *elementary transformation* is meant the operation of replacing one of them, say  $c$ , by a ray  $c'$  not collinear with  $c$  and such that neither  $a$  nor  $b$  is between  $c$  and  $c'$ . The class consisting of all ordered triads into which  $abc$  is transformable by finite sequences of elementary transformations is called a *sense-class* and is denoted by  $S(abc)$ .

An elementary transformation of  $abc$  into  $abc'$  is said to be *restricted* with respect to a set of rays  $a_1, a_2, \dots, a_n$  of the pencil if none of the rays  $a_1, a_2, \dots, a_n$  is between  $c$  and  $c'$ .

THEOREM 31. *Let  $a_1, a_2, \dots, a_n$  be an arbitrary set of rays of a pencil. Two ordered triads of rays of the pencil are in the same sense-class if and only if one can be transformed into the other by a sequence of elementary transformations which are restricted with respect to  $a_1, a_2, \dots, a_n$ .*

THEOREM 32. *Let  $a_1, a_2, a_3$  be three distinct rays of a pencil such that no one of the three is between the other two. There exists a one-to-one reciprocal correspondence  $\Gamma$  between the rays of the pencil and the points of a projective line such that to each elementary transformation of the rays which is restricted with respect to  $a_1, a_2, a_3$  there corresponds an elementary transformation on the projective line which is restricted with respect to the points corresponding to  $a_1, a_2, a_3$ .*

The correspondence  $\Gamma$  required in this theorem may be set up as follows: Let three arbitrary collinear points  $A_1, A_2, A_3$  be the correspondents of  $a_1, a_2, a_3$  respectively; let  $\Gamma_1$  be a projectivity which carries the lines which contain the rays between  $a_1$  and  $a_2$  to the points of the segment complementary to  $\overline{A_1 A_2 A_3}$  and carries the line containing  $a_1$  to  $A_1$ ; for the rays between  $a_1$  and  $a_3$  let  $\Gamma$  be the correspondence in which each ray between  $a_1$  and  $a_3$  corresponds to the point to which the line containing it is carried by  $\Gamma_1$ ; let  $\Gamma_2$  be the projectivity which carries the lines which contain the rays between  $a_2$  and  $a_3$  to the points of the segment complementary to  $\overline{A_2 A_1 A_3}$  and carries the line containing  $a_2$  to  $A_2$ ; for the rays between  $a_2$  and  $a_3$  let  $\Gamma$  be the correspondence in which each ray corresponds to the point to which the line containing it is carried by  $\Gamma_2$ ; let  $\Gamma_3$  be a projectivity which carries the lines which contain the rays between  $a_3$  and  $a_1$  to the points of the segment complementary to  $\overline{A_3 A_2 A_1}$  and carries the line containing  $a_3$  to  $A_3$ ; for the rays between  $a_3$  and  $a_1$  let  $\Gamma$  be the correspondence in which each ray corresponds to the point to which the line containing it is carried by  $\Gamma_3$ .

**COROLLARY.** *There is a one-to-one reciprocal correspondence between the points of a projective line and the rays of a pencil such that two ordered triads of rays of the pencil are in the same sense-class if and only if the corresponding triads of points are in the same sense-class on the line.*

**THEOREM 33.** *If  $a, b, c$  are three rays of a pencil and  $a', b', c'$  are the respectively opposite rays,  $S(abc) = S(a'b'c')$ .*

**DEFINITION.** If  $a$  and  $b$  are any two noncollinear rays of a pencil, by an *elementary transformation* of the ordered pair  $ab$  is meant the operation of replacing one of them, say  $b$ , by another ray,  $b'$ , of the pencil, such that no ray of the line containing  $a$  is between  $b$  and  $b'$  or coincident with  $b'$ . The set of all ordered pairs (i.e. angles) into which an ordered pair of rays  $ab$  can be carried by sequences of elementary transformations is called a *sense-class* and is denoted by  $S(ab)$ .

**THEOREM 34.** *If  $O$  is the center of a pencil of rays and  $A, B, C, D$  are points of rays  $a, b, c, d$  respectively of the pencil, then  $S(ab) = S(cd)$  if and only if  $S(OAB) = S( OCD)$ .*

THEOREM 35. If  $a$  and  $b$  are any two noncollinear rays of a pencil,  $S(ab) \neq S(ba)$ . Every ordered pair of noncollinear rays in the pencil is either in  $S(ab)$  or in  $S(ba)$ . If  $a'$  is the ray opposite to  $a$ ,  $S(ab) \neq S(a'b)$ .

THEOREM 36. If  $a, b$  and  $a', b'$  are two ordered pairs of rays of a pencil and  $c$  and  $c'$  are the rays opposite to  $a$  and  $a'$  respectively, then  $S(ab) = S(a'b')$  if and only if  $S(abc) = S(a'b'c')$ . The same conclusion holds if  $c$  is any ray between  $a$  and  $b$  and  $c'$  any ray between  $a'$  and  $b'$ .

THEOREM 37. DEFINITION. The points not on the sides or vertex of an angle  $\angle ab$  fall into two classes having the sides and vertex as boundary and such that any segment joining a point of one class to a point of the other contains a point of the sides or the vertex. If the angle is a straight angle, both of these classes of points are convex regions. If not, one and only one of them is convex and is called the interior of the angle; the other is called the exterior of the angle.

THEOREM 38. If  $A'$  is any point of the side  $OA$  of an angle  $\angle AOB$ , and  $B'$  is any point of the side  $OB$ , then  $S(OAB) = S(OA'B')$ . If  $C$  is any point interior to the angle,  $S(OAB) = S(OAC) = S(OCB)$ , and any point  $C$  satisfying these conditions is interior to the angle.

THEOREM 39. Any ray having the vertex of an angle as origin, and not itself a side of the angle, is entirely in one or the other of the two classes of points described in Theorem 37. If it is in the interior it contains one and only one point on each segment joining a point of one side of the angle to a point on the other side.

DEFINITION. Two rays  $a, b$  of a pencil are said to be separated by two other rays  $h, k$  of the same pencil (or by the angle  $\angle hk$ ) if and only if  $a$  is in one and  $b$  in the other of the classes of points determined according to Theorem 37 by  $\angle hk$ . A set of rays having a common origin are said to be in the order  $\{a_1 a_2 a_3 a_4 \dots a_n\}$  if no two of the rays are separated by any of the angles  $\angle a_1 a_2, \angle a_2 a_3, \dots, \angle a_{n-1} a_n, \angle a_n a_1$ .

THEOREM 40. A set of rays in the order  $\{a_1 a_2 a_3 \dots a_{n-1} a_n\}$  are also in the orders  $\{a_2 a_3 \dots a_n a_1\}$  and  $\{a_n a_{n-1} \dots a_2 a_1\}$ .

COROLLARY. Any two rays  $a, b$  having a common origin are in the orders  $\{ab\}$  and  $\{ba\}$ . Any three rays  $a, b, c$  having a common origin are in the orders  $\{abc\}, \{bca\}, \{cab\}, \{acb\}, \{bac\}, \{cba\}$ .

**THEOREM 41.** *To any finite number  $n \equiv 2$  of rays having a common origin may be assigned a notation so that they are in the order  $\{a_1 a_2 a_3 \cdots a_n\}$ .*

**\*171. Pencils of segments and directions.** The notion of a ray belongs essentially with that of a convex region, but the theorems of the last section may easily be put into a form which is not limited to convex regions. The proofs are all omitted for the same reasons as in the section above.

**DEFINITION.** A set of all segments having a common end and lying in the same plane is called a *pencil of segments*. The common end is called the *center* of the pencil. Two segments or intervals having a common end  $A$  are said to be *similarly directed at  $A$*  if either of them is entirely contained in the other. The set of all segments similarly directed at a given point with a given segment is called a *direction-class* or, more simply, a *direction*. The set of all directions of the segments of a pencil at its center is called a *pencil of directions*. The directions of two collinear segments having a common end  $A$  and not similarly directed are said to be *opposite*, and the two segments are said to be *oppositely directed at  $A$* .

Thus if  $ABCD$  are four collinear points in the order  $\{ABCD\}$  the segments  $\overline{ABC}$  and  $\overline{ABD}$  are similarly directed, while  $\overline{ABC}$  and  $\overline{ADC}$  are oppositely directed. At a given point on a given line there are obviously two and only two directions, and these are opposite to each other. Two noncollinear segments with a common end are contained in one and only one pencil, namely, the one having the common end as center and lying in the plane of the two segments.

**DEFINITION.** A segment  $\sigma$  is said to be *between* two noncollinear segments  $\sigma_1, \sigma_2$  if the three segments are in the same pencil and  $\sigma$  is similarly directed with a segment which is in the pencil and contained entirely in the triangular region determined by  $\sigma_1$  and  $\sigma_2$  (Theorem 12). A direction  $d$  is said to be *between* two noncollinear directions  $d_1, d_2$  if there exist three segments  $\sigma, \sigma_1, \sigma_2$  in the directions  $d, d_1, d_2$  respectively such that  $\sigma$  is between  $\sigma_1$  and  $\sigma_2$ .

This extension of the notion of betweenness to directions is justified by the following theorem.

**THEOREM 42.** *If  $\alpha$  and  $\beta$  are two noncollinear segments with a common end  $O$ , and  $\alpha'$  and  $\beta'$  are similarly directed with  $\alpha$  and  $\beta$*

respectively at  $O$ , the segments between  $\alpha$  and  $\beta$  are similarly directed with the segments between  $\alpha'$  and  $\beta'$ .

DEFINITION. Let  $\sigma_1, \sigma_2, \sigma_3$  be three segments of a pencil no two of them being similarly directed. By an *elementary transformation* is meant the operation of replacing one of them, say  $\sigma_3$ , by a segment  $\sigma_4$ , which is in the pencil and such that neither  $\sigma_1$  nor  $\sigma_2$  is between  $\sigma_3$  and  $\sigma_4$  or similarly directed with  $\sigma_4$ . A class consisting of all ordered triads into which  $\sigma_1\sigma_2\sigma_3$  is transformable by finite sequences of elementary transformations is called a *sense-class* and is denoted by  $S(\sigma_1\sigma_2\sigma_3)$ . If  $d_1, d_2, d_3$  are three directions of a pencil, and  $\sigma_1, \sigma_2, \sigma_3$  three segments in the directions  $d_1, d_2, d_3$  respectively, the *sense-class*  $S(d_1d_2d_3)$  is the class of all triads of directions which are the directions of triads of segments in the sense-class  $S(\sigma_1\sigma_2\sigma_3)$ .

THEOREM 43. If  $\sigma_1, \sigma_2, \sigma_3$  are three segments of a pencil, no two of them being similarly directed, and  $\sigma'_3$  is similarly directed with  $\sigma_3$ ,  $S(\sigma_1\sigma_2\sigma_3) = S(\sigma_1\sigma_2\sigma'_3)$ .

THEOREM 44. There is a one-to-one reciprocal correspondence between the directions of a pencil and the points of a line such that two triads of directions are in the same sense if and only if the corresponding triads of points have the same sense.

We now take from §§ 21-23 of Chap. II the definitions of separation, order, etc., and on account of Theorem 44 we have at once

COROLLARY 1. The Theorems of §§ 21-23 remain valid when applied to the directions of a pencil instead of to the points of a line.

COROLLARY 2. Two pairs of opposite directions separate each other.

DEFINITION. Let  $\sigma_1$  and  $\sigma_2$  be two noncollinear segments of a pencil; by an *elementary transformation* is meant the operation of replacing one of them, say  $\sigma_2$ , by any segment  $\sigma_3$  of the pencil such that no segment collinear with  $\sigma_1$  is between  $\sigma_2$  and  $\sigma_3$ . The set of all ordered pairs of segments into which  $\sigma_1\sigma_2$  is transformable by sequences of elementary transformations is called a *sense-class* and is denoted by  $S(\sigma_1\sigma_2)$ .

THEOREM 45. If a pair of segments  $\sigma_1\sigma_2$  is transformable by elementary transformations into a pair  $\sigma'_1\sigma'_2$ , then  $\sigma'_1\sigma'_2$  is transformable by elementary transformations into  $\sigma_1\sigma_2$ .

**THEOREM 46.** *If a segment  $\sigma_2$  is similarly directed with a segment  $\sigma'_2$  and not collinear with a segment  $\sigma_1$  which has the same origin as  $\sigma_2$ ,  $S(\sigma_1\sigma_2) = S(\sigma_1\sigma'_2)$ .*

**THEOREM 47.** *If  $\sigma_1, \sigma_2, \sigma_3, \sigma_4$  are segments of a pencil and  $\sigma_1$  is not collinear with  $\sigma_2$ , nor  $\sigma_3$  with  $\sigma_4$ , then either  $S(\sigma_1\sigma_2) = S(\sigma_3\sigma_4)$  or  $S(\sigma_2\sigma_1) = S(\sigma_4\sigma_3)$ .  $S(\sigma_1\sigma_2) \neq S(\sigma_2\sigma_1)$ . If  $\sigma'_1$  is opposite to  $\sigma_1$ , and  $\sigma'_3$  to  $\sigma_3$ ,  $S(\sigma_1\sigma_2) = S(\sigma_3\sigma_4)$  if and only if  $S(\sigma'_1\sigma_1\sigma_2) = S(\sigma'_3\sigma_3\sigma_4)$ .*

**DEFINITION.** Let  $d_1$  and  $d_2$  be two directions of a pencil and let  $\sigma_1$  and  $\sigma_2$  be two segments in the directions  $d_1$  and  $d_2$  respectively. By the *sense-class*  $S(d_1d_2)$  is meant the class of all ordered pairs of directions which are the directions of ordered pairs of segments in the sense-class  $S(\sigma_1\sigma_2)$ .

It is evident that the last two theorems may be restated, without material change, in terms of directions instead of segments.

**\*172. Bundles of rays, segments, and directions.** **DEFINITION.** The set of all rays in a three-dimensional convex region which have a common origin  $O$  is called a *bundle* of rays. The point  $O$  is called the *center* of the bundle.

Let  $a, b, c$  be three noncoplanar rays of a bundle. By an *elementary transformation* is meant the operation of replacing one of the rays, say  $a$ , by a ray  $a'$  such that no ray of the plane containing  $b$  and  $c$  is between  $a$  and  $a'$ . The set of all ordered triads of rays into which  $abc$  can be carried by sequences of elementary transformations is called a *sense-class* and is denoted by  $S(abc)$ .

**THEOREM 48.** *If  $abc$  and  $a'b'c'$  are two ordered triads of noncoplanar rays having a common origin  $O$ , and  $A, B, C, A', B', C'$  are points of the rays  $a, b, c, a', b', c'$  respectively, then  $S(abc) = S(a'b'c')$  if and only if  $S(OABC) = S(OA'B'C')$ .*

**THEOREM 49.** *If  $a, b, c$  are three noncoplanar rays of a bundle,  $S(abc) = S(bca) \neq S(acb)$ . If  $a', b', c'$  are any other three noncoplanar rays of the bundle, either  $S(a'b'c') = S(abc)$  or  $S(a'b'c') = S(acb)$ .*

**THEOREM 50.** *If  $a, b, c$  are three noncoplanar rays of a bundle and  $a'$  is the ray opposite to  $a$ ,  $S(abc) \neq S(a'bc)$ .*

**THEOREM 51.** *If  $abc$  are three noncoplanar rays of a bundle, the set  $[x]$  of rays of the bundle which satisfy the relation  $S(xab) = S(xbc) = S(xca)$  are in such a one-to-one reciprocal correspondence  $\Gamma$  with the points of a triangular region that if rays  $x_1, x_2, x_3$ ,*

$x_4, x_5, x_6$  correspond to points  $X_1, X_2, X_3, X_4, X_5, X_6$  respectively,  $S(x_1x_2x_3) = S(x_4x_5x_6)$  if and only if  $S(X_1X_2X_3) = S(X_4X_5X_6)$ . If  $A, B, C$  are points of the rays  $a, b, c$  respectively, and the triangular region is the interior of the triangle  $ABC$ ,  $\Gamma$  may be taken as the correspondence in which each  $x$  corresponds to the point in which it meets the triangular region.

**THEOREM 52.** If  $a, b, c, d$  are four rays of a bundle such that any plane containing two of them contains a ray between the other two, any other ray of the bundle is between two rays of the set  $a, b, c, d$  or in one of four sets  $[x], [y], [z], [w]$  such that  $[x]$  satisfies the condition  $S(xbc) = S(xcd) = S(xdb)$ ,  $[y]$  satisfies  $S(yac) = S(ycd) = S(yda)$ ,  $[z]$  satisfies  $S(zab) = S(zbd) = S(zda)$ ,  $[w]$  satisfies  $S(wab) = S(wbc) = S(wca)$ .

**COROLLARY.** Under the conditions of the theorem if  $A, B, C, D$  are points of the rays  $a, b, c, d$  respectively, the center of the bundle is interior to the tetrahedron  $ABCD$ .

**DEFINITION.** A set of all segments having a common end is called a *bundle of segments*. The set of all directions of the segments of a bundle is called a *bundle of directions*.

**DEFINITION.** Let  $\sigma_1, \sigma_2, \sigma_3$  be three segments of the same bundle, but not in the same pencil; the operation of replacing any one of them, say  $\sigma_3$ , by a segment  $\sigma_4$  of the bundle such that no segment of the pencil containing  $\sigma_1$  and  $\sigma_2$  is between  $\sigma_3$  and  $\sigma_4$  or coincident with  $\sigma_4$  is called an *elementary transformation*. A class consisting of all ordered triads of segments into which  $\sigma_1\sigma_2\sigma_3$  can be carried by finite sequences of elementary transformations is called a *sense-class* and is denoted by  $S(\sigma_1\sigma_2\sigma_3)$ .

The generalization of Theorems 48-52 to the corresponding theorems for a bundle of segments presents no difficulty.

**\*173. One- and two-sided regions.** A discussion of the order relations in projective spaces which is closely analogous both to § 168 and to § 169 may be made according to the following outline. The details are left as an exercise for the reader.

Let  $O$  be any point of a planar region  $R$ . Let  $A, B, C$  be the vertices of a triangular region  $T$  containing  $O$  and contained in  $R$ , and let  $\alpha, \beta, \gamma$  be the segments in  $R$  joining  $O$  to  $A, B, C$  respectively. Then  $S(\alpha\beta) = S(\beta\gamma) = S(\gamma\alpha)$ .

If  $O'$  is any other point of  $T$ , and  $\alpha', \beta'$  the segments of  $R$  joining  $O'$  to  $A$  and  $B$  respectively,  $S(\alpha\beta)$  is said to be *like*  $S(\alpha'\beta')$ ; and

if  $S(\alpha\beta)$  is like  $S(\alpha'\beta')$ , and  $S(\alpha'\beta')$  like  $S(\alpha''\beta'')$ , then  $S(\alpha\beta)$  is said to be *like*  $S(\alpha''\beta'')$ . A region for which a given sense-class at one point is like the other sense-class at that point is said to be *one-sided*. Any other region is said to be *two-sided*.

A convex region is two-sided. A projective plane is a one-sided region.

Let  $O$  be any point of a three-dimensional region  $R$ . Let  $A, B, C, D$  be the vertices of a tetrahedral region  $T$  containing  $O$  and contained in  $R$ , and let  $\alpha, \beta, \gamma, \delta$  be the segments in  $R$  joining  $O$  to  $A, B, C, D$  respectively. Then  $S(\alpha\beta\gamma) = S(\beta\alpha\delta) = S(\delta\gamma\beta) = S(\gamma\delta\alpha)$ .

If  $O'$  is any other point of  $T$ , and  $\alpha', \beta', \gamma'$  are the segments of  $R$  joining  $O'$  to  $A, B, C$  respectively,  $S(\alpha\beta\gamma)$  is said to be *like*  $S(\alpha'\beta'\gamma')$ ; if  $S(\alpha\beta\gamma)$  is like  $S(\alpha'\beta'\gamma')$ , and  $S(\alpha'\beta'\gamma')$  is like  $S(\alpha''\beta''\gamma'')$ , then  $S(\alpha\beta\gamma)$  is said to be *like*  $S(\alpha''\beta''\gamma'')$ .

One- and two-sided regions are defined as in the two-dimensional case.

Any region in a three-dimensional projective space is two-sided.

**174. Sense-classes on a sphere.** The theorems in § 172 can be regarded as defining the order relations among the points of a sphere if carried over to the sphere by letting each point of the sphere correspond to the ray joining it to the center of the sphere. Another way of treating the order relations on a sphere and one which connects directly with § 97 is as follows:

**DEFINITION.** Let  $A, B, C, D$  be four points of a sphere not all on the same circle. By an *elementary transformation* is meant the operation of replacing one of them, say  $A$ , by a point  $A'$  on the same side of the circle  $BCD$ . The set of all ordered tetrads into which  $ABCD$  is transformable by sequences of elementary transformations is called a *sense-class* and is denoted by  $S(ABCD)$ .

**THEOREM 53.** *There are two and only two sense-classes on a sphere.  $S(ABCD) \neq S(ABDC)$ .*

**THEOREM 54.**  $S(ABCD) = S(A'B'C'D')$  if and only if  $bb' > 0$ , where  $R(AB, CD) = a + b\sqrt{-1}$ ,  $R(A'B', C'D') = a' + b'\sqrt{-1}$ , and  $a, a', b, b'$  are real.

**175. Order relations on complex lines.** In view of the isomorphism between the geometry of the real sphere and the complex projective line (cf. §§ 91, 95, and 100) the theorems of the section above and of § 97 determine the order relations on any complex line.

One very important difference between the situation as to order in the real and the complex spaces is the following: In a real plane or space one sense-class on a line is carried by projectivities of a continuous group into both sense-classes on any other line. So that fixing a particular sense-class on one line as positive does not determine a positive sense-class on all other lines. On a complex line, however, an ordered set of four points  $ABCD$  is in one sense-class or the other according as  $b$  is positive or negative, where  $a + b\sqrt{-1} = R(AB, CD)$  and  $a$  and  $b$  are real (Theorem 54). In consequence of the invariance of cross ratios under projection, a given sense-class on one line goes by projectivities into one and only one sense-class on any other line. Hence if one sense-class is called positive on one line, the positive sense-class can be determined on every other line as being that sense-class which is projective with the positive sense-class on the initial line.

This connects very closely with the convention for purposes of analytic geometry that by  $\sqrt{c}$  is meant that one of the square roots of  $c$  which takes the form  $a + b\sqrt{-1}$ , where  $a$  and  $b$  are real and  $b > 0$ , or if  $b = 0$ ,  $a > 0$ . The symbol  $\sqrt{-1}$  is taken to represent that one of the square roots of  $-1$  for which  $S(\infty \ 0 \ 1 \ \sqrt{-1})$  is positive.

**176. Direct and opposite collineations in space.** From the algebraic definition of direct collineation in terms of the sign of a determinant, we obtain at once

**THEOREM 55.** *Any collineation of a real three-dimensional projective space which leaves a Euclidean space invariant is direct if and only if the collineation which it effects in the Euclidean space is direct.*

In a Euclidean space a point  $D$  is on the same side of a plane  $ABCD$  with a point  $E$  if and only if  $S(ABCD) = S(ABCE)$ . Hence a homology whose center is at infinity is direct or opposite according as a point not on its plane of fixed points is transformed to a point on the same or the opposite side of this plane. Extending this result to the projective space by the aid of the theorem above we have

**THEOREM 56.** *A homology which carries a point  $A$  to a point  $A'$ , distinct from  $A$ , is opposite or direct according as  $A$  and  $A'$  are separated or not separated by the center of the homology and the point in which its plane of fixed points is met by the line  $AA'$ .*

COROLLARY 1. *A harmonic homology is opposite.*

COROLLARY 2. *The inverse of a direct homology is direct.*

Since any collineation is expressible as a product of homologies (§ 29, Vol. I), it follows that

COROLLARY 3. *The inverse of a direct collineation is direct.*

Since an elation is a product of two harmonic homologies having the same plane of fixed points it follows from Cor. 1 that

COROLLARY 4. *An elation is direct.*

Since a line reflection (§ 101) is a product of two harmonic homologies,

COROLLARY 5. *A line reflection is direct.*

THEOREM 57. *A collineation leaving three skew lines invariant is direct.*

*Proof.* Denote the lines by  $l_1, l_2, l_3$  and the collineation by  $\Gamma$ . The projectivity on  $l_1$  which is effected by  $\Gamma$  is a product of two or three hyperbolic involutions (§ 74). Each involution on  $l_1$  is effected by a line reflection whose directrices are the lines which pass through the double points of the involution and meet  $l_2$  and  $l_3$ . The product,  $\Pi$  of these line reflections leaves  $l_1, l_2, l_3$  invariant and effects the same transformation on  $l_1$  as  $\Gamma$ . Hence  $\Pi^{-1}\Gamma$  leaves  $l_2, l_3$  and all points on  $l_1$  invariant. It also leaves invariant any line meeting  $l_1, l_2$ , and  $l_3$ , and hence leaves all points on  $l_2$  and  $l_3$  invariant. Hence  $\Pi^{-1}\Gamma$  is the identity, and hence  $\Gamma = \Pi$ . Since the line reflections are all direct,  $\Gamma$  is direct.

COROLLARY 1. *Any collineation leaving all points of two skew lines invariant is direct.*

*Proof.* Such a collineation leaves invariant three skew lines meeting the given pair of invariant lines.

COROLLARY 2. *Any collineation transforming a regulus into itself is direct.*

*Proof.* Such a collineation is a product of a collineation leaving all lines of the given regulus invariant by one leaving all lines of the conjugate regulus invariant. Hence it is direct, by the theorem.

Corollary 2 is also a direct consequence of Cor. 5, above, and Theorem 34, Chap. VI.

COROLLARY 3. *Any collineation carrying a regulus into its conjugate regulus is opposite.*

*Proof.* The two reguli are interchanged by a harmonic homology whose center and axis are pole and polar with regard to the regulus. This harmonic homology is opposite by Cor. 1, Theorem 56, and since its product by any collineation  $\Gamma$  interchanging the two reguli leaves them both invariant and hence is direct by Cor. 2, it follows that  $\Gamma$  is opposite.

DEFINITION. By a *doubly oriented line* is meant a line  $l$  associated with one sense-class among the points on  $l$  and one sense-class among the planes on  $l$ . The doubly oriented line is said to be *on* any point, line, or plane on  $l$ .

A doubly oriented line may be denoted by the symbol  $(ABC, \alpha\beta\gamma)$  if  $A, B, C$  denote collinear points and  $\alpha, \beta, \gamma$  planes on the line  $AB$ . For this symbol determines the line  $AB$  and the sense-classes  $S(ABC)$  and  $S(\alpha\beta\gamma)$  uniquely. Since there are two sense-classes  $S(ABC)$  and  $S(ACB)$  among the points on a line  $AB$  and two sense-classes  $S(\alpha\beta\gamma)$  and  $S(\alpha\gamma\beta)$  among the planes on  $AB$ , there are four doubly oriented lines,

$$\begin{aligned} &(ABC, \alpha\beta\gamma), \\ &(ACB, \alpha\gamma\beta), \\ &(ABC, \alpha\gamma\beta), \\ &(ACB, \alpha\beta\gamma), \end{aligned}$$

into which  $AB$  enters.

THEOREM 58. *The collineations which transform a doubly oriented line into itself are all direct.*

*Proof.* Let  $(ABC, \alpha\beta\gamma)$  be a doubly oriented line,  $\Gamma$  a collineation leaving it invariant,  $l$  any line not meeting  $AB$ , and  $l' = \Gamma(l)$ . The line  $l'$  cannot meet  $AB$ , because  $AB$  is transformed into itself by  $\Gamma$ . If  $l'$  does not intersect  $l$ , let  $m$  be the line harmonically separated from  $AB$  by  $l$  and  $l'$  in the regulus containing  $AB, l$ , and  $l'$ . If  $l'$  meets  $l$  let  $m$  be the line harmonically separated by  $l$  and  $l'$  from the point in which the plane  $l'$  is met by  $AB$ . In either case  $AB$  does not intersect  $m$ , and if  $\Lambda$  is the line reflection whose directrices are  $AB$  and  $m$ ,  $\Lambda(l') = l$ . Hence  $\Lambda\Gamma$  leaves both  $AB$  and  $m$  invariant. Since  $\Lambda$  and  $\Gamma$  preserve sense both in the pencil of points  $AB$  and in the pencil of planes  $\alpha\beta$ ,  $\Lambda\Gamma$  preserves sense both on  $AB$  and on  $m$ . Hence by § 74,  $\Lambda\Gamma$  effects a projectivity on  $AB$  which is a

product of two hyperbolic involutions,  $\{P_4P_3\} \cdot \{P_2P_1\}$ , and it effects a projectivity on  $m$  which is a product of two hyperbolic involutions,  $\{Q_4Q_3\} \cdot \{Q_2Q_1\}$ . Let  $l_1, l_2, l_3, l_4$  be the lines  $P_1Q_1, P_2Q_2, P_3Q_3, P_4Q_4$  respectively. The product

$$\{l_1l_2\} \cdot \{l_3l_4\} \cdot \Lambda \cdot \Gamma$$

leaves all points on  $AB$  and on  $m$  invariant and is therefore direct by Cor. 1, Theorem 57. All the collineations in this product except  $\Gamma$  are direct by Cor. 5, Theorem 56. Hence  $\Gamma$  is direct.

**COROLLARY 1.** *Any collineation which reverses both sense-classes of a doubly oriented line is direct.*

*Proof.* Let  $\Gamma$  be a collineation reversing both sense-classes of a doubly oriented line  $(ABC, \alpha\beta\gamma)$ . Let  $a$  and  $b$  be two lines meeting  $AB$  but not intersecting each other. The line reflection  $\{ab\}$  reverses both sense-classes of  $(ABC, \alpha\beta\gamma)$  and is direct. Hence  $\{ab\} \cdot \Gamma$  leaves them both invariant and is direct by the theorem. Hence  $\Gamma$  is direct.

**COROLLARY 2.** *Any collineation which transforms each of two skew lines into itself and effects a direct projectivity on each is direct.*

**COROLLARY 3.** *Any collineation which transforms each of two skew lines into itself and effects an opposite projectivity on each is direct.*

**177. Right- and left-handed figures.** The theorems of the last section can be used in showing that other figures than the ordered pentads of points may be classified as right-handed and left-handed. For this purpose the following theorem is fundamental.

**THEOREM 59.** *If the collineations carrying a figure  $F_0$  into itself are all direct, the figures equivalent to  $F_0$  under the group of all collineations fall into two classes such that any collineation carrying a figure of one class into a figure of the same class is direct and any collineation carrying it into a figure of the other class is opposite.*

*Proof.* Let  $[F]$  be the set of all figures into which  $F_0$  can be carried by direct collineations. There is no opposite collineation carrying  $F_0$  into an  $F$ ; for suppose  $\Gamma$  were such an opposite collineation, let  $P$  be one of the direct collineations which by definition of  $[F]$  carry  $F_0$  into  $F$ ; then  $P^{-1}\Gamma$  would be an opposite collineation carrying  $F_0$  into itself. In like manner it follows that any collineation carrying any  $F$  into itself or any other  $F$  is direct.

Let  $[F']$  be the set of all figures into which  $F_0$  is carried by opposite collineations. An argument like that above shows (1) that any collineation carrying  $F_0$  into an  $F'$  is opposite and (2) that any collineation carrying an  $F'$  into itself or another  $F'$  is direct. It follows at once that any collineation carrying an  $F$  into an  $F'$  or an  $F'$  into an  $F$  is opposite.

Since the direct collineations form a continuous family of transformations, we have

**COROLLARY.** *The figures conjugate to  $F_0$  under the group of direct collineations form a continuous family.*

The propositions about the sense-classes of ordered tetrads of non-collinear points are corollaries of this theorem because the only collineation carrying an ordered pentad of noncollinear points into itself is the identity.

By Theorems 57 and 59 all triads of noncollinear lines fall into two classes such that any collineation carrying a triad of one class into a triad of the same class is direct and any collineation carrying a triad of one class into a triad of the other class is opposite. It is to be noted particularly that the triads of lines here considered need not be ordered triads, since by Cor. 2, Theorem 57, the collineation effecting any permutation of a set of three noncollinear lines is direct.

Similar propositions hold with regard to doubly oriented lines, reguli, congruences, and complexes (cf. § 178).

Let us now suppose that a particular sense-class  $S(ABCD)$  in a Euclidean space has been designated as *right-handed* (cf. § 162). Any ordered tetrad of points in this sense-class is also called *right-handed* and any ordered tetrad in the other sense-class is called *left-handed*.

Let  $P$  be a point interior to the triangular region  $BCD$ ,  $Q$  the point at infinity of the line  $AP$ ,  $\beta$  the plane  $APB$ ,  $\gamma$  the plane  $APC$ , and  $\delta$  the plane  $APD$ . All doubly oriented lines into which  $(APQ, \beta\gamma\delta)$  is carried by direct collineations shall be called *right-handed* and all others shall be called *left-handed*.

The set of points  $ABCDQ$  and the sense-class  $S(ABCDQ)$  in the projective space  $ABCD$  shall be called *right-handed* and all other ordered pentads of noncollinear points and the other sense-class shall be called *left-handed*.

These conventions give the same determination of right-handed doubly oriented lines and ordered pentads of points no matter what point of the triangular region  $BCD$  is taken as  $P$ , because any collineation leaving  $A, B, C, D$  invariant and carrying one such  $P$  into another is direct. In like manner these conventions are independent of the choice of  $ABCD$ , so long as  $S(ABCD)$  is direct.

A triad of skew lines  $l_1, l_2, l_3$  shall be said to be *right-handed* or *left-handed* according as the doubly oriented line  $(ABC, \alpha\beta\gamma)$  is right-handed or left-handed, provided that  $m$  is a line meeting  $l_1, l_2, l_3$ , and  $A, B, C$  are the points  $ml_1, ml_2, ml_3$  respectively, and  $\alpha, \beta, \gamma$  are the planes  $ml_1, ml_2, ml_3$  respectively.

This convention is independent of the choice of  $m$ , by Theorems 57 and 58. By the same theorems any collineation carrying a right-handed triad of noncollinear lines into a right-handed triad of lines is direct, and any collineation carrying a right-handed triad of lines into a left-handed triad is opposite.

The reader should verify that a pair of skew lines  $lm$  in a Euclidean space is right-handed or left-handed in the sense of § 163 according as  $lml_\infty$  is right-handed or left-handed,  $l_\infty$  being the line at infinity which is the absolute polar of the point at infinity of  $l$ . If  $m_\infty$  is the absolute polar of the point at infinity of  $m$ ,  $lmm_\infty$  is right-handed if and only if  $lml_\infty$  is right-handed.

Let  $A$  be a point of the axis of a twist  $\Gamma$  in a Euclidean space, let  $B = \Gamma(A)$ , and let  $C$  be the point at infinity of the line  $AB$ ; let  $\alpha$  be any plane on the line  $AB$ ,  $\beta = \Gamma(\alpha)$ , and  $\gamma$  the plane on  $AB$  perpendicular to  $\alpha$ . Then  $\Gamma$  is right-handed in the sense of § 163 if and only if the doubly oriented line  $(ABC, \alpha\beta\gamma)$  is right-handed. This is easily verified.

**178. Right- and left-handed reguli, congruences, and complexes.** By Cor. 2, Theorem 57, every triad of lines in a regulus is right-handed or every triad is left-handed. In the first case the regulus shall be said to be *right-handed* and in the second case to be *left-handed*.

**THEOREM 60.** *The collineations which leave an elliptic linear congruence invariant are all direct.*

*Proof.* An elliptic congruence has a pair of conjugate imaginary lines as its directrices (§ 109), and there is one real line of the

congruence through each point of a directrix. Any collineation  $\Gamma$  which carries each directrix of the congruence into itself effects a projectivity on that directrix. This projectivity is a product of two involutions (§ 78, Vol. I). Each involution may be effected by a line reflection whose lines of fixed points are the (real) lines of the congruence through the (imaginary) double points of the involution; since such a line reflection leaves both directrices invariant, it leaves the congruence invariant. Hence there exist two line reflections  $\Lambda_1, \Lambda_2$ , each transforming the congruence into itself, such that  $\Lambda_2\Lambda_1\Gamma$  leaves all points on a directrix invariant. Hence  $\Lambda_2\Lambda_1\Gamma$  transforms each line of the congruence into itself. By Theorem 57,  $\Lambda_2\Lambda_1\Gamma$  is direct, and by Theorem 56, Cor. 5,  $\Lambda_1$  and  $\Lambda_2$  are direct. Hence  $\Gamma$  is direct.

If  $l$  is any real line not in the congruence, the lines of the congruence meeting  $l$  form a regulus, and the directrices are double lines of an involution in the lines of the conjugate regulus. If  $l'$  is the line conjugate to  $l$  in this involution, the line reflection  $\{l'\}$  must interchange the two directrices. Hence if  $\Gamma'$  is any collineation interchanging the directrices,  $\{l'\} \cdot \Gamma'$  is a collineation which leaves each of them invariant. Hence by the paragraph above  $\{l'\} \cdot \Gamma'$  is direct. Hence  $\Gamma'$  is direct. Hence any real collineation leaving an elliptic linear congruence invariant is direct.

**COROLLARY 1.** *The triads of lines of an elliptic linear congruence are all right-handed or all left-handed.*

For any triad can be carried into any other triad by a direct collineation.

**COROLLARY 2.** *If four linearly independent lines are such that all sets of three of them are right-handed or such that all sets of three of them are left-handed, the linear congruence which contains them is elliptic.*

An elliptic congruence shall be said to be *right-handed* if every triad of lines in it is right-handed; otherwise it is said to be *left-handed*.

A pair of conjugate imaginary lines of the second kind (§ 109) is said to be *right-handed* or *left-handed* according as it is determined by a right-handed or a left-handed congruence.

A pair of Clifford parallels (§ 142) is said to be *right-handed* or *left-handed* according as the congruence of Clifford parallels to which

they belong is right-handed or left-handed. This distinction is in agreement with that introduced in § 142, because according to both definitions a collineation carrying a system of right-handed Clifford parallels into a system of right-handed ones is direct, and a collineation carrying a system of right-handed Clifford parallels into a system of left-handed ones is opposite.

**THEOREM 61.** *The collineations which leave a nondegenerate linear complex invariant are all direct.*

*Proof.* Let  $\Gamma$  be a collineation leaving a complex  $C$  invariant, and let  $l$  be any line of  $C$  and  $l' = \Gamma(l)$ . Let  $l''$  be any line of  $C$  not meeting  $l$  or  $l'$ . The lines of  $C$  which meet  $l$  and  $l''$  constitute a regulus, and three lines of this regulus together with  $l$  and  $l''$  constitute a set of five linearly independent lines (§ 106, Vol. I) upon which, therefore, all the lines of  $C$  are linearly dependent. Hence a collineation  $\Gamma'$  which leaves this regulus invariant and interchanges  $l$  and  $l''$  leaves  $C$  invariant. Let  $\Gamma''$  be a collineation, similarly obtained, which interchanges  $l''$  and  $l'$  and leaves  $C$  invariant. The product  $\Gamma'\Gamma''\Gamma$  leaves  $C$  and  $l$  invariant, and  $\Gamma'$  and  $\Gamma''$  are direct.

Any collineation leaving  $C$  and  $l$  invariant leaves invariant the projectivity  $\Pi$  between the points on  $l$  and the planes corresponding to them in the null system determined by  $C$ . The projectivity  $\Pi$  transforms an arbitrary sense-class among the points on  $l$  into an arbitrary sense-class among the planes on  $l$ . These two sense-classes determine a doubly oriented line,  $\bar{l}$ . The other sense-class of the points on  $l$  is carried by  $\Pi$  into the other sense-class of planes on  $l$ , and these two sense-classes determine a doubly oriented line  $\bar{\bar{l}}$ . Since any collineation leaving  $C$  and  $l$  invariant leaves  $\Pi$  invariant, it either transforms this doubly oriented line into itself or into the one obtained by reversing both its sense-classes. Hence any such collineation is direct by Theorem 58 and its first corollary. In particular  $\Gamma'\Gamma''\Gamma$  is direct, and since  $\Gamma'$  and  $\Gamma''$  are direct, it follows that  $\Gamma$  is direct.

By Theorem 61 all the doubly oriented lines analogous to  $\bar{l}$  which are determined by  $C$  are all right-handed or all left-handed. In the first case  $C$  shall be called *right-handed*, and in the second case  $C$  shall be called *left-handed*.

The algebraic criteria in the exercises below are taken from the article by E. Study referred to in § 162. See also F. Klein, *Auto-graphierte Vorlesungen über nicht-Euclidische Geometrie*, Vol. II, Chap. I, Göttingen, 1890.

## EXERCISES

1. Classify parabolic congruences (§ 107, Vol. I) as right-handed and left-handed.

2. For two lines  $p$  and  $p'$  let

$$(p, p') = p_{12}p'_{34} + p_{13}p'_{42} + p_{14}p'_{23} + p_{34}p'_{12} + p_{42}p'_{13} + p_{23}p'_{14},$$

where  $p_{ij}$  are the Plücker coördinates (§ 109, Vol. I) of  $p$ , and  $p'_{ij}$  those of  $p'$ . Three lines  $p, p', p''$  are right-handed or left-handed according as

$$(p, p') \cdot (p', p'') \cdot (p'', p)$$

is positive or negative.

3. A pair of conjugate imaginary lines of the second kind whose Plücker coördinates are  $p_{ij}$  and  $\bar{p}_{ij}$  respectively are right-handed or left-handed according as

$$p_{12}\bar{p}_{13} + p_{13}\bar{p}_{14} + p_{14}\bar{p}_{23} + p_{34}\bar{p}_{34} + p_{42}\bar{p}_{42} + p_{23}\bar{p}_{23}$$

is positive or negative.

4. The linear line complex whose equation in Plücker coördinates is (§ 110, Vol. I)

$$a_{12}p_{12} + a_{13}p_{13} + a_{14}p_{14} + a_{34}p_{34} + a_{42}p_{42} + a_{23}p_{23} = 0$$

is right-handed or left-handed according as

$$a_{12}a_{34} + a_{13}a_{42} + a_{14}a_{23}$$

is positive or negative.

5. A twist given by the parameters of § 130 is right-handed if  $\alpha_0\beta_0 > 0$  and the coördinate system is right-handed.

6. The linear complex  $C$  determined by a twist according to Ex. 7, § 122, is right-handed or left-handed according as the twist is right-handed or left-handed.

**\*179. Elementary transformations of triads of lines.** Let  $F_0$  be a figure such that all the collineations which transform it into itself are direct, and let  $[F]$  be the set of all figures equivalent to  $F$  under direct transformations. From the fact that the group of all direct collineations is continuous, it can be proved that  $[F]$  is a continuous family of figures.

This can also be put into evidence by generalizing the notion of elementary transformation to other figures. This is essentially what has been done in §§ 169 and 173. For triads of skew lines the following theorem is fundamental.

**THEOREM 62.** *If  $l_1, l_2, l_3$  are three skew lines, and  $l_4$  is a line coplanar with  $l_3$  and such that the points in which  $l_1$  and  $l_2$  meet the plane  $l_3l_4$  are not separated by the lines  $l_3$  and  $l_4$ , then  $l_1l_2l_3$  can be carried to  $l_1l_2l_4$  by a direct collineation.*

*Proof.* Let  $\alpha$  be a line meeting  $l_1, l_3$ , and  $l_4$  (fig. 86) in points  $A_1, A_3, A_4$  respectively, which are all distinct. Let  $\alpha$  be the plane containing  $l_3$  and the point  $B$  of intersection of  $l_3$  and  $l_4$ . If  $A_1$  is in  $\alpha$ , an elation with  $A_1$  as center,  $\alpha$  as plane of fixed points, and carrying  $A_3$  to  $A_4$  will carry  $l_1, l_2, l_3$  into  $l_1, l_2, l_4$  respectively. By Theorem 56, Cor. 4, this elation is direct.

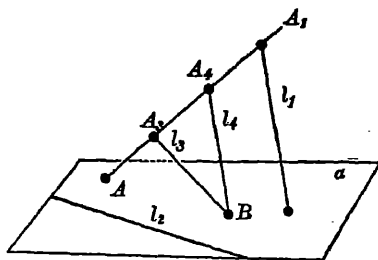


FIG. 86

If  $A_1$  is not in  $\alpha$ , the points  $A_3$  and  $A_4$  are not separated by  $A_1$  and the point  $A$  in which  $\alpha$  meets  $\alpha$ ; for by hypothesis  $A_1$  and the point in which  $l_2$  meets the line  $BA$  are not separated by the lines  $l_3$  and  $l_4$ . Hence the homology with  $A_1$  as center and  $\alpha$  as plane of fixed points which carries  $l_3$  to  $l_4$  is direct (Theorem 56). This homology carries  $l_1, l_2, l_3$  into  $l_1, l_2, l_4$  respectively.

An *elementary transformation* of a triad of skew lines  $l_1l_2l_3$  may be defined as the operation of replacing one of them, say  $l_3$ , by a line  $l_4$  which is coplanar with  $l_3$  and such that  $l_3$  and  $l_4$  do not separate the points in which their plane is met by  $l_1$  and  $l_2$ .

By Theorem 62 an elementary transformation may be effected by a direct collineation. A sequence of elementary transformations therefore carries a right-handed triad into a right-handed triad and a left-handed triad into a left-handed triad.

Conversely, it can be proved that any right-handed triad can be carried into any right-handed triad by a sequence of elementary transformations and that the two classes of lines determined by a pair of skew lines  $ab$  according to Ex. 2, § 25, are the lines  $[x]$  such that  $abx$  is right-handed and the lines  $[y]$  such that  $aby$  is left-handed. These propositions are left to the reader.

**\*180. Doubly oriented lines.** The theory of sense-classes in three dimensions could be based entirely on that of doubly oriented lines (§ 176). We shall prove the earliest theorems of such a theory in

this section. The proofs are based on Assumptions A, E, S and do not make use of the preceding discussions of order in three-space.

**DEFINITION.** Two doubly oriented lines are said to be *doubly perspective* if they can be given the notation  $(ABC, \alpha\beta\gamma)$  and  $(A'B'C', \alpha'\beta'\gamma')$  respectively in such a way that  $A, B, C, \alpha, \beta, \gamma$  are on  $\alpha', \beta', \gamma', A', B', C'$  respectively. Two doubly oriented lines  $l_0$  and  $l$  are said to be *similarly oriented* if and only if there exists a sequence of doubly oriented lines  $l_1, l_2, \dots, l_n$  such that  $l_0$  is doubly perspective with  $l_1$ ,  $l_1$  with  $l_2$ ,  $l_2$  with  $l_3$ ,  $l_3$  with  $l_4$ ,  $l_4$  with  $l_5$ ,  $l_5$  with  $l_6$ ,  $l_6$  with  $l_7$ ,  $l_7$  with  $l_8$ ,  $l_8$  with  $l_9$ ,  $l_9$  with  $l_{10}$ ,  $l_{10}$  with  $l_{11}$ ,  $l_{11}$  with  $l_{12}$ ,  $l_{12}$  with  $l_{13}$ ,  $l_{13}$  with  $l_{14}$ ,  $l_{14}$  with  $l_{15}$ ,  $l_{15}$  with  $l_{16}$ ,  $l_{16}$  with  $l_{17}$ ,  $l_{17}$  with  $l_{18}$ ,  $l_{18}$  with  $l_{19}$ ,  $l_{19}$  with  $l_{20}$ ,  $l_{20}$  with  $l_{21}$ ,  $l_{21}$  with  $l_{22}$ ,  $l_{22}$  with  $l_{23}$ ,  $l_{23}$  with  $l_{24}$ ,  $l_{24}$  with  $l_{25}$ ,  $l_{25}$  with  $l_{26}$ ,  $l_{26}$  with  $l_{27}$ ,  $l_{27}$  with  $l_{28}$ ,  $l_{28}$  with  $l_{29}$ ,  $l_{29}$  with  $l_{30}$ ,  $l_{30}$  with  $l_{31}$ ,  $l_{31}$  with  $l_{32}$ ,  $l_{32}$  with  $l_{33}$ ,  $l_{33}$  with  $l_{34}$ ,  $l_{34}$  with  $l_{35}$ ,  $l_{35}$  with  $l_{36}$ ,  $l_{36}$  with  $l_{37}$ ,  $l_{37}$  with  $l_{38}$ ,  $l_{38}$  with  $l_{39}$ ,  $l_{39}$  with  $l_{40}$ ,  $l_{40}$  with  $l_{41}$ ,  $l_{41}$  with  $l_{42}$ ,  $l_{42}$  with  $l_{43}$ ,  $l_{43}$  with  $l_{44}$ ,  $l_{44}$  with  $l_{45}$ ,  $l_{45}$  with  $l_{46}$ ,  $l_{46}$  with  $l_{47}$ ,  $l_{47}$  with  $l_{48}$ ,  $l_{48}$  with  $l_{49}$ ,  $l_{49}$  with  $l_{50}$ ,  $l_{50}$  with  $l_{51}$ ,  $l_{51}$  with  $l_{52}$ ,  $l_{52}$  with  $l_{53}$ ,  $l_{53}$  with  $l_{54}$ ,  $l_{54}$  with  $l_{55}$ ,  $l_{55}$  with  $l_{56}$ ,  $l_{56}$  with  $l_{57}$ ,  $l_{57}$  with  $l_{58}$ ,  $l_{58}$  with  $l_{59}$ ,  $l_{59}$  with  $l_{60}$ ,  $l_{60}$  with  $l_{61}$ ,  $l_{61}$  with  $l_{62}$ ,  $l_{62}$  with  $l_{63}$ ,  $l_{63}$  with  $l_{64}$ ,  $l_{64}$  with  $l_{65}$ ,  $l_{65}$  with  $l_{66}$ ,  $l_{66}$  with  $l_{67}$ ,  $l_{67}$  with  $l_{68}$ ,  $l_{68}$  with  $l_{69}$ ,  $l_{69}$  with  $l_{70}$ ,  $l_{70}$  with  $l_{71}$ ,  $l_{71}$  with  $l_{72}$ ,  $l_{72}$  with  $l_{73}$ ,  $l_{73}$  with  $l_{74}$ ,  $l_{74}$  with  $l_{75}$ ,  $l_{75}$  with  $l_{76}$ ,  $l_{76}$  with  $l_{77}$ ,  $l_{77}$  with  $l_{78}$ ,  $l_{78}$  with  $l_{79}$ ,  $l_{79}$  with  $l_{80}$ ,  $l_{80}$  with  $l_{81}$ ,  $l_{81}$  with  $l_{82}$ ,  $l_{82}$  with  $l_{83}$ ,  $l_{83}$  with  $l_{84}$ ,  $l_{84}$  with  $l_{85}$ ,  $l_{85}$  with  $l_{86}$ ,  $l_{86}$  with  $l_{87}$ ,  $l_{87}$  with  $l_{88}$ ,  $l_{88}$  with  $l_{89}$ ,  $l_{89}$  with  $l_{90}$ ,  $l_{90}$  with  $l_{91}$ ,  $l_{91}$  with  $l_{92}$ ,  $l_{92}$  with  $l_{93}$ ,  $l_{93}$  with  $l_{94}$ ,  $l_{94}$  with  $l_{95}$ ,  $l_{95}$  with  $l_{96}$ ,  $l_{96}$  with  $l_{97}$ ,  $l_{97}$  with  $l_{98}$ ,  $l_{98}$  with  $l_{99}$ ,  $l_{99}$  with  $l_{100}$ ,  $l_{100}$  with  $l_{101}$ ,  $l_{101}$  with  $l_{102}$ ,  $l_{102}$  with  $l_{103}$ ,  $l_{103}$  with  $l_{104}$ ,  $l_{104}$  with  $l_{105}$ ,  $l_{105}$  with  $l_{106}$ ,  $l_{106}$  with  $l_{107}$ ,  $l_{107}$  with  $l_{108}$ ,  $l_{108}$  with  $l_{109}$ ,  $l_{109}$  with  $l_{110}$ ,  $l_{110}$  with  $l_{111}$ ,  $l_{111}$  with  $l_{112}$ ,  $l_{112}$  with  $l_{113}$ ,  $l_{113}$  with  $l_{114}$ ,  $l_{114}$  with  $l_{115}$ ,  $l_{115}$  with  $l_{116}$ ,  $l_{116}$  with  $l_{117}$ ,  $l_{117}$  with  $l_{118}$ ,  $l_{118}$  with  $l_{119}$ ,  $l_{119}$  with  $l_{120}$ ,  $l_{120}$  with  $l_{121}$ ,  $l_{121}$  with  $l_{122}$ ,  $l_{122}$  with  $l_{123}$ ,  $l_{123}$  with  $l_{124}$ ,  $l_{124}$  with  $l_{125}$ ,  $l_{125}$  with  $l_{126}$ ,  $l_{126}$  with  $l_{127}$ ,  $l_{127}$  with  $l_{128}$ ,  $l_{128}$  with  $l_{129}$ ,  $l_{129}$  with  $l_{130}$ ,  $l_{130}$  with  $l_{131}$ ,  $l_{131}$  with  $l_{132}$ ,  $l_{132}$  with  $l_{133}$ ,  $l_{133}$  with  $l_{134}$ ,  $l_{134}$  with  $l_{135}$ ,  $l_{135}$  with  $l_{136}$ ,  $l_{136}$  with  $l_{137}$ ,  $l_{137}$  with  $l_{138}$ ,  $l_{138}$  with  $l_{139}$ ,  $l_{139}$  with  $l_{140}$ ,  $l_{140}$  with  $l_{141}$ ,  $l_{141}$  with  $l_{142}$ ,  $l_{142}$  with  $l_{143}$ ,  $l_{143}$  with  $l_{144}$ ,  $l_{144}$  with  $l_{145}$ ,  $l_{145}$  with  $l_{146}$ ,  $l_{146}$  with  $l_{147}$ ,  $l_{147}$  with  $l_{148}$ ,  $l_{148}$  with  $l_{149}$ ,  $l_{149}$  with  $l_{150}$ ,  $l_{150}$  with  $l_{151}$ ,  $l_{151}$  with  $l_{152}$ ,  $l_{152}$  with  $l_{153}$ ,  $l_{153}$  with  $l_{154}$ ,  $l_{154}$  with  $l_{155}$ ,  $l_{155}$  with  $l_{156}$ ,  $l_{156}$  with  $l_{157}$ ,  $l_{157}$  with  $l_{158}$ ,  $l_{158}$  with  $l_{159}$ ,  $l_{159}$  with  $l_{160}$ ,  $l_{160}$  with  $l_{161}$ ,  $l_{161}$  with  $l_{162}$ ,  $l_{162}$  with  $l_{163}$ ,  $l_{163}$  with  $l_{164}$ ,  $l_{164}$  with  $l_{165}$ ,  $l_{165}$  with  $l_{166}$ ,  $l_{166}$  with  $l_{167}$ ,  $l_{167}$  with  $l_{168}$ ,  $l_{168}$  with  $l_{169}$ ,  $l_{169}$  with  $l_{170}$ ,  $l_{170}$  with  $l_{171}$ ,  $l_{171}$  with  $l_{172}$ ,  $l_{172}$  with  $l_{173}$ ,  $l_{173}$  with  $l_{174}$ ,  $l_{174}$  with  $l_{175}$ ,  $l_{175}$  with  $l_{176}$ ,  $l_{176}$  with  $l_{177}$ ,  $l_{177}$  with  $l_{178}$ ,  $l_{178}$  with  $l_{179}$ ,  $l_{179}$  with  $l_{180}$ ,  $l_{180}$  with  $l_{181}$ ,  $l_{181}$  with  $l_{182}$ ,  $l_{182}$  with  $l_{183}$ ,  $l_{183}$  with  $l_{184}$ ,  $l_{184}$  with  $l_{185}$ ,  $l_{185}$  with  $l_{186}$ ,  $l_{186}$  with  $l_{187}$ ,  $l_{187}$  with  $l_{188}$ ,  $l_{188}$  with  $l_{189}$ ,  $l_{189}$  with  $l_{190}$ ,  $l_{190}$  with  $l_{191}$ ,  $l_{191}$  with  $l_{192}$ ,  $l_{192}$  with  $l_{193}$ ,  $l_{193}$  with  $l_{194}$ ,  $l_{194}$  with  $l_{195}$ ,  $l_{195}$  with  $l_{196}$ ,  $l_{196}$  with  $l_{197}$ ,  $l_{197}$  with  $l_{198}$ ,  $l_{198}$  with  $l_{199}$ ,  $l_{199}$  with  $l_{200}$ ,  $l_{200}$  with  $l_{201}$ ,  $l_{201}$  with  $l_{202}$ ,  $l_{202}$  with  $l_{203}$ ,  $l_{203}$  with  $l_{204}$ ,  $l_{204}$  with  $l_{205}$ ,  $l_{205}$  with  $l_{206}$ ,  $l_{206}$  with  $l_{207}$ ,  $l_{207}$  with  $l_{208}$ ,  $l_{208}$  with  $l_{209}$ ,  $l_{209}$  with  $l_{210}$ ,  $l_{210}$  with  $l_{211}$ ,  $l_{211}$  with  $l_{212}$ ,  $l_{212}$  with  $l_{213}$ ,  $l_{213}$  with  $l_{214}$ ,  $l_{214}$  with  $l_{215}$ ,  $l_{215}$  with  $l_{216}$ ,  $l_{216}$  with  $l_{217}$ ,  $l_{217}$  with  $l_{218}$ ,  $l_{218}$  with  $l_{219}$ ,  $l_{219}$  with  $l_{220}$ ,  $l_{220}$  with  $l_{221}$ ,  $l_{221}$  with  $l_{222}$ ,  $l_{222}$  with  $l_{223}$ ,  $l_{223}$  with  $l_{224}$ ,  $l_{224}$  with  $l_{225}$ ,  $l_{225}$  with  $l_{226}$ ,  $l_{226}$  with  $l_{227}$ ,  $l_{227}$  with  $l_{228}$ ,  $l_{228}$  with  $l_{229}$ ,  $l_{229}$  with  $l_{230}$ ,  $l_{230}$  with  $l_{231}$ ,  $l_{231}$  with  $l_{232}$ ,  $l_{232}$  with  $l_{233}$ ,  $l_{233}$  with  $l_{234}$ ,  $l_{234}$  with  $l_{235}$ ,  $l_{235}$  with  $l_{236}$ ,  $l_{236}$  with  $l_{237}$ ,  $l_{237}$  with  $l_{238}$ ,  $l_{238}$  with  $l_{239}$ ,  $l_{239}$  with  $l_{240}$ ,  $l_{240}$  with  $l_{241}$ ,  $l_{241}$  with  $l_{242}$ ,  $l_{242}$  with  $l_{243}$ ,  $l_{243}$  with  $l_{244}$ ,  $l_{244}$  with  $l_{245}$ ,  $l_{245}$  with  $l_{246}$ ,  $l_{246}$  with  $l_{247}$ ,  $l_{247}$  with  $l_{248}$ ,  $l_{248}$  with  $l_{249}$ ,  $l_{249}$  with  $l_{250}$ ,  $l_{250}$  with  $l_{251}$ ,  $l_{251}$  with  $l_{252}$ ,  $l_{252}$  with  $l_{253}$ ,  $l_{253}$  with  $l_{254}$ ,  $l_{254}$  with  $l_{255}$ ,  $l_{255}$  with  $l_{256}$ ,  $l_{256}$  with  $l_{257}$ ,  $l_{257}$  with  $l_{258}$ ,  $l_{258}$  with  $l_{259}$ ,  $l_{259}$  with  $l_{260}$ ,  $l_{260}$  with  $l_{261}$ ,  $l_{261}$  with  $l_{262}$ ,  $l_{262}$  with  $l_{263}$ ,  $l_{263}$  with  $l_{264}$ ,  $l_{264}$  with  $l_{265}$ ,  $l_{265}$  with  $l_{266}$ ,  $l_{266}$  with  $l_{267}$ ,  $l_{267}$  with  $l_{268}$ ,  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$m_0$  is doubly perspective with  $m_1$ ,  $m_1 = (A_1B_1C_1, \alpha_1\beta_1\gamma_1)$ ; and since  $m_1$  is doubly perspective with  $m_2$ ,  $m_2 = (A_2B_2C_2, \alpha_2\beta_2\gamma_2)$ . But by construction  $(A_2B_2C_2, \alpha_2\beta_2\gamma_2)$  is doubly perspective with  $(ABC, \alpha\beta\gamma)$ , i.e.  $m_2$  is doubly perspective with  $m_0$ .

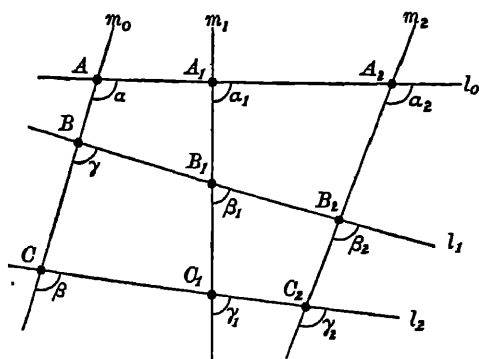


FIG. 88

In the second case  $(ABC, \alpha\beta\gamma)$  is doubly perspective with  $(A_1C_1B_1, \alpha_1\beta_1\gamma_1)$  and this with  $(A_2B_2C_2, \alpha_2\gamma_2\beta_2)$ . Since  $m_0 = (ABC, \alpha\beta\gamma)$ , and  $m_1$  is doubly perspective with  $m_0$ ,  $m_1 = (A_1C_1B_1, \alpha_1\beta_1\gamma_1)$ ; and since  $m_1$  is doubly perspective with  $m_2$ ,  $m_2 = (A_2B_2C_2, \alpha_2\gamma_2\beta_2)$ . But by construction  $(A_2B_2C_2, \alpha_2\gamma_2\beta_2)$  is doubly perspective with  $(ACB, \alpha\gamma\beta)$ ; i.e.  $m_2$  is doubly perspective with the doubly oriented line obtained by changing both sense-classes of  $m_0$ .

**THEOREM 65.** *A doubly oriented line  $(ABC, \alpha\beta\gamma)$  is similarly oriented with  $(ACB, \alpha\gamma\beta)$  and oppositely oriented to  $(ABC, \alpha\gamma\beta)$  and  $(ACB, \alpha\beta\gamma)$ .*

*Proof.* Let  $l_0, l_1, l_2$  be three lines distinct from  $AB$  and such that  $l_0$  is on  $A$  and  $\alpha$ ,  $l_1$  on  $B$  and  $\gamma$ ,  $l_2$  on  $C$  and  $\beta$ . Let  $m_1$  and  $m_2$  be two lines distinct from  $AB$ , each of which meets  $l_0, l_1$ , and  $l_2$ . Let  $A_1, B_1, C_1$  be the points  $l_0m_1, l_1m_1, l_2m_1$  respectively and  $\alpha_1, \beta_1, \gamma_1$  the planes  $l_0m_1, l_1m_1, l_2m_1$  respectively; let  $A_2, B_2, C_2$  be the points  $l_0m_2, l_1m_2, l_2m_2$  respectively and  $\alpha_2, \beta_2, \gamma_2$  the planes  $l_0m_2, l_1m_2, l_2m_2$  respectively. Then by construction (fig. 88) and definition the oriented line  $(ABC, \alpha\beta\gamma)$  is doubly perspective with  $(A_1C_1B_1, \alpha_1\beta_1\gamma_1)$ , and this with  $(A_2B_2C_2, \alpha_2\gamma_2\beta_2)$ , and this with  $(ACB, \alpha\gamma\beta)$ . Hence  $(ABC, \alpha\beta\gamma)$  is similarly oriented with  $(ACB, \alpha\gamma\beta)$ . By a change of notation it is evident that  $(ABC, \alpha\gamma\beta)$  is similarly oriented with  $(ACB, \alpha\beta\gamma)$ . It remains, therefore, to prove that  $(ABC, \alpha\beta\gamma)$  is not similarly oriented with  $(ACB, \alpha\beta\gamma)$ .

If these two oriented lines were similarly oriented, there would be a sequence of doubly oriented lines  $m_0, m_1, m_2, \dots, m_n$  such that

$m_0 = (ABC, \alpha\beta\gamma)$  and  $m_n = (ACB, \alpha\beta\gamma)$ , and such that each oriented line of the sequence would be doubly perspective with the next one in the sequence. Let  $m$  be a doubly oriented line not coplanar with any of  $m_0, m_1, \dots, m_n$ , and doubly perspective with  $m_0$ ; let  $\bar{m}$  be the doubly oriented line obtained by changing both sense-classes on  $m$ . By Theorem 64  $m_1$  is doubly perspective with  $m$  or  $\bar{m}$ . By a second application of this theorem  $m_2$  is doubly perspective with  $m$  or  $\bar{m}$ , and by repeating this process  $n$  times we find that  $m_n$  is doubly perspective with  $m$  or  $\bar{m}$ . But this means that  $m_n$  is  $(ABC, \alpha\beta\gamma)$  or  $(ACB, \alpha\gamma\beta)$ .

**THEOREM 66.** *There are two and only two classes of doubly oriented lines such that any two doubly oriented lines of the same class are similarly oriented and any two of different classes are oppositely oriented.*

*Proof.* Let  $(ABC, \alpha\beta\gamma)$  be an arbitrary fixed doubly oriented line and let  $K$  be the class of doubly oriented lines similarly oriented to it. This class contains (Theorem 65)  $(ACB, \alpha\gamma\beta)$  but not  $(ACB, \alpha\beta\gamma)$  or  $(ABC, \alpha\gamma\beta)$ . If  $l$  is any line and  $m$  any line not meeting  $l$  or  $AB$ ,  $(ABC, \alpha\beta\gamma)$  is doubly perspective with one of the doubly oriented lines determined by  $m$  and this with one of those determined by  $l$ . Hence  $K$  contains two of the four doubly oriented lines determined by any line of space. Let  $K'$  be the class of doubly oriented lines similarly oriented with  $(ACB, \alpha\beta\gamma)$ . It also contains two of the four doubly oriented lines determined by any line of space.  $K$  and  $K'$  cannot have a doubly oriented line in common, because this would imply that  $(ABC, \alpha\beta\gamma)$  and  $(ACB, \alpha\beta\gamma)$  were similarly oriented. Hence every doubly oriented line is either in  $K$  or in  $K'$ .

There can be no other pair of classes of similarly oriented doubly oriented lines including all doubly oriented lines of space, because one class of such a pair would contain elements both of  $K$  and of  $K'$ , and this would imply, by Theorem 63, that  $(ABC, \alpha\beta\gamma)$  was similarly oriented with  $(ACB, \alpha\beta\gamma)$ .

From the construction which determines whether two doubly oriented lines are similarly oriented or not, it is evident that any collineation carries any two doubly oriented lines which are similarly oriented into two which are similarly oriented. Hence, if a collineation carries one doubly oriented line into a similarly oriented one, it carries every doubly oriented line into a similarly oriented one; and

if it carries one into an oppositely oriented one, it carries every doubly oriented line into an oppositely oriented one.

Any collineation which carries a doubly oriented line into a similarly oriented one is said to be *direct*, and any collineation which carries a doubly oriented line into an oppositely oriented one is said to be *opposite*. This definition of direct and opposite collineations is easily seen to be equivalent to that in § 32.

**\*181. More general theory of sense.** The theory of sense-classes in the preceding pages can be extended to analytic transformations by means of simple limiting considerations. For example, consider a transformation of a part of a Euclidean plane

$$\begin{aligned}x' &= f(xy) = a_{00} + a_{10}x + a_{01}y + a_{20}x^2 + \dots, \\y' &= g(xy) = b_{00} + b_{10}x + b_{01}y + b_{20}x^2 + \dots,\end{aligned}$$

where both series are convergent for all points in a region including the point  $(0, 0)$ . If the determinant

$$\begin{vmatrix} a_{10} & a_{01} \\ b_{10} & b_{01} \end{vmatrix} = \begin{vmatrix} \frac{\partial f(0, 0)}{\partial x} & \frac{\partial f(0, 0)}{\partial y} \\ \frac{\partial g(0, 0)}{\partial x} & \frac{\partial g(0, 0)}{\partial y} \end{vmatrix} = J$$

is not zero, it can be shown that there is a region including  $(0, 0)$  which is transformed into a region including  $(a_{00}, b_{00})$  in such a way that all ordered point triads of a sense-class in the first region go into ordered point triads of one sense-class in the second region; and if  $(x', y')$  is in the same plane as  $(x, y)$ , the two sense-classes will be the same if and only if  $J > 0$ .

By a similar limiting process the notions of right- and left-handedness can be extended to curves, ruled surfaces, and other figures having analytic equations. A discussion of some of the cases which arise will be found in the article by Study referred to in § 162.

This sort of theory of sense relations belongs essentially to differential geometry, although the domain to which it applies may be extended by methods of the type used in §§ 168 and 173.

The theory of sense may, however, be extended in a different way so as to apply to the geometry of all continuous transformations instead of merely to the projective geometry or to the geometry of analytic transformations. The main theorems are as follows:

Any one-to-one reciprocal continuous transformation of a curve into itself transforms each sense-class on the curve either into itself or into the other sense-class. A transformation of the first kind is called *direct* and one of the second kind *opposite*. A direct transformation is a *deformation* (§ 157), and an opposite transformation is not a deformation.

Any simple closed curve consisting of points in or on the boundary of a 2-cell  $R$  (§ 155) is the boundary of a unique 2-cell which consists entirely of points of  $R$ .

A 2-cell can be deformed into itself in such a way as to transform an arbitrary simple closed curve of the cell into an arbitrary simple closed curve of the cell. Any one-to-one reciprocal continuous transformation of a 2-cell and its boundary into themselves is a deformation if and only if it effects a deformation on the boundary; i.e. if and only if it transforms a sense-class on the boundary into itself.

If the sense-classes on one curve  $j_1$  of a 2-cell and its boundary are designated as *positive* and *negative* respectively, any sense-class on any other curve  $j_1$  is called *positive* or *negative* according as it is the transform of the positive or of the negative sense-class on  $j_1$  by a deformation of  $j_1$  into  $j$  through intermediate positions which are all simple closed curves on the 2-cell and its boundary. By the theorems above, this gives a unique determination of the positive and negative sense-classes on any curve of the given convex region. A curve associated with its positive sense-class is called a *positively oriented curve*, and a curve associated with its negative sense-class is called a *negatively oriented curve*.

Any transformation of a 2-cell which is one-to-one, reciprocal, and continuous either transforms all positively oriented curves into positively oriented curves or transforms all positively oriented curves into negatively oriented curves. In the first case the transformation is said to be *direct* and in the second case to be *opposite*. The transformation is a deformation if and only if it is direct. A 2-cell associated with its positively oriented curves or with its negatively oriented curves is called an *oriented 2-cell*.

The oriented 2-cells of a simple surface fall into two classes such that any oriented 2-cell of one class can be carried by a continuous deformation of the surface into any other oriented 2-cell of the

same class, but not into any oriented 2-cell of the other class. The two oriented 2-cells determined by a given 2-cell are in different classes. A simple surface associated with one of these classes of oriented 2-cells is said to be *oriented*.

A similar theorem does not hold for the oriented 2-cells of a projective plane. Instead we have the theorem that every continuous one-to-one reciprocal transformation of a projective plane is a deformation. Consequently any oriented 2-cell can be carried into any other oriented 2-cell by a deformation.

The oriented simple surfaces in a 3-cell and its boundary fall into two classes such that any member of either class can be deformed into any other member of the same class through a set of intermediate positions which are all oriented simple surfaces, but cannot be deformed in this way into any member of the other class. A continuous one-to-one reciprocal correspondence which carries a 3-cell and its boundary into themselves either interchanges the two classes of oriented simple surfaces or leaves them invariant. In the second case the transformation is a deformation and in the first case it is not. A 3-cell associated with one of its classes of oriented surfaces is called an *oriented 3-cell*.

The oriented 3-cells of a projective space fall into two classes such that any member of one class can be carried by a continuous deformation of the projective space into any member of the same class but not into any member of the other class. A continuous one-to-one reciprocal transformation of the projective space either transforms each class of oriented 3-cells into itself or into the other class. In the first case it is a deformation and in the second it is not. A projective space associated with one of its classes of oriented 3-cells is called an *oriented projective space*.

The one-dimensional theorems outlined above are easily proved on the basis of the discussion of the sense-classes on a line in §§ 159 and 165. The two-dimensional ones, though more difficult, are consequences of known theorems of analysis situs. They involve, however, such theorems as that of Jordan, that a simple closed curve separates a convex region into two regions; and the theorems of this class do not belong (§§ 34, 39, 110) to projective geometry. The Jordan theorem in the special case of a simple closed polygon does, however, belong to projective geometry and is proved below (§ 187).

The three-dimensional propositions outlined here have not all been proved as yet, but are (in form) direct generalizations of the one- and two-dimensional ones.

Let us note that an ordered triad of points as treated in § 160 may be regarded as determining an oriented 2-cell. For the triangular region having the points as vertices is a 2-cell, and a sense-class is determined on its boundary by the order of the vertices. This sense-class determines a sense-class on every curve of the 2-cell and thus determines an oriented 2-cell.

In like manner an ordered tetrad of points as treated in § 160 determines an oriented 3-cell. For the tetrahedral region having points  $ABCD$  as vertices is a 3-cell. The triangular region  $BCD$  is a 2-cell which does not contain  $A$ , and is oriented in view of the order of the points on its boundary. This oriented 2-cell determines an orientation of the boundary of the 3-cell, and thus of the 3-cell.

Likewise an ordered tetrad  $ABCD$  of a projective plane determines a 2-cell, i.e. that one of the triangular regions  $BCD$  which contains  $A$ ; and this 2-cell is oriented by the order of the points  $BCD$ . Similarly, an ordered pentad  $ABCDE$  of points in a projective space determines an oriented 3-cell, i.e. that one of the tetrahedral regions  $BCDE$  which contains  $A$ , oriented according to the order of the points  $BCDE$ .

**182. Broken lines and polygons.** DEFINITION. A set of  $n$  points  $A_1, A_2, \dots, A_n$ , together with a set of  $n$  segments joining  $A_1$  to  $A_2$ ,  $A_2$  to  $A_3, \dots, A_{n-1}$  to  $A_n$ , is called a *broken line* joining  $A_1$  to  $A_n$ . The points  $A_1, \dots, A_n$  are called the *vertices* and the segments joining them the *edges* of the broken line. If the vertices are all distinct and no edge contains a vertex or a point on another edge, the broken line is said to be *simple*. If  $A_1 = A_n$  the broken line is said to be *closed*, otherwise it is said to be *open*. The set of all points on a closed broken line is called a *polygon*. If the vertices of a polygon are all distinct and no edge contains a vertex or a point on another edge, the polygon is said to be *simple*.

A broken line whose vertices are  $A_1, A_2, \dots, A_n$ , and whose edges are the segments joining  $A_1$  to  $A_2$ ,  $A_2$  to  $A_3, \dots, A_{n-1}$  to  $A_n$ , is called the broken line  $A_1A_2 \dots A_n$ , and its edges are denoted by  $A_1A_2, A_2A_3, \dots, A_{n-1}A_n$  respectively. If  $A_1 = A_n$  the corresponding polygon is denoted by  $A_1A_2 \dots A_{n-1}A_1$ ; the vertex  $A_1$  is sometimes denoted by  $A_n, A_2$  by  $A_{n+1}$ , etc.

The following theorem is an obvious consequence of the definition.

**THEOREM 67.** *The polygon  $A_1A_2 \dots A_nA_1$  is the same as  $A_2A_3 \dots A_nA_1A_2$  and  $A_1A_n \dots A_2A_1$ . If  $P$  is any point of the edge  $A_1A_2$  of*

a simple polygon  $A_1A_2 \dots A_nA_1$ , this polygon is the same as a polygon  $A_1PA_2 \dots A_nA_1$  in which the edge  $A_1P$ , the vertex  $P$ , and the edge  $PA_2$  constitute the same set of points as the edge  $A_1A_2$ . If a simple polygon  $A_1A_2A_3 \dots A_nA_1$  is such that  $A_1A_2A_3$  are collinear and  $A_3 \neq A_1$ , this polygon is the same as the polygon  $A_1A_3 \dots A_nA_1$  in which all the edges but  $A_1A_3$  are the same as before and  $A_1A_3$  is the segment  $A_1A_2A_3$ .

DEFINITION. If  $A, B, C$  are any three points on a simple polygon, an *elementary transformation* is the operation of replacing any one of these points, say  $C$ , by a point  $C'$  such that  $C$  and  $C'$  are joined by a segment consisting of points of the polygon and not containing either of the other two points. A class consisting of all ordered triads each of which is transformable by a finite number of elementary transformations into a fixed triad  $ABC$  is called the *sense-class*  $ABC$  and is denoted by  $S(ABC)$ .

THEOREM. 68. *There exists a one-to-one and reciprocal correspondence between the points of any simple polygon and the points of any line such that two triads of points on the polygon are in the same sense-class with respect to the polygon if and only if the corresponding triads of points on the line are in the same sense-class.*

*Proof.* Let the vertices of the polygon be denoted by  $A_1, A_2, \dots, A_{n-1}$  and let  $A_1$  be also denoted by  $A_n$ . Let  $B_1, B_2, \dots, B_{n-1}$  be  $n$  arbitrary points of a line  $l$  in the order  $\{B_1, B_2, \dots, B_{n-1}\}$  and let  $B_n$  also denote  $B_1$ . Let  $\beta_i$  denote that segment  $B_iB_{i+1}$ , which contains none of the other points  $B$ . Let the edge joining  $A_i$  to  $A_{i+1}$  correspond projectively to the segment  $\beta_i$  in such a way that  $A_i$  and  $A_{i+1}$  are homologous with  $B_i$  and  $B_{i+1}$  respectively. (In general the projectivities by which two sides of the polygon correspond to two segments on the line will be different.) If we also let  $A_i$  correspond to  $B_i$  ( $i = 1, \dots, n-1$ ), there is evidently determined a one-to-one and reciprocal correspondence  $\Gamma$  between the polygon and the line which is such that each side of the polygon with its two ends corresponds with preservation of order relations to a segment of the line and its two ends.

Let  $P_1, P_2, P_3, P_4$  denote points of the polygon and  $L_1, L_2, L_3, L_4$  the points of  $l$  to which they respectively correspond under  $\Gamma$ . The correspondence  $\Gamma$  is so defined that if  $P_1P_2P_3$  goes into  $P_1P_2P_4$  by an

elementary transformation with respect to the polygon, then  $L_1L_2L_3$  goes into  $L_1L_2L_4$  by an elementary transformation restricted with respect to  $B_1, B_2, \dots, B_n$  (cf. § 165), and conversely. Hence the theorem follows at once from the corollary of Theorem 22.

**COROLLARY 1.** *The theorem above remains true if the words "broken line with distinct ends" be substituted for polygon, and "interval" for line.*

The definitions of separation and order given in § 21 for the points on a line may now be applied word for word to the points on a simple polygon, and in view of the correspondence established in Theorem 68, the theorems about order relations on a line may be applied without change to polygons.

By comparison with the proof of Theorem 15 we obtain immediately

**COROLLARY 2.** *A simple polygon is a simple closed curve.*

**COROLLARY 3.** *A simple broken line joining two distinct points  $A_1, A_n$  is a simple curve joining  $A_1$  and  $A_n$ .*

The order relations on a broken line which is not simple may be studied by the method given above with the aid of a simple device. Suppose we associate an integer with each point of a broken line  $A_1A_2 \dots A_n$  as follows: With  $A_1$  and every point of the segment joining  $A_1$  to  $A_2$  the number 1; with  $A_2$  and every point of the segment joining  $A_2$  to  $A_3$  the number 2; and so on, and, finally, with  $A_n$  the number  $n$ .

**DEFINITION.** The object formed by a point of the broken line and the number associated with it by the above process shall be called a *numbered point*; and the numbered point is said to be on any segment, line, plane, etc. which the point is on. If  $A, B, C$  are any three numbered points on a polygon, an *elementary transformation* is the operation of replacing any one of these numbered points, say  $C$ , by a point  $C'$  such that  $C$  and  $C'$  are joined by a segment of numbered points all having the same number. A class consisting of all ordered triads of numbered points each of which is transformable by a finite sequence of elementary transformations into a fixed triad  $ABC$  is called the *sense-class*  $ABC$  and is denoted by  $S(ABC)$ .

By the proof given for Theorem 68 we now have

**THEOREM 69.** *There exists a one-to-one and reciprocal correspondence between the numbered points of any broken line and the points*

*of any interval such that two triads of numbered points are in the same sense-class if and only if the corresponding triads of points on the interval are in the same sense-class.*

We are therefore justified in applying the theorems and definitions about order relations on an interval to the numbered points of a broken line.

#### EXERCISE

\*Any two points of a region can be joined by a broken line consisting entirely of points of the region.

**183. A theorem on simple polygons.** In the last section a polygon was defined as the set of points contained in a sequence of points and linear segments. This is the most usual definition and doubtless the most natural. With a view to generalizing so as to obtain the theory of polyhedra in spaces of three and more dimensions, however, we shall find it more convenient to use the property of a simple polygon stated in the following theorem.\*

**THEOREM 70.** *A set of points  $[P]$  is a simple polygon if and only if the following conditions are satisfied: (1)  $[P]$  consists of a set of distinct points, called vertices, and of distinct segments, called edges, such that the ends of each edge are vertices and each vertex is an end of an even number of edges; (2) if any points of  $[P]$  are omitted, the remaining subset of  $[P]$  does not have the property (1).*

*Proof.* It is obvious that a simple polygon, as defined in § 182, satisfies Conditions (1) and (2), because no edge has a point in common with any other edge or vertex and each vertex is an end of exactly two edges.

Let us now consider a set of points  $[P]$  satisfying (1) and (2). If two or more edges have a point in common, this point divides each edge into two segments. Hence the point may be regarded as a vertex at which an even number of edges meet. In like manner, if an edge contains a vertex the two segments into which the edge is divided by the vertex may be regarded as edges. Since there are originally given only a finite number of vertices and edges, this process determines a finite number of vertices and edges such that no edge contains a vertex or any point of another edge.

\*This form of the definition of a polygon and a corresponding definition of a polyhedron are due to N. J. Lennes, *American Journal of Mathematics*, Vol. XXXIII (1911), p. 37.

Now let  $e_1$  be any edge and  $P_1$  one of its ends. Since there are an even number of segments having  $P_1$  as an end, there exists another distinct from  $e_1$ ; let this be denoted by  $e_2$ . Let  $P_2$  be its other end, and let  $e_3$  be a second segment having  $P_2$  as an end, and so on. By this process we obtain a sequence of points and segments

$$e_1, P_1, e_2, P_2, e_3, \dots$$

Since the number of vertices is finite, this process must lead by a finite number of steps to a point  $P_n$  which coincides with one of the previous points, say  $P_i$ . The set of points included in the points and segments

$$P_i, e_{i+1}, P_{i+1}, \dots, P_{n-1}, e_n$$

satisfies the definition of a simple polygon and has the property that each  $P_j$  ( $j = i, i+1, \dots, P_{n-1}$ ) is an end of two and only two  $e$ 's. Hence it satisfies Condition (1). By Condition (2) it must include all points of the set  $[P]$ .

**COROLLARY.** *A set of points satisfying Condition (1) of Theorem 70 consists of a finite number of simple polygons no two of which have any point in common which is not a vertex.*

*Proof.* In the proof of the second part of the theorem above, Condition (2) is not used before the last sentence. If Condition (2) be not satisfied, the set of points remaining when the segments  $e_{i+1}, \dots, e_n$  (and those of the points  $P_i, \dots, P_{n-1}$  which are not ends of the remaining segments) are removed continues to satisfy Condition (1). For on removing two segments from an even number, an even number remains. Hence the process by which the simple polygon  $P_i, e_{i+1}, \dots, P_{n-1}, e_n$  was obtained may be repeated and another simple polygon removed. Since the total number of edges is finite, this step can be repeated only a finite number of times.

**184. Polygons in a plane.** In the next three sections we shall prove that the polygons in a projective plane are of two kinds, a polygon of the first kind being such that all points not on it constitute two regions, and a polygon of the second kind being such that all points not on it constitute a single region. The boundary of a triangular region is a polygon of the first kind, and a projective line a polygon of the second kind. In proving that the points not on a polygon constitute one or two regions, we shall need the following:

**THEOREM 71.** *Any point coplanar with but not on a polygon  $p$  in a plane  $\alpha$  is in a triangular region of  $\alpha$  containing no point of  $p$ .*

*Proof.* Let the polygon be denoted by  $A_1A_2 \dots A_nA_1$  and the point by  $P$ . By an obvious construction (the details of which are left to the reader; cf. § 149) a triangular region  $T_1$  may be found containing  $P$  and not containing  $A_1$  or  $A_2$  or any point of the edge  $A_1A_2$ . In like manner a triangular region  $T_2$  may be constructed which contains  $P$ , is contained in  $T_1$ , and does not contain  $A_2$  or any point of the edge  $A_2A_3$ . By repeating this construction we obtain a sequence of triangular regions  $T_1, T_2, \dots, T_n$ , each contained in all the preceding ones, containing  $P$ , and such that  $T_k$  does not contain any point of the broken line  $A_1A_2 \dots A_{k+1}$ . Thus  $T_n$  contains  $P$  and contains no point of the polygon  $A_1A_2 \dots A_nA_1$ .

**COROLLARY.** *Any point of space not on a polygon  $p$  is in a tetrahedral region containing no point of  $p$ .*

Let the set of lines containing the edges of a simple polygon in a plane be denoted by  $l_1, l_2, \dots, l_n$ . Since more than one edge may be on the same line,  $n$  is less than or equal to the number of edges. According to Theorem 67 we can first suppose that the notation is so assigned that no two edges having a common end are collinear except in the case of a polygon of two sides (which is a projective line), for two collinear edges and their common end can be regarded as a single edge. In the second place, according to the same theorem, we can introduce as a vertex any point in which an edge is met by one of the lines  $l_1, l_2, \dots, l_n$  which does not contain it.

Under these conventions the polygon may be denoted by  $A_1A_2 \dots A_mA_1$ , where each point  $A_i (i = 1, 2, \dots, m)$  is a point of intersection of two of the lines  $l_1, l_2, \dots, l_n$ , and each edge is a segment joining two vertices and containing points of only one of the lines  $l_1, l_2, \dots, l_n$ .

In like manner, when two or more simple polygons are under consideration, let us denote the set of lines containing all their edges by  $l_1, l_2, \dots, l_n$ . We may first arrange that no two edges of the same polygon which have an end in common are collinear, and then introduce new vertices at every point in which an edge is met by one of the lines  $l_1, l_2, \dots, l_n$  which is not on it. Thus in this case also the polygons may be taken to have all their vertices at points

of intersection of the  $n$  lines  $l_1, l_2, \dots, l_n$  and to have no edge which contains such a point of intersection.

We are thus led to study the points of intersection of a set of  $n$  coplanar lines and the segments of these lines which join the points of intersection.

**185. Subdivision of a plane by lines.** Consider a set of  $n$  lines  $l_1, l_2, \dots, l_n$  all in the same plane  $\pi$ . The number  $\alpha_0$  of their points of intersection is subject to the condition

$$1 \leq \alpha_0 \leq \frac{n(n-1)}{2},$$

the two extreme cases being the case where all  $n$  lines are concurrent and the case where no three are concurrent. According to § 22, Chap. II, and the definition of boundary (§ 150), the points of intersection bound a number  $\alpha_1$  of linear convex regions upon the lines. The number  $\alpha_1$  is subject to the condition

$$n \leq \alpha_1 \leq n(n-1),$$

the two extreme cases being the same as before.

**THEOREM 72.** *The points of a plane which are not on any one of a finite set of lines  $l_1, l_2, \dots, l_n$  fall into a number  $\alpha_2$  of convex regions such that any segment joining two points of different regions contains at least one point of  $l_1, l_2, \dots, l_n$ . The number  $\alpha_2$  satisfies the inequality*

$$n \leq \alpha_2 \leq \frac{n(n-1)}{2} + 1.$$

*Proof.* The proof may be made by induction. If  $n=1$  the theorem follows directly from the definition of a convex region. We suppose that it is true for  $n=k$ , and prove it for  $n=k+1$ .

We are given  $k+1$  lines  $l_1, l_2, \dots, l_{k+1}$ . The lines  $l_1, l_2, \dots, l_k$  determine a number  $N_k$  not less than  $k$  and not more than  $\frac{k(k-1)}{2} + 1$ , of convex regions. The line  $l_{k+1}$  meets the remaining  $k$  lines in at least one point and not more than  $k$  points. The remaining points of  $l_{k+1}$  therefore form at least one and at most  $k$  linear convex regions, each of which is the set of all points common to  $l_{k+1}$ , and one of the planar convex regions (Theorem 3). By Theorem 8 each convex region which contains points of  $l_{k+1}$  is divided into two convex regions such that any segment joining two points of different

regions meets  $l_{k+1}$  if it does not meet one of the lines  $l_1, l_2, \dots, l_k$ . Hence the  $k+1$  lines determine a number  $N_{k+1}$  of convex regions of the required kind such that  $N_k + 1 \leq N_{k+1} \leq N_k + k$ . Since

$$k \leq N_k \leq \frac{k(k-1)}{2} - 1,$$

it follows that

$$k+1 \leq N_{k+1} \leq \frac{k(k-1)}{2} - 1 + k = \frac{(k+1)k}{2} - 1.$$

COROLLARY 1. *If  $n$  lines of a plane pass through a point, they determine  $n$  convex regions in the plane; if no three of them are concurrent, they determine  $\frac{n(n-1)}{2} + 1$  convex regions.*

Let us denote the  $\alpha_0$  points of intersection of the lines  $l_1, l_2, \dots, l_n$  by

$$\alpha_1^0, \alpha_2^0, \dots, \alpha_{\alpha_0}^0,$$

or any one of them by  $\alpha^0$ ; the  $\alpha_1$  linear convex regions which these points determine upon the lines by

$$\alpha_1^1, \alpha_2^1, \dots, \alpha_{\alpha_1}^1,$$

or any one of them by  $\alpha^1$ ; and the  $\alpha_2$  planar convex regions by

$$\alpha_1^2, \alpha_2^2, \dots, \alpha_{\alpha_2}^2,$$

or any one of them by  $\alpha^2$ .

COROLLARY 2. *If the lines  $l_1, l_2, \dots, l_n$  are not concurrent, any line coplanar with and containing a point of an  $\alpha^2$  has a segment of points in common with it. The ends of this segment are on the boundary of the  $\alpha^2$ , and no other point of the line is on this boundary.*

*Proof.* The given line, which we shall call  $l$ , meets the lines  $l_1, l_2, \dots, l_n$  in at least two points, and, as seen in the proof of the theorem, one and only one of the mutually exclusive segments having these points as ends is composed entirely of points of the  $\alpha^2$ . Let  $\sigma$  denote this segment. Its ends are boundary points of the  $\alpha^2$  by Theorem 10. Let  $l_i$  and  $l_j$  be lines of the set  $l_1, l_2, \dots, l_n$  such that  $l_i$  contains one end of  $\sigma$  and  $l_j$  the other. All points of the  $\alpha^2$  are separated from the points of the segment complementary to  $\sigma$  by the lines  $l_i$  and  $l_j$ . Hence any point of the complementary segment is in a triangular region containing no point of the  $\alpha^2$  and is therefore not a boundary point of the  $\alpha^2$ .

This argument carries with it the proof of

COROLLARY 3. *Any interval joining a point of an  $a^2$  to a point not in the  $a^2$  contains a point of the boundary of the  $a^2$ .*

THEOREM 73. *If the lines  $l_1, l_2, \dots, l_n$  are not all concurrent, the boundary of each  $a^2$  is a simple polygon whose vertices are  $a^n$ 's and whose edges are  $a^1$ 's.*

*Proof.* The theorem is a direct consequence of § 151 in case  $n=3$ . Let us prove the general theorem by induction; i.e. we assume it true for  $n=k$  and prove it for  $n=k+1$ .

Let the notation be so assigned that  $l_1, l_2, l_3$  are not concurrent. Then any one of the convex regions, say  $R$ , determined by  $l_1, l_2, \dots, l_k$  is contained in a triangular region determined by  $l_1, l_2$ , and  $l_3$ , because no two points of  $R$  are separated by any two of the lines  $l_1, l_2, l_3$ . Let  $m$  be a line containing no point of this triangular region nor any of its vertices. The segments  $A_i A_j$  etc. referred to below do not contain any point of  $m$ .

If  $l_{k+1}$  contains a point of  $R$ , it contains, by Cor. 2, above, a segment of points of  $R$  such that the ends of this segment are on the boundary of  $R$ . By Theorem 67, the ends of this segment may be taken as vertices of the polygon  $p$  which by hypothesis bounds  $R$ . Thus we may denote this polygon by  $A_1 A_2 \dots A_i \dots A_j A_1$ , where  $A_1$  and  $A_i$  are the points in which  $l_{k+1}$  meets the polygon.

There are just two simple polygons which are composed of the segment  $A_1 A_i$  and of sides and vertices of  $p$ . For any such polygon which contains  $A_1 A_i$  contains  $A_1 A_2$  or  $A_1 A_j$ ; if it contains  $A_1 A_2$  it must contain  $A_2 A_3$  and therefore  $A_3 A_4, \dots, A_{i-1} A_i$ , and since it contains  $A_i A_1$  it must be the polygon  $A_1 A_2 A_3 \dots A_i A_1$ ; if it contains  $A_1 A_j$  it must contain  $A_j A_{j-1}$  and therefore  $A_{j-1} A_{j-2}, \dots, A_{i+1} A_i$ , and since it contains  $A_i A_1$  it must be  $A_1 A_j A_{j-1} \dots A_i A_1$ .

Neither of the lines  $l_{k+1}$  and  $m$  meets any edge of the polygon  $A_1 A_2 A_3 \dots A_i A_1$  except  $A_i A_1$ , which is contained in  $l_{k+1}$ . Hence all points of this polygon except  $A_i, A_1$  and those on the edge  $A_i A_1$  are in one of the two regions, which we shall call  $R'$  and  $R''$ , bounded by  $l_{k+1}$  and  $m$ . In like manner all points of the polygon  $A_1 A_j A_{j-1} \dots A_i A_1$  except  $A_i, A_1$  and those on the edge  $A_i A_1$  are in one of the two regions  $R'$  and  $R''$ .

The points of  $R$  on any line coplanar with  $R$  and meeting the segment  $A_i A_1$  in one point form a segment  $\sigma$  (Cor. 2, above) which

does not contain any point of  $m$ . Hence the ends  $P, Q$  of  $\sigma$  are separated by  $l_{k+1}$  and  $m$ . But  $P$  and  $Q$  are boundary points of  $R$  by Cor. 2, above. Hence the boundary of  $R$  has points in both of the regions  $R'$  and  $R''$  bounded by  $l_{k+1}$  and  $m$ . By the paragraph above, the points of the boundary of  $R$  in the one region, say  $R'$ , must be the points, exclusive of the interval  $A_i A_1$ , of the polygon  $A_1 A_2 \cdots A_i A_1$ ; and those in the other,  $R''$ , must be the points, exclusive of the interval  $A_i A_1$ , of the polygon  $A_1 A_j \cdots A_i A_1$ .

Let  $R_1$  and  $R_2$  be the two convex regions formed by the points of  $R$  not on  $l_{k+1}$ . Since these two regions are separated by  $l_{k+1}$  and  $m$ , we may assume that  $R_1$  is in  $R'$  and  $R_2$  in  $R''$ . Every boundary point of  $R_1$  which is not a point of  $l_{k+1}$  is in  $R'$ . For if  $B$  is a point of the boundary of  $R_1$  it is not on  $m$ , by construction, and if it is not on  $l_{k+1}$  it can be enclosed in a triangular region containing no point of  $l_{k+1}$  or  $m$ . Such a triangular region must contain points of  $R_1$  and hence can contain no point of  $R''$ , since any segment joining a point of  $R'$  to a point of  $R''$  contains a point of  $l_{k+1}$  or of  $m$ . Hence  $B$  is in  $R'$ . In like manner any boundary point of  $R_2$  not on  $l_{k+1}$  is in  $R''$ . But by Theorem 10 every point  $B$  of the boundary of  $R$  is on the boundary of  $R_1$  or  $R_2$ . Hence the boundary of  $R_1$  contains all points of the boundary of  $R$  in  $R'$ ; and by Theorem 10 it contains no other points not on  $l_{k+1}$ . Hence it is the polygon  $A_1 A_2 \cdots A_i A_1$ . In like manner the polygon  $A_1 A_j \cdots A_i A_1$  is the boundary of  $R_2$ .

Hence the boundaries of the two planar convex regions into which any one of the planar convex regions determined by  $l_1, l_2, \dots, l_k$  is separated by  $l_{k+1}$  are simple polygons. The other planar convex regions determined by  $l_1, l_2, \dots, l_{k+1}$  are identical with regions determined by  $l_1, l_2, \dots, l_k$ .

COROLLARY 1. *Each  $a^1$  is on the boundaries of two and only two  $a^2$ s.*

COROLLARY 2. *In case all the lines  $l_1, l_2, \dots, l_n$  are concurrent, there is only one  $a^0$ , the common point of the lines; there are  $n$   $a^1$ s, each consisting of all points except  $a^0$  of one of the lines  $l_i$ ; and there are  $n$   $a^2$ s, each having a pair of the lines as its boundary.*

THEOREM 74. *The numbers  $\alpha_0, \alpha_1, \alpha_2$  satisfy the relation*

$$\alpha_0 - \alpha_1 + \alpha_2 = 1.$$

*Proof.* We shall make the proof by mathematical induction. The theorem is obvious if  $n = 2$ , for in this case  $\alpha_0 = 1, \alpha_1 = 2, \alpha_2 = 2$ .

Let us now assume it to be true for  $n=k$  and prove that it follows for  $n=k+1$ .

The lines  $l_1, l_2, \dots, l_k$  determine a set of  $\alpha'_0$  points,  $\alpha'_1$  linear convex regions, and  $\alpha'_2$  planar convex regions subject to the relation  $\alpha'_0 - \alpha'_1 + \alpha'_2 = 1$ . The line  $l_{k+1}$  meets a number, say  $r$ , of the planar convex regions and separates each of these into two planar convex regions. Hence  $\alpha'_2$  is increased to  $\alpha'_2 + r$ . The number of one-dimensional convex regions is increased by  $r$  for the number of convex regions on  $l_{k+1}$  and also by a number\*  $s$  equal to the number of linear convex regions of the lines  $l_1, l_2, \dots, l_k$  which are met by  $l_{k+1}$ . The number of points of intersection of  $l_1, l_2, \dots, l_{k+1}$  also exceeds  $\alpha'_0$  by  $s$ . Hence for  $l_1, l_2, \dots, l_{k+1}$  the numbers  $\alpha_0, \alpha_1, \alpha_2$  are  $\alpha'_0 + s, \alpha'_1 + r + s, \alpha'_2 + r$ . Hence  $\alpha_0 - \alpha_1 + \alpha_2 = (\alpha'_0 + s) - (\alpha'_1 + r + s) + (\alpha'_2 + r) = 1$ .

**186. The modular equations and matrices.** The relations among the points, linear convex regions, and planar convex regions may be described by means of two matrices of which those given in § 151 for the triangle are special cases. The first matrix, which we shall denote by  $H_1$ , is an array of  $\alpha_0$  rows and  $\alpha_1$  columns, each row being associated with an  $a^0$  and each column with an  $a^1$ . The element of the  $i$ th row and  $j$ th column is 1 or 0 according as  $a_i^0$  is or is not an end of  $a_j^1$ . The second matrix,  $H_2$ , has  $\alpha_1$  rows and  $\alpha_2$  columns associated respectively with the  $a^1$ 's and  $a^2$ 's. The element of the  $i$ th row and  $j$ th column is 1 or 0 according as  $a_i^1$  is or is not on the boundary of  $a_j^2$ .

Since every segment  $a^1$  has two and only two ends, each column of  $H_1$  contains just two 1's; and since each  $a^1$  is on the boundary of two and only two  $a^2$ 's (Theorem 73, Cor. 1), each row of  $H_2$  contains just two 1's.

For each of the  $a^1$ 's let us introduce a variable which can take on only the values 0 and 1, these being regarded as marks of the field obtained by reducing modulo 2. We denote these variables by  $x_1, x_2, \dots, x_{\alpha_1}$  respectively. There are  $2^{\alpha_1}$  sets of values which can be given to the symbol†  $(x_1, x_2, \dots, x_{\alpha_1})$ .

\*The number  $s$  is less than  $r$  if  $l_{k+1}$  contains points of intersection of  $l_1, l_2, \dots, l_k$ .

†Excluding the one in which all the variables are zero, these symbols constitute the points of a finite projective space of  $\alpha_1 - 1$  dimensions in which there are three points on every line (cf. § 72, Vol. I).

Every one of these symbols  $(x_1, x_2, \dots, x_{a_1})$  corresponds to a way of labeling each segment  $a^1$  of the original  $n$  lines with a 0 or a 1, the segment  $a_i^1$  being labeled with the value of  $x_i$ . We shall regard the symbol as the notation for the set of edges labeled with 1's. By the *sum* of two symbols  $(x_1, x_2, \dots, x_{a_1})$  and  $(y_1, y_2, \dots, y_{a_1})$  we shall mean  $(x_1 + y_1, x_2 + y_2, \dots, x_{a_1} + y_{a_1})$ , the addition being performed modulo 2. According to our convention the sum represents the set of  $a^1$ 's which are in either of the sets represented by  $(x_1, x_2, \dots, x_{a_1})$  and  $(y_1, y_2, \dots, y_{a_1})$  but not in both. By a repetition of these considerations it follows that the sum of  $n$  symbols of the form  $(x_1, x_2, \dots, x_{a_1})$  for sets of edges is the symbol for a set of edges each of which is in an odd number of the  $n$  sets of edges.

In the sequel we shall say that a polygon  $p$  is the sum, modulo 2, of a set of polygons  $p_1, p_2, \dots, p_n$  if it is represented by a symbol  $(x_1, x_2, \dots, x_{a_1})$  which is the sum of the symbols for  $p_1, p_2, \dots, p_n$ . Let us now inquire what is the condition on a symbol  $(x_1, x_2, \dots, x_{a_1})$  that it shall represent a polygon?

At every vertex of a polygon there meet two and only two edges. Hence, if we add all the  $x$ 's that correspond to the  $a$ 's meeting in any point, this sum must be zero, modulo 2. This gives  $a_0$  equations, one for each  $a^0$ , of the form

$$(4) \quad x_p + x_q + \dots + x_k = 0 \quad (\text{mod. } 2)$$

( $a_p^0, a_q^0, \dots, a_k^0$  being the edges which meet at a given vertex), which must be satisfied by the symbol for any polygon. Obviously the matrix of the coefficients of these equations is  $H_1$ . For example, in the case of the triangle these equations are (cf. § 151)

$$(5) \quad \begin{aligned} x_5 + x_4 + x_6 + x_8 &= 0, \\ x_1 + x_2 + x_6 + x_8 &= 0, \\ x_1 + x_2 + x_5 + x_4 &= 0. \end{aligned} \quad (\text{mod. } 2)$$

We shall denote the set of equations (4) by  $(H_1)$ . Since each column of  $H_2$  gives the notation for a polygon bounding an  $a^2$ , the columns of  $H_2$  are solutions of the equations  $(H_1)$ . For example, the columns of the matrix  $H_2$  in § 151 are solutions of  $(H_1)$ .

Any solution whatever of these equations corresponds to a labeling of the  $a$ 's with 0's and 1's in such a way that there are an even number of 1's on the  $a$ 's meeting at each  $a^0$ . Hence, by the corollary of Theorem 70 the  $a$ 's labeled with 1's must constitute

one or more simple polygons. *Hence every solution of the equations  $(H_1)$  represents a simple polygon or a set of simple polygons.*

Since each column of the matrix  $H_1$  contains exactly two 1's, any one of the equations is obtained by adding all the rest. Since the only marks of our field are 0 and 1, any linear combination of the equations  $(H_1)$  would be merely the sum of a subset of these equations. Consider such a subset and the points  $\alpha^0$  which correspond to the equations in the subset. Every  $\alpha^1$  joining two points of the subset is represented in two equations, and the corresponding variable disappears in the sum. There remain in this sum the variables corresponding to the  $\alpha$ 's joining the points of the subset to the remaining points of the figure. These cannot all pass through the same point unless the subset consists of all points but one (since any two of the original  $n$  lines have a point in common). Hence while any one of the equations is linearly dependent\* on all the rest, it is not linearly dependent on any smaller subset. *Hence  $\alpha_0 - 1$  of the equations  $(H_1)$  are linearly independent.*

Since the number of variables is  $\alpha_1$ , the number of solutions in a set of linearly independent solutions on which all other solutions are linearly dependent is  $\alpha_1 - \alpha_0 + 1$ . By Theorem 74 this number is  $\alpha_2$ .† Thus the total number of polygons and sets of polygons is  $2^{\alpha_2} - 1$ .

The simple polygons which bound the regions  $\alpha^2$  are a set of solutions, namely, the columns of the matrix  $H_2$ . Since each row of the matrix  $H_2$  contains just two 1's, it follows that if we add all the columns we obtain a solution of  $(H_2)$  in which all the variables are 0. On the other hand, if we add any subset of the columns of  $H_2$  the sum will be a solution in which not all the variables are zero. For consider a segment joining an interior point  $A$  of the region  $\alpha^2$  corresponding to one of the columns in the subset to an interior point  $B$  of a region  $\alpha^2$  corresponding to one of the columns not in the subset; this segment may be chosen so as not to pass through a point of intersection of two of the lines  $l_1, l_2, \dots, l_n$ . Hence it contains a finite number of points on the polygons corresponding to the columns in the subset. The first one of these in the sense from  $B$

\*Since the only coefficients which can enter are 0 and 1, the statement that one solution is linearly dependent on a set of others is equivalent to saying that it is a sum of a number of them.

†In the modular space of  $\alpha_1 - 1$  dimensions this means that the  $\alpha_0 - 1$  independent  $(\alpha_1 - 2)$ -spaces intersect in an  $(\alpha_2 - 1)$ -space.

to  $A$  is on an  $\alpha_2^1$  which is on the boundary of a region in the subset and a region not in the subset. The variable corresponding to this interval therefore appears in only one of the  $\alpha$ 's in the subset and so does not drop out in the sum. Hence any  $\alpha_2 - 1$  of the boundaries of the  $\alpha_2$  convex regions correspond to a set of linearly independent solutions of (4). In other words,  $2^{\alpha_2 - \alpha_2} - 1$ , or one less than half of all the solutions of  $(H_1)$ , are linearly dependent on the solutions corresponding to the columns of  $H_2$ . The solutions of  $H_1$  are thus divided into two classes, those linearly dependent on the columns of  $H_2$  and those not so dependent.

Since each of the lines  $l_1, l_2, \dots, l_n$  is a polygon, it corresponds to a solution of the equations  $(H_1)$ , but it does not correspond to a solution which is linearly dependent on the columns of the matrix  $H_2$ . This is a corollary of the argument used in showing that the sum of any subset of the columns of  $H_2$  is not a solution in which all the variables are zero. For in that argument we showed that a certain segment  $AB$  contains a point on the polygon represented by the sum of such a subset. The same argument applies to the complementary segment. Hence the line  $AB$  has two points, at least, in common with the polygon or polygons represented by the sum of the subset of columns. Hence this sum cannot represent a line.

Thus, if we take the solution of the equations  $(H_1)$  corresponding to any one of the lines  $l_1, l_2, \dots, l_n$  together with any  $\alpha_2 - 1$  of the columns of the matrix  $H_2$ , we have a linearly independent set of solutions. But since this set contains  $\alpha_2$  independent solutions, all solutions are linearly dependent on this set.

**187. Regions determined by a polygon.** If  $p$  is any polygon it can, by § 184, be regarded as one whose vertices are  $\alpha^0$ 's and whose edges are  $\alpha^1$ 's of a set of lines  $l_1, l_2, \dots, l_n$ .

Two cases arise according as  $p$  is represented by a symbol which is or is not a sum of a subset of the columns of the matrix  $H_2$ . In the first case  $p$  corresponds also to the sum of all the remaining columns, because the sum of all the columns is  $(0, 0, \dots, 0)$ . It cannot correspond to a third set of columns, for the sum of the columns in the second and third sets, which is also the sum of the columns not common to these two sets, would be  $(0, 0, \dots, 0)$ . Hence there would be a linear relation among a subset of the columns of  $H_2$  contrary to what has been proved above.

Let us denote the two sets of columns of  $H_2$ , whose sums are the symbol for  $p$ , by  $c_1, c_2, \dots, c_k$  and  $c_{k+1}, \dots, c_{a_2}$  respectively, and suppose the notation so assigned that they represent the boundaries of  $a_1^2, a_2^2, \dots, a_k^2$  and  $a_{k+1}^2, \dots, a_{a_2}^2$  respectively. Let the points of the plane in  $a_1^2, a_2^2, \dots, a_k^2$ , together with such points of the boundaries as are not points of  $p$ , be denoted by  $[P]$ . Let the set of the points analogously related to  $a_{k+1}^2, \dots, a_{a_2}^2$  be denoted by  $[Q]$ . Clearly, the sets of points  $[P]$ ,  $[Q]$ , and  $p$  are mutually exclusive and include all points of the plane.

Consider any point  $P_0$  of the convex region  $a_1^2$  corresponding to  $c_1$ . It is connected by a segment consisting entirely of points  $P$  to every point  $P$  in or on the boundary of  $a_1^2$ . If  $k > 1$ ,  $c_1$  has an edge in common with at least one of  $c_2, \dots, c_k$ , and the notation may be assigned so that  $c_1$  has an edge in common with  $c_2$ . Then  $P_0$  can be joined to any point  $P_1$  of the common edge by a segment of  $P$ -points, and  $P_1$  by another segment of  $P$ -points to every  $P$ -point of the region  $a_2^2$  and its boundary. If  $k > 2$  there is a solution which may be called  $c_3$ , with an edge in common with  $c_1$  or  $c_2$ ; for if not, the solution  $c_1 + c_2$  would be one in which all the 1's correspond to the edges of  $p$ , and as no subset of the edges of  $p$  forms a polygon,  $c_1 + c_2$  would correspond to  $p$  itself. As before, every point of the region  $a_3^2$  and its boundary can be joined to  $P_0$  by a broken line of at most three edges. Since there is no subset of  $c_1, \dots, c_k$  whose sum corresponds to  $\pi$ , this process can be continued till we have any point  $R$  of the convex regions  $a_1^2, a_2^2, \dots, a_k^2$  and their boundaries joined by a broken line  $b$  to  $P_0$ . If  $R$  is on  $\pi$  the process of constructing  $b$  is such that all points of  $b$  except  $R$  are in  $[P]$ , whereas if  $R$  is in  $[P]$  all points of  $b$  are in  $[P]$ .

Hence any two points of  $[P]$  can be joined by a broken line consisting only of such points; and, since every point of  $p$  is on the boundary of one of  $a_1^2, a_2^2, \dots, a_k^2$ , any  $P$  can be joined to any point  $R$  of  $p$  by a broken line every point of which, except  $R$ , is in  $[P]$ . A precisely similar statement is true of  $[Q]$ .

Consider now any broken line  $b'$  joining a point  $P$  to a point  $Q$ . The points which are on this broken line and also on any  $a^1$  and its ends constitute a finite number of points and segments. Hence  $b'$  meets the lines  $l_1, l_2, \dots, l_n$  in a finite number of points and segments, each of the segments being contained entirely in an  $a^1$ .

These points and the ends of these segments we shall denote by  $A_1, A_2, \dots, A_k$  taken in the sense on the broken line from  $P$  to  $Q$ . Since  $P$  and  $A_1$  are within or on the boundary of the same convex region,  $A_1$  is either in  $[P]$  or on  $p$ . If  $A_1$  is in  $[P]$  the same consideration shows that  $A_2$  is in  $[P]$  or on  $p$ . If none of the  $A$ 's were on  $p$ , this process would lead to the result that  $A_k$  is in  $[P]$ , and hence  $Q$  would also be in  $[P]$ , contrary to hypothesis. Hence one of the  $A$ 's is on  $p$ , and hence any broken line joining a point  $P$  to a point  $Q$  contains a point on  $p$ .

It now follows that  $[P]$  and  $[Q]$  are both regions. For we have seen that any point  $P$  can be joined to any other  $P$  by a broken line consisting entirely of points of  $P$ . By Theorem 71 any point  $P$  is contained in a triangular region containing no points of  $p$ . This triangular region contains no  $Q$ , because if it did a segment joining it to  $P$  would, by the argument just made, contain a point of  $p$ . Hence  $[P]$  satisfies the definition of a two-dimensional region given in § 155. A similar argument applies to  $[Q]$ . Hence we have

**THEOREM 75.** *Any simple polygon  $p$  which corresponds to a symbol  $(x_1, x_2, \dots, x_n)$  which is the sum of a set of columns of  $H_2$  is the boundary of two mutually exclusive regions which include all points of the plane not on  $p$  and are such that any two points of the same region can be joined by a broken line which is in the region. Any broken line joining a point of the one region to a point of the other region contains a point of the polygon.*

**COROLLARY 1.** *Any point  $R$  of  $p$  can be joined to any point not on  $p$  by a broken line containing no other point of  $p$ .*

**COROLLARY 2.** *If a segment  $ST$  meets  $p$  in a single point  $O$  which is not a vertex of  $p$ ,  $S$  and  $T$  are in different regions with respect to  $p$ .*

*Proof.* Let  $S'$  and  $T'$  be two points in the order  $\{SS'OT'T\}$  and such that the segment  $\bar{S'OT'}$  contains no point of  $l_1, l_2, \dots, l_n$  except  $O$ . By § 185,  $S'$  and  $T'$  are in two convex regions  $a^2$  which have an edge in common. Since this edge is an edge of  $p$ , the columns of  $H_2$  corresponding to these two  $a^2$ 's must be one in the set  $c_1, c_2, \dots, c_k$ , and the other in the set  $c_{k+1}, \dots, c_n$ . Hence, if  $S'$  and  $S$  are in  $[P]$ ,  $T'$  and  $T$  are in  $[Q]$ , and vice versa.

**THEOREM 76.** *Any simple polygon  $p$  which corresponds to a symbol  $(x_1, x_2, \dots, x_n)$  which is not the sum of a set of columns of  $H_2$*

is the boundary of a region which includes all points of the plane not on  $p$ . Any two points not on  $p$  can be joined by a broken line not meeting  $p$ .

*Proof.* By Theorem 71 any point not on  $p$  can be enclosed in a triangular region containing no point of  $p$ . Hence the theorem will be proved if we can show that any two points not on  $p$  are joined by a broken line consisting only of such points. If this were not so, we could let  $P_0$  be any point not on  $p$  and let  $[P]$  be the set of all points not on  $p$  which can be joined to  $P_0$  by broken lines not meeting  $p$ . As in the proof of Theorem 75,  $[P]$  would have to consist of a number of regions  $\alpha^2$ , together with those points of their boundaries which were not on  $p$ ; and the boundary of  $[P]$  could consist only of points of  $p$ . But the boundary of  $[P]$  must consist of the polygon or polygons whose symbol is obtained by adding the columns of  $H_2$  corresponding to the  $\alpha^2$ 's in  $[P]$ . By § 183 no subset of the points of  $p$  can be a simple polygon. Hence  $p$  would be the the boundary of  $[P]$  and be expressible linearly in terms of the boundaries of  $\alpha^2$ 's, contrary to hypothesis.

Every polygon whose edges are on  $l_1, l_2, \dots, l_n$  corresponds to a symbol  $(x_1, x_2, \dots, x_n)$  which either is or is not expressible linearly in terms of the columns of  $H_2$ . Hence the arbitrary simple polygon  $p$  with which this section starts and which determines the lines  $l_1, l_2, \dots, l_n$  is described either in Theorem 75 or in Theorem 76. Hence we have

**THEOREM 77. DEFINITION.** *The polygons of a plane  $\alpha$  fall into two classes the individuals of which are called odd and even respectively. A polygon of the first class is the boundary of a single region comprising all points of  $\alpha$  not on the polygon. A polygon of the second class is the boundary of each of two regions which contain all points of  $\alpha$  not on the polygon, have no point in common, and are such that any broken line joining a point of one region to a point of the other contains a point of the polygon.*

The odd polygons are also called *unicursal*, and the even polygons are also called *bounding*. A line is an example of an odd polygon, and the boundary of a triangular region is an example of an even one. The segments  $\bar{\alpha}, \bar{\beta}, \bar{\gamma}$  as defined in § 26 are the edges of an odd polygon.

**THEOREM 78.** *Two polygons of which one is even and which are such that neither polygon has a vertex on the other have an even (or zero) number of points in common.*

*Proof.* Let  $p_1$  be an even polygon, let  $p_2$  be any other polygon, and let the points of intersection of the polygons be  $R_1, \dots, R_n$  in the order  $\{R_1 R_2 \dots R_n\}$  with respect to  $p_2$ . If  $n = 0$  the theorem is verified. If  $n$  were 1 the edge of  $p_2$  containing  $R_1$  would have its ends in different regions with respect to  $p_1$ , and hence the broken line composed of all  $p_2$  except the side containing  $R_1$  would have to contain a point of  $p_1$ , contrary to hypothesis. If  $n > 1$  the interval of  $p_2$  which has  $R_1$  and  $R_2$  as ends and contains no other points  $R$  is a broken line which belongs (except for its ends) entirely to one of the two regions  $[P]$  and  $[Q]$  determined by  $p_1$ ; and by Cor. 2, Theorem 75, the interval of  $p_2$  similarly determined by  $R_2$  and  $R_3$  belongs entirely to the other of the two regions  $[P]$  and  $[Q]$ . Thus, if  $S_1, S_2, \dots, S_n$  are a set of points of  $p_2$  in the order  $\{R_1 S_1 R_2 S_2 R_3 \dots S_{n-1} R_n S_n\}$ , and  $S_1$  is in  $[P]$ , all the  $S$ 's with odd subscripts are in  $[P]$  and all the  $S$ 's with even subscripts are in  $[Q]$ . But by Cor. 2, Theorem 75,  $S_n$  is in  $[Q]$  since  $S_1$  is in  $[P]$ . Hence  $n$  is even.

**COROLLARY.** *A line coplanar with and containing no vertex of an even polygon meets it in an even (or zero) number of points.*

**THEOREM 79.** *Two odd polygons such that neither has a vertex on the other meet in an odd number of points.*

*Proof.* Let the polygons be  $p_1$  and  $p_2$ , let the lines containing the sides of  $p_1$  be  $l_1, \dots, l_{n-1}$ , and let  $l_n$  be a line containing no vertex of either polygon. According to the results stated at the end of the last section,  $p_1$  is expressible by addition, modulo 2, as the sum of  $l_n$  and a number of boundaries of  $a^2$ 's. The latter combine into a number of even polygons, the edges of which are either edges of  $p_1$  or of  $l_n$ . Hence these even polygons have no vertices on  $p_2$  and contain no vertices of  $p_2$ . Hence by Theorem 78 they have an even (or zero) number of points in common with  $p_2$ . Thus our theorem will follow if we can show that  $l_n$  has an odd number of points in common with  $p_2$ .

By the argument just used  $p_2$  can be expressed as the sum, modulo 2, of a line  $m$  and a number of even polygons which have no vertices on  $l_n$ . The latter meet  $l_n$  in an even (or zero) number

of points, and  $m$  meets  $l_n$  in one point. Hence  $p_2$  meets  $l_n$  in an odd number of points.

COROLLARY 1. *Two odd polygons always have at least one point in common.*

COROLLARY 2. *If  $p$  is a simple polygon and there exists an odd polygon  $p_1$  meeting  $p$  in an even (or zero) number of points and such that neither polygon has a vertex on the other, then  $p$  is even.*

Since the plane of a convex region always contains at least one line not having a point in common with the region, the last result has the following special case, which, on account of its importance, we shall list as a theorem.

THEOREM 80. *Any simple polygon lying entirely in a convex region is even.*

To complete the theory of the subdivision of the plane by a polygon, there are needed a number of other theorems which can be handled by methods analogous to those already developed. They are stated below as exercises.

### EXERCISES

1. If a simple polygon  $p$  lies entirely in a convex region  $R$ , the points of  $R$  not on  $p$  fall into two regions such that any broken line joining a point of one region to a point of the other has a point on  $p$ . One of these regions, called the *interior* of the polygon, has the property that any ray (with respect to  $R$ ) whose origin is a point of this region meets  $p$  in an odd number of points, provided it contains no vertex of  $p$ . The other region, called the *exterior* of the polygon with respect to  $R$ , has the property that any ray whose origin is one of the points of this region meets  $p$  in an even (or zero) number of points, provided it contains no vertex of  $p$ .

2. If  $p$  is an even polygon in a plane  $\alpha$ , one of the two regions determined by  $p$ , according to Theorem 77, contains no odd polygon and is called the *interior* of  $p$ . The other contains an infinity of odd polygons and is called the *exterior* of  $p$ .

3. If one line coplanar with and not containing a vertex of a simple polygon meets it in an odd (even or zero) number of points, every line not containing a vertex and coplanar with it meets it in an odd (even or zero) number of points.

4. If the boundary of a convex region consists of a finite number of linear segments, together with their ends, it is a simple polygon.

5. A simple polygon which is met by every line not containing a vertex in two or no points is the boundary of a convex region.

6. For any simple polygon  $A_1A_2 \cdots A_nA_1$ , there exists a set of  $n-2$  triangular regions such that (1) every point of the interior of the polygon is in or on the boundary of one of the triangular regions, (2) every vertex of one of the triangular regions is a vertex of the polygon, and (3) no two of the triangular regions have a point in common.

\*7. By use of convex regions and matrices analogous to  $H_1$  and  $H_2$ , prove Theorem 77 for any curve made up of analytic pieces (i.e. 1-cells which satisfy analytic equations).

**188. Polygonal regions and polyhedra.** DEFINITION. A *planar polygonal region* is a two-dimensional region  $R$  for which there exists a finite number of points and linear regions such that any interval joining a point of  $R$  to a point not in  $R$ , but coplanar with it, meets one of these points or linear regions. A (three-dimensional) *polyhedral region* is a three-dimensional region  $R$  for which there exists a finite number of points, linear regions, and planar polygonal regions such that any interval joining a point of  $R$  to a point not in  $R$  meets one of these points, linear regions, or planar polygonal regions.

Let  $R$  be a planar polygonal region and let  $l_1, l_2, \dots, l_n$  be a set of lines coplanar with  $R$  and containing all the points and linear regions such that any interval joining a point in  $R$  to a point not in  $R$  meets one of these points or one of these linear regions. Let us adopt the notation of § 185.

If a point  $P$  of one of the two-dimensional convex regions  $\alpha^2$  is in  $R$ , all points of the  $\alpha^2$  are in  $R$ , for all such points are joined to  $P$  by intervals not meeting  $l_1, l_2, \dots, l_n$ .

Since any point not on  $l_1, l_2, \dots, l_n$  is interior to a triangular region containing no points of  $l_1, l_2, \dots, l_n$ , no such point can be a boundary point of  $R$ .

Let  $\alpha_1^2, \dots, \alpha_k^2$  be the  $\alpha^2$ 's which have points in  $R$ . As we have seen, all points of these  $\alpha^2$ 's are in  $R$ . All points of their boundaries are either in  $R$  or on its boundary; for every point of the boundary of an  $\alpha_r^2$  ( $r=1, \dots, k$ ) may be joined to a point of  $\alpha_k^2$ , that is, to a point of  $R$ , by a segment of points of  $R$ , and hence is either a point of  $R$  or of its boundary.

Any point  $B$  of the boundary of  $R$  is on the boundary of one of  $\alpha_1^2, \dots, \alpha_k^2$ . For any triangular region  $T$  containing  $B$  contains points of  $R$  and hence contains a triangular region  $T'$  of points of  $R$ . The region  $T'$  must have points in common with at least one  $\alpha^2$ . If  $T$  be chosen so as to contain no points of any  $\alpha^2$  which does

not have  $B$  on its boundary, any  $a^2$  having a point in common with  $T'$  is one of  $a_{i_1}^2, \dots, a_{i_k}^2$ . Hence every boundary point of  $R$  is on the boundary of one of  $a_{i_1}^2, \dots, a_{i_k}^2$ . Hence the set of points of  $R$  and its boundary is identical with the set of all points of  $a_{i_1}^2, \dots, a_{i_k}^2$  and their boundaries. In other words,

**THEOREM 81.** *For any planar polygonal region  $R$  there is a finite set of convex polygonal regions  $R_1, \dots, R_n$  such that the set of all points of  $R_1, \dots, R_n$  and their boundaries is identical with  $R$  and its boundary.*

As a consequence, any set of points which consists of planar polygonal regions and their boundaries can be described as a set of points in a set of convex polygonal regions and their boundaries. Therefore no generality is lost in the following definition of a polyhedron by stating it in terms of convex polygonal regions.

**DEFINITION.** A set of points  $[P]$  is called a *polyhedron* if it satisfies the following conditions and contains no subset which satisfies them:  $[P]$  consists of a set of distinct points  $a_1^0, a_2^0, \dots, a_{\alpha}^0$ , segments  $a_1^1, a_2^1, \dots, a_{\alpha}^1$ , and convex planar polygonal regions  $a_1^2, a_2^2, \dots, a_{\alpha}^2$ , such that each  $a^1$  is bounded by two  $a^0$ 's and each  $a^2$  by a simple polygon whose vertices are  $a^0$ 's and whose edges are  $a^1$ 's; no  $a^1$  or  $a^2$  contains an  $a^0$  and no two of the  $a^1$ 's or  $a^2$ 's have a point in common; each  $a^1$  is on the boundary of an even number of  $a^2$ 's. The points  $a^0$  are called the *vertices*, the segments  $a^1$  the *edges*, and the planar regions  $a^2$  the *faces* of the polyhedron.

Just as any point of a polygon can be regarded as a vertex, so any point of an edge of a polyhedron can be regarded as a vertex, and any segment contained in a face and joining two of its vertices can be regarded as an edge.

The relations among the vertices, edges, and faces of a polyhedron can be described by means of matrices  $H_1$  and  $H_2$  analogous to those of § 186. In the first matrix,

$$H_1 = (\eta_{ij}^1),$$

the element  $\eta_{ij}^1$  is 0 or 1 according as  $a_i^0$  is not or is an end of  $a_j^1$ . In the second matrix,

$$H_2 = (\eta_{ij}^2),$$

the element  $\eta_{ij}^2$  is 0 or 1 according as  $a_i^1$  is not or is on the boundary of  $a_j^2$ . The theory of the polyhedron can be derived from a

discussion of these matrices just as that of the projective plane (a special polyhedron) has been derived in the sections above.

Thus the polygons which can be formed from the vertices and edges of the polyhedron are denoted by symbols of the form  $(x_1, x_2, \dots, x_{a_i})$  as in § 186. They are all expressible as sums, modulo 2, of the boundaries of the faces together with  $P - 1$  other polygons. The number  $P$  is called the *connectivity* of the polyhedron and is the same no matter how the polyhedron is subdivided into faces, edges, and vertices. It is determined by the following relation:

$$\alpha_0 - \alpha_1 + \alpha_2 = 3 - P.$$

### EXERCISES

1. Any polygonal region can be regarded as composed of a finite set of triangular regions together with portions of their boundaries, no two of the triangular regions having a point in common.

2. If  $R$  is a polygonal region, every broken line joining a point of  $R$  to a point not in  $R$  has a point on the boundary.

3. For any three-dimensional polyhedral region  $R$  there is a finite set of polyhedral regions  $R_1, R_2, \dots, R_n$  such that the set of all points of  $R_1, R_2, \dots, R_n$  and their boundaries is identical with  $R$  and its boundary.  $R_1, R_2, \dots, R_n$  may be so chosen as all to be tetrahedral regions.

4. If a polyhedron is the boundary of a convex region, each edge of the polyhedron is on the boundaries of two and only two of its faces.

189. Subdivision of space by planes. The theorems of § 185 generalize at once into the following. The proofs (with one exception) are left to the reader.

**THEOREM 82.** *The points of space which are not upon any one of a finite set of planes  $\pi_1, \pi_2, \dots, \pi_n$  fall into a finite number  $\alpha_s$  of convex regions such that any segment joining two points of different regions contains at least one point of  $\pi_1, \pi_2, \dots, \pi_n$ . The number  $\alpha_s$  satisfies the inequality  $n \leq \alpha_s \leq \frac{n(n-1)(n-2)}{6} + n$ .*

As in § 185, we indicate the  $\alpha_0$  points of intersection of  $n$  planes  $\pi_1, \pi_2, \dots, \pi_n$  by  $\alpha_1^0, \alpha_2^0, \dots, \alpha_{a_0}^0$ ,

or any one of them by  $a^0$ ; the  $\alpha_1$  linear convex regions determined by these points upon the lines of intersection, by

$$a_1^1, a_2^1, \dots, a_{a_1}^1,$$

or any one of them by  $a^1$ ; the  $\alpha_2$  planar convex regions determined by the lines of intersection upon the planes, by

$$a_1^2, a_2^2, \dots, a_{\alpha_2}^2,$$

or any one of them by  $a^2$ ; and the  $\alpha_3$  spatial convex regions determined by the planes, by  $a_1^3, a_2^3, \dots, a_{\alpha_3}^3,$

or any one of them by  $a^3$ .

**THEOREM 83.** *If the planes  $\pi_1, \pi_2, \dots, \pi_n$  are not all coaxial, the boundary of each  $a^3$  is composed of a finite number of  $a^2$ 's and of those  $a^1$ 's and  $a^0$ 's which bound the  $a^2$ 's in question. Each  $a^2$  is upon the boundary of two and only two  $a^1$ 's.*

**COROLLARY 1.** *If the planes  $\pi_1, \pi_2, \dots, \pi_n$  are coaxial,  $\alpha_0 = 0, \alpha_1 = 0$ , and the boundary of each  $a^3$  is composed of two  $a^2$ 's together with the common line of the planes.*

**COROLLARY 2.** *If the planes are not all concurrent, any line through a point  $I$  of one of the regions  $a^3$  meets the boundary in two points  $P, Q$ . The segment  $\overline{PIQ}$  consists entirely of points of the  $a^3$ , and the complementary segment entirely of points not in the  $a^3$ .*

**THEOREM 84.** *If an  $a^1$  is on the boundary of an  $a^3$ , it is on the boundaries of two and only two  $a^2$ 's of the boundary of the  $a^3$ . Any plane section of an  $a^3$  is a two-dimensional convex region bounded by a simple polygon which is a plane section of the boundary of the  $a^3$ .*

**COROLLARY.** *The boundary of each  $a^3$  is a polyhedron.*

**THEOREM 85.** *The numbers  $\alpha_0, \alpha_1, \alpha_2, \alpha_3$  are subject to the relation  $\alpha_0 - \alpha_1 + \alpha_2 - \alpha_3 = 0$ .*

*Proof.* The proof is made by induction. In the case of two planes,  $\alpha_0 = 0, \alpha_1 = 0, \alpha_2 = 2, \alpha_3 = 2$ . Assuming that the theorem is true for  $n$  planes, let us see what is the effect of introducing a plane  $\pi_{n+1}$ . This plane is divided by the other planes into a number of convex two-dimensional regions equal to the number of  $a^2$ 's in which it has points; but it divides each of these  $a^2$ 's into two  $a^2$ 's. Hence the adjunction of these new  $a^2$ 's and  $a^3$ 's increases  $\alpha_2$  and  $\alpha_3$  by equal amounts. The plane  $\pi_{n+1}$ , according to Theorem 8, Cor. 1, divides in two each  $a^2$  which it meets; but it has a new  $a^1$  in common with each such region. Here, therefore,  $\alpha_2$  and  $\alpha_1$  are increased by equal amounts. The plane  $\pi_{n+1}$  divides in two each  $a^1$  which it meets; but it has a point in common with each such region. Hence,

in this case,  $\alpha_1$  and  $\alpha_0$  are increased by equal amounts. Hence, if the formula is true for  $n$  planes, it is true for  $n+1$ .

**COROLLARY.** *The number of  $a^1$ 's for  $\pi_1, \pi_2, \dots, \pi_n$  which are not on lines of intersection of pairs of the planes  $\pi_1, \pi_2, \dots, \pi_{n-1}$  is the number by which  $\alpha_1 - \alpha_0$  for the planes  $\pi_1, \pi_2, \dots, \pi_n$  exceeds  $\alpha_1 - \alpha_0$  for the planes  $\pi_1, \pi_2, \dots, \pi_{n-1}$ .*

*Proof.* New  $a^1$ 's are produced by the introduction of  $\pi_n$  in two ways: (1)  $\pi_n$  may meet an  $a^1$  of  $\pi_1, \pi_2, \dots, \pi_{n-1}$  in a point; if so, this  $a^1$  is separated into two  $a^1$ 's and a new  $a^0$  is introduced; (2)  $\pi_n$  may meet an  $a^2$  of  $\pi_1, \pi_2, \dots, \pi_{n-1}$  in a new  $a^1$ . The only new  $a^0$ 's produced by the introduction of  $\pi_n$  are accounted for under (1). Hence (2) accounts for the increase of  $\alpha_1 - \alpha_0$ , as stated above.

**190. The matrices  $H_1, H_2$ , and  $H_3$ .** The relations among the convex regions determined by  $n$  planes which are not coaxial may be described by means of three matrices, which we shall call  $H_1, H_2$ , and  $H_3$ . In the first matrix,

$$H_1 = (\eta_{ij}^1),$$

$i=1, 2, \dots, \alpha_0$ ;  $j=1, 2, \dots, \alpha_1$ ; and  $\eta_{ij}^1=1$  or 0 according as  $a_i^0$  is or is not an end of  $a_j^1$ . In the second matrix,

$$H_2 = (\eta_{ij}^2),$$

$i=1, 2, \dots, \alpha_1$ ;  $j=1, 2, \dots, \alpha_2$ ; and  $\eta_{ij}^2=1$  or 0 according as  $a_i^1$  is or is not on the boundary of  $a_j^2$ . In the third matrix,

$$H_3 = (\eta_{ij}^3),$$

$i=1, 2, \dots, \alpha_2$ ;  $j=1, 2, \dots, \alpha_3$ ; and  $\eta_{ij}^3=1$  or 0 according as  $a_i^2$  is or is not on the boundary of  $a_j^3$ . Examples of these three matrices are those given in § 152 to describe the tetrahedron. It will be noted that  $H_1$  has two 1's in each column, and  $H_3$  two 1's in each row.

Corresponding to the matrix  $H_1$ , there is a set of  $\alpha_0$  linear equations (modulo 2)

$$(H_1) \quad \sum_{j=1}^{\alpha_1} \eta_{ij}^1 x_j \quad (i=1, 2, \dots, \alpha_0).$$

Let the symbol  $(x_0, x_1, \dots, x_{\alpha_1})$ , where the  $x_k$ 's are 0 or 1, be taken to represent a set of  $a^1$ 's containing  $a_k^1$  if  $x_k=1$  and not containing it if  $x_k=0$ . Just as in § 186, this set of  $a^1$ 's will be the edges of a polygon or set of polygons if and only if  $(x_0, x_1, \dots, x_{\alpha_1})$  is a solution of  $(H_1)$ .

Just as in § 186, the sum of two sets of polygons (modulo 2) will be

taken to be the set of polygons represented by the sum of the symbols  $(x_1, x_2, \dots, x_{a_1})$  for the two sets of polygons. The sum, modulo 2, of two sets of polygons  $p_1$  and  $p_2$  is therefore the set of polygons whose edges appear either in  $p_1$  or in  $p_2$  but not in both  $p_1$  and  $p_2$ .

By the reasoning in § 186,  $\alpha_0 - 1$  of the equations  $(H_1)$  are linearly independent, and the other one is linearly dependent on these. The columns of  $H_2$  are the symbols  $(x_0, x_1, \dots, x_{a_1})$  for the boundaries of the  $a^2$ 's and hence are solutions of  $(H_1)$ .

Corresponding to the matrix  $H_2$ , there is a set of  $\alpha_1$  linear equations (modulo 2)

$$(H_2) \quad \sum_{j=1}^{\alpha_2} \eta_{ij}^2 x_j \quad (i=1, 2, \dots, \alpha_1).$$

Let the symbol  $(x_1, x_2, \dots, x_{a_1})$ , where the  $x_k$ 's are 0 or 1, be taken to represent a set of  $a^2$ 's containing  $a_k^2$  if  $x_k = 1$  and not containing it if  $x_k = 0$ . If this symbol is a solution of  $(H_2)$ , it represents a set of  $a^2$ 's such that each  $a^1$  is on the boundaries of an even number (or zero) of them; i.e. it represents the faces of a polyhedron or a set of polyhedra.

The columns of  $H_3$  represent the boundaries of the  $a^3$ 's. By Theorem 84 any  $a^1$  of the boundary of an  $a^3$  is on the boundaries of two and only two  $a^2$ 's of this boundary. Hence the columns of  $H_3$  are solutions of  $(H_2)$ .

Corresponding to the matrix  $H_3$ , there is a set of  $\alpha_2$  linear equations (modulo 2)

$$(H_3) \quad \sum_{j=1}^{\alpha_3} \eta_{ij}^3 x_j \quad (i=1, 2, \dots, \alpha_2)$$

Let the symbol  $(x_1, x_2, \dots, x_{a_2})$ , where the  $x_k$ 's are 0 or 1, be taken to represent a set of  $a^3$ 's containing  $a_k^3$  if  $x_k = 1$  and not containing it if  $x_k = 0$ . If this symbol is a solution of  $(H_3)$ , it represents a set of  $a^3$ 's of which there is an even number on each  $a^2$ . It is easily seen that the only such set of  $a^3$ 's is the set of all  $a^3$ 's in space. Hence the only solutions of  $(H_3)$  are  $(0, 0, \dots, 0)$  and  $(1, 1, \dots, 1)$ . Hence there are  $\alpha_3 - 1$  linearly independent equations in  $(H_3)$  on which all the rest are linearly dependent.

Let the ranks\* of the matrices  $H_1, H_2, H_3$  be  $r_1, r_2, r_3$  respectively.

\*The rank of a matrix is the number of rows (or columns) in a set of linearly independent rows (or columns) on which all the other rows (or columns) are linearly dependent.

By what has been seen above

$$r_1 = \alpha_0 - 1,$$

$$r_3 = \alpha_3 - 1.$$

The discussion in the next section will establish that

$$r_2 = \alpha_1 - \alpha_0.$$

**191. The rank of  $H_2$ .** Let us now suppose that  $\pi_1, \pi_2, \dots, \pi_n$  are not all on the same point and that the notation is so assigned that  $\pi_1, \pi_2, \pi_3, \pi_4$  are the faces of a tetrahedron. By inspection of the matrices given in § 152, it is clear that for the case  $n=4$ ,  $\alpha_0=4$ ,  $\alpha_1=12$ ,  $\alpha_2=16$ ,  $\alpha_3=8$ , and  $r_2=8$  (a set of linearly independent columns of  $H_2$  upon which the rest depend linearly is the set of columns corresponding to  $\tau_{11}, \tau_{12}, \tau_{13}, \tau_{21}, \tau_{22}, \tau_{23}, \tau_{31}$ , and  $\tau_{32}$ ). The number of solutions of  $(H_1)$  in a linearly independent set upon which all the other solutions depend is  $\alpha_1 - \alpha_0 + 1 = 9$ . Hence one solution which does not depend linearly upon the columns of  $H_2$ , together with a set of eight linearly independent columns of  $H_2$ , constitute a set of linearly independent solutions of  $(H_1)$  upon which all the others depend linearly. Any solution representing a projective line, e.g.  $(1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$ , will serve this purpose.

In case  $n > 4$ , the columns of  $H_2$  fall into three classes: (1) those representing the boundaries of  $a^3$ 's in  $\pi_n$ ; (2) those representing the boundaries of  $a^2$ 's which are not in  $\pi_n$  but have an  $a^1$  in  $\pi_n$ ; and (3) those representing the boundaries of  $a^2$ 's which have no  $a^1$  in  $\pi_n$ .

Any column of Class (1) is expressible as a sum of columns of Classes (2) and (3). For the  $a^2$  whose boundary it represents is on the boundary of an  $a^3$  whose boundary has no other  $a^2$  in common with  $\pi_n$  (cf. § 150). Since each  $a^1$  on the boundary of an  $a^3$  is on the boundary of two and only two  $a^2$ 's of the boundary of the  $a^3$  (Theorem 84), it follows that the given column is the sum of the columns which represent the boundaries of the other  $a^2$ 's on the boundary of the  $a^3$ . These columns are all of Classes (2) or (3).

Each  $a^1$  which is not on a line of intersection of two of the planes  $\pi_1, \pi_2, \dots, \pi_{n-1}$  is the linear segment in which one of the  $a^2$ 's determined by  $\pi_1, \pi_2, \dots, \pi_{n-1}$  is met by  $\pi_n$ . Hence the row of  $H_2$  corresponding to this  $a^1$  contains just two 1's in columns of Class (2), and the sum of these two columns of Class (2) is the symbol for

the boundary of one of the  $\alpha^2$ 's determined by  $\pi_1, \pi_2, \dots, \pi_{n-1}$ . Moreover, the columns of  $H_2$  of Class (2) form a set of pairs of this sort, since every  $\alpha^1$  of  $\pi_n$  is either on a line of intersection of two of the planes  $\pi_1, \pi_2, \dots, \pi_{n-1}$  or is an edge of two and only two  $\alpha^2$ 's not in  $\pi_n$ .

No one of such a pair of columns of  $H_2$  can enter into a linear relation among a set of columns of Classes (2) and (3) unless the other does. For this column would be the only column of the set containing a 1 in the row corresponding to the  $\alpha^1$  common to the boundaries of the  $\alpha^2$ 's represented by the two columns, and hence the sum of columns could not reduce to  $(0, 0, \dots, 0)$ .

Let  $H'_2$  be the matrix consisting of the columns of Class (3) of  $H_2$  and the sums of the pairs of columns of Class (2) discussed in the last two paragraphs. According to the last paragraph the rank of  $H'_2$  is less than the rank of  $H_2$  by the number of these pairs of columns; and by the corollary of Theorem 85 this number is the difference between the values of  $\alpha_1 - \alpha_0$  for  $\pi_1, \pi_2, \dots, \pi_n$  and for  $\pi_1, \pi_2, \dots, \pi_{n-1}$ .

The columns of  $H'_2$  are the symbols in terms of the  $\alpha^1$ 's determined by  $\pi_1, \pi_2, \dots, \pi_n$  for the boundaries of the  $\alpha^2$ 's determined by  $\pi_1, \pi_2, \dots, \pi_{n-1}$ . Hence any two rows of this matrix which correspond to a pair of  $\alpha^1$ 's into which an  $\alpha^1$  determined by  $\pi_1, \pi_2, \dots, \pi_{n-1}$  is separated by  $\pi_n$  must be identical; and if one of each such pair of rows is omitted,  $H'_2$  reduces to the  $H_2$  for  $\pi_1, \pi_2, \dots, \pi_{n-1}$ . Hence  $H'_2$  has the same rank as the  $H_2$  for  $\pi_1, \pi_2, \dots, \pi_{n-1}$ .

Since the difference in the ranks of  $H_2$  for  $\pi_1, \pi_2, \dots, \pi_n$  and of  $H'_2$  is the same as the difference between the values of  $\alpha_1 - \alpha_0$  for  $\pi_1, \pi_2, \dots, \pi_n$  and for  $\pi_1, \pi_2, \dots, \pi_{n-1}$ , it follows that the introduction of  $\pi_n$  increases the rank of  $H_2$  by the same amount that it increases  $\alpha_1 - \alpha_0$ . Since  $\alpha_1 - \alpha_0 = r_2$  for  $n = 4$ , the same relation holds for all values of  $n$ . Hence we have

**THEOREM 86.** *For a set of planes  $\pi_1, \pi_2, \dots, \pi_n$  which are not all concurrent,*

$$\alpha_1 - \alpha_0 = r_2.$$

By Theorem 85 this relation is equivalent to

$$\alpha_2 - \alpha_3 = r_2.$$

**192. Polygons in space.** **THEOREM 87.** *The symbol  $(x_1, x_2, \dots, x_{n-1})$  for a line is not linearly dependent on the columns of  $H_2$ .*

*Proof.* Let  $\pi$  be any plane not containing any of the points  $a^0$ . The boundary of any  $a^2$  is an even polygon in the sense of § 187 and is met by  $\pi$  in two points or none, the two points being on different edges, if existent. The sum, modulo 2, of two sets of polygons  $p_1, p_2$  each of which is met by  $\pi$  in an even number (regarding zero as even) of points is a set of polygons  $p$  met by  $\pi$  in an even number of points; for if  $\pi$  meets  $p_1$  in  $2k_1$  points and  $p_2$  in  $2k_2$  points, and if  $k_3$  of these points are on edges common to  $p_1$  and  $p_2$ ,  $\pi$  must meet  $p$  in  $2k_1 + 2k_2 - 2k_3$  points. Hence any polygon which is a sum of the boundaries of the  $a^2$ 's is met by  $\pi$  in an even number of points; i.e. any polygon represented by a symbol  $(x_1, x_2, \dots, x_{\alpha_1})$  linearly dependent on the columns of  $H_2$  is met by  $\pi$  in an even number of points. Since no line is met by  $\pi$  in an even number of points, the symbol representing it cannot be a sum of any number of columns of  $H_2$ .

**THEOREM 88.** *All solutions of  $(H_1)$  are linearly dependent on a set of  $r_2$  (i.e.  $\alpha_1 - \alpha_0$ ) linearly independent columns of  $H_2$  and the symbol  $(x_1, x_2, \dots, x_{\alpha_1})$  for one line.*

*Proof.* It has been shown that the rank of  $H_1$  is  $\alpha_0 - 1$ . The number of variables in the equations  $(H_1)$  is  $\alpha_1$ . The number of linearly independent solutions in a set on which all the rest are linearly dependent is therefore  $\alpha_1 - \alpha_0 + 1$ . Since the rank of  $H_2$  is  $\alpha_1 - \alpha_0$ , and the columns of  $H_2$  are solutions of  $(H_1)$ , there are  $\alpha_1 - \alpha_0$  linearly independent columns of  $H_2$  which are solutions of  $(H_1)$ ; and since the solution of  $(H_1)$  which represents a line is not linearly dependent on these, the statement in the theorem follows.

In the proof of Theorem 87 it appeared that any polygon which is a sum, modulo 2, of a set of polygons bounding  $a^n$ 's is met by a plane which contains none of its vertices in an even number of points. Since a line is met by a plane not containing it in one point, an argument of the same type shows that any polygon which is a sum, modulo 2, of a line and a number of polygons bounding  $a^n$ 's is met by a plane containing none of its vertices in an odd number of points. Thus we have, taking Theorem 88 into account:

**THEOREM 89. DEFINITION.** *A polygon which is the sum, modulo 2, of a number of polygons which bound convex planar regions is met by any plane not containing a vertex in an even number of points*

and is called an even polygon. A polygon which is the sum, modulo 2, of a line and a number of polygons which bound convex planar regions is met by any plane not containing a vertex in an odd number of points and is called an odd polygon. Any polygon is either odd or even.

Suppose a polygon  $p$  is the sum, modulo 2, of the boundaries of a set of convex regions  $a_1^2, \dots, a_m^2$ . The set of points  $[P]$  in  $a_1^2, \dots, a_m^2$  or on their boundaries is easily seen (by an argument analogous to that given in the proof of Theorem 75) to be a connected set. By an extension of the definition in § 150  $p$  may be said to be the boundary of  $[P]$ . From this point of view an even polygon is a bounding polygon and an odd polygon is not.

193. Odd and even polyhedra. It has been seen in § 190 that the solutions of  $(H_2)$  represent polyhedra or sets of polyhedra. The converse is also true, as is obvious on reference to the definition of a polyhedron. The sum of two symbols  $(x_1, x_2, \dots, x_n)$  which represent sets of polyhedra is a symbol representing a set of polyhedra. This is obvious either geometrically or from the algebraic consideration that the sum of two solutions of  $(H_2)$  is a solution of  $(H_2)$ .

The set of polyhedra  $p$  represented by the symbol which is the sum of the symbols for two sets of polyhedra  $p_1$  and  $p_2$  is called the sum, modulo 2, of  $p_1$  and  $p_2$ . As in the analogous case of polygons,  $p$  is a set of polyhedra whose faces are in  $p_1$  or in  $p_2$  but not in both  $p_1$  and  $p_2$ .

The number of variables in  $(H_2)$  is  $\alpha_2$  and the rank of  $H_2$  is  $\alpha_2 - \alpha_3$  by Theorems 86 and 85. Hence the solutions of  $(H_2)$  are linearly dependent on a set of  $\alpha_3$  linearly independent solutions. Since any  $\alpha_3 - 1$  of the columns of  $H_2$  are linearly independent, such a set of columns, together with one other solution linearly independent of them, will furnish a set of linearly independent solutions of  $(H_2)$ .

The symbol for any plane is a solution of  $(H_2)$  linearly independent of the columns of  $H_2$ . For let  $l$  be any line meeting no  $a^0$  or  $a^1$ . Any column of  $H_2$  represents the polyhedron bounding an  $a^3$ , and such a polyhedron is met by  $l$  in two points or none. By reasoning analogous to that used in the proof of Theorem 87, it follows that  $l$  meets the sum, modulo 2, of the boundaries of any number of  $a^3$ 's in an even number of points or none. Since  $l$  meets each plane  $\pi_1$  in one point, the symbol for  $\pi_1$  is not linearly dependent on the columns of  $H_2$ . By the last paragraph we now have

**THEOREM 90.** *Any solution of  $(H_n)$  is linearly dependent on  $\alpha_3 - 1$  columns of  $H_3$  and the symbol for any one of the planes  $\pi_1, \pi_2, \dots, \pi_n$ .*

**COROLLARY 1.** *Any polyhedron is the sum, modulo 2, of a subset of a set of polyhedra consisting of one plane and all polyhedra which bound convex regions.*

*Proof.* Let  $\pi_1, \pi_2, \dots, \pi_n$  be a set of planes containing all vertices, edges, and faces of a given polyhedron and such that  $\pi_1, \pi_2, \pi_3, \pi_4$  are not concurrent. By the theorem the given polyhedron is either expressible as a sum of the boundaries of some of the  $\alpha$ 's determined by  $\pi_1, \pi_2, \dots, \pi_n$  or as a sum of one of these planes and some of the  $\alpha$ 's.

In the course of the argument above it was shown that any polyhedron expressible in terms of the boundaries of the  $\alpha$ 's was met in an even number of points by any line not meeting an  $\alpha^0$  or an  $\alpha^1$ . One of the planes  $\pi_1, \pi_2, \dots, \pi_n$  is met by such a line in one point. Hence any polyhedron which is the sum of such a plane and a number of the boundaries of  $\alpha$ 's is met by this line in an odd number of points. Hence

**COROLLARY 2. DEFINITION.** *A polyhedron which is the sum, modulo 2, of a number of boundaries of convex three-dimensional regions is met in an even number of points by any line not meeting a vertex or an edge. Such a polyhedron is said to be even. A polyhedron which is the sum, modulo 2, of a plane and a number of boundaries of convex three-dimensional regions is met in an odd number of points by any line not meeting a vertex or an edge. Such a polygon is said to be odd.*

#### EXERCISE

Let  $p$  be a polygon and  $\pi$  a polyhedron such that  $\pi$  contains no vertex of  $p$  and  $p$  contains no vertex or edge of  $\pi$ . If  $p$  and  $\pi$  are both odd they have an odd number of points in common. If one of them is even they have an even number (or zero) of points in common.

**194. Regions bounded by a polyhedron.** An even polyhedron  $p$  is the sum of the boundaries of a set of convex three-dimensional polyhedral regions, and we may assign the notation so that these regions are denoted by  $\alpha_1^3, \alpha_2^3, \dots, \alpha_k^3$ .

The polyhedron  $p$  is also the sum of the boundaries of

$$\alpha_{k+1}^3, \alpha_{k+2}^3, \dots, \alpha_{\alpha_k}^3$$

because the sum of all the columns of  $H_g$  is  $(0, 0, \dots, 0)$ . There is no other linear expression for  $p$  in terms of the boundaries of the  $\alpha^g$ 's, because there is only one linear relation among the columns of  $H_g$ .

This is all a direct generalization of what is said at the beginning of § 187. As in § 187, it is easily seen that the points of  $\alpha_1^g, \alpha_2^g, \dots, \alpha_k^g$ , together with those points of their boundaries which are not on  $p$ , constitute a region bounded by  $p$ ; and that the points of  $\alpha_k^g, \alpha_{k+1}^g, \dots, \alpha_n^g$ , together with those points of their boundaries which are not on  $p$ , constitute a second region bounded by  $p$ . With a few additional details (which are generalizations of those given in the proof of Theorem 75) this constitutes the proof of the following theorem:

**THEOREM 91.** *Any even polyhedron is the boundary of each of two and only two regions which contain all points of space not on the polyhedron. These regions are such that any broken line joining a point of one region to a point of the other contains a point of the polyhedron. Any two points of the same region can be joined by a broken line consisting entirely of points of the region.*

By a similar generalization of Theorem 76, we obtain

**THEOREM 92.** *Any odd polyhedron is the boundary of a single region containing all points not on the polyhedron. Any two points of this region can be joined by a broken line not containing any point of the polyhedron.*

**COROLLARY.** *Any point  $P$  on a polyhedron can be joined to any point not on it by a broken line containing no point of the polyhedron except  $P$ .*

**195. The matrices  $E_1$  and  $E_2$  for the projective plane.** **DEFINITION.** A segment, interval, broken line, polygon, two-dimensional convex region, or three-dimensional convex region associated with a sense-class among its points is called an *oriented* or *directed* segment, interval, broken line, polygon, two-dimensional convex region, or three-dimensional convex region.

**DEFINITION.** Let  $a^1$  be any segment which, with its ends  $A$  and  $B$ , is contained in a segment  $s$ , and let  $s^1$  denote the oriented segment obtained by associating  $a^1$  with one of its sense-classes. The sense-class of  $s^1$  is contained in a sense-class of  $s$  which is either  $S(AO)$  or  $S(OA)$  if  $O$  is any point of  $a^1$ . In the first case  $A$  is said to be

*positively related* to  $s^1$  and in the second case  $A$  is said to be *negatively related* to  $s^1$ .

To aid the intuition, we may think of an oriented segment as marked with an arrow, the head of which is at the end which is positively related to the oriented segment.

Obviously, if one end of an oriented segment is positively related to it, the other end is negatively related to it, and vice versa.

DEFINITION. The sense-class  $S(A_1A_2A_3)$  of a polygon  $A_1A_2\cdots A_nA_1$  and the sense-class  $S(AB)$  on the edge  $A_1A_2$  are said to *agree* in case of the order  $\{A_1ABA_2\}$  and to *disagree* in case of the order  $\{A_1BAA_2\}$ .

Returning to the notation of § 185, the segments  $\alpha_1^1, \dots, \alpha_n^1$  may each be associated with two senses. They thus give rise to  $2\alpha_1$  directed segments. Assigning an arbitrary one of the two senses to each  $\alpha^1$ , we have  $\alpha_1^1$  oriented segments to which we may assign the notation  $s_1^1, \dots, s_n^1$ . We shall denote the oriented segment obtained by changing the sense-class of  $s_i^1$  by  $-s_i^1$  and call it the *negative* of  $s^1$ .

The relations of the  $s^1$ 's to the points  $\alpha_1^0, \dots, \alpha_n^0$  may be indicated by means of a matrix which we shall call  $E_1$ . In the matrix  $E_1$  the element of the  $i$ th row and  $j$ th column shall be 1,  $-1$ , or 0, according as the point  $\alpha_i^0$  is positively related to, negatively related to, or not an end of, the oriented segment  $s_j^1$ .

It is clear that the signs 1 and  $-1$  are interchanged in the  $j$ th column of this matrix if the sense-class of  $s_j^1$  is changed. Since the sense-class of each segment is arbitrary, a matrix equivalent to  $E_1$  can be obtained from the matrix  $H_1$ , § 186, by arbitrarily changing one and only one 1 in each column to  $-1$ .

In the case of the triangle, by letting the segments  $\alpha, \bar{\alpha}, \beta, \bar{\beta}, \gamma, \bar{\gamma}$  give rise to  $s_1, s_2, \dots, s_6$  respectively, we derive the following matrix from  $H_1$  of § 151:

$E_1 :$

	$s_1$	$s_2$	$s_3$	$s_4$	$s_5$	$s_6$
$A$	0	0	1	1	1	1
$B$	1	1	0	0	-1	-1
$C$	-1	-1	-1	-1	0	0

The elements of the matrix  $E_1$  may be regarded as the coefficients of a set of linear equations analogous to the equations  $(H_1)$  of § 186,

where, however, the variables and coefficients are not reduced with respect to any modulus. These equations arise as follows:

Let  $(x_1, x_2, \dots, x_n)$  be a symbol in which the  $x$ 's can take on any integral values, positive, negative, or zero, and let this symbol represent a set of oriented segments comprising  $s_i^1$  counted  $x_i$  times if  $x_i$  is positive,  $-s_i^1$  counted  $-x_i$  times if  $x_i$  is negative, and neither  $s_i^1$  nor  $-s_i^1$  if  $x_i$  is zero,  $i$  taking on the values  $1, 2, \dots, \alpha_i$ .

The sense-class of an oriented polygon agrees with a definite sense-class of each of its sides and thus determines a set of oriented segments. The symbol  $(x_1, x_2, \dots, x_n)$  for this set of oriented segments may also be regarded as a symbol for the oriented polygon. Each vertex of the polygon is positively related to one of the oriented segments represented by  $(x_1, x_2, \dots, x_n)$  and negatively related to another. Thus if  $s_i^1$  and  $s_j^1$  meet at a certain vertex to which they are both positively related according to the matrix  $E_1$ , we have that  $x_i=1$  and  $x_j=-1$  or that  $x_i=-1$  and  $x_j=1$  in the symbol  $(x_1, \dots, x_n)$  for any directed polygon containing the sides  $\alpha_i$  and  $\alpha_j$ . The  $x$ 's corresponding to the segments not in the polygon must of course be zero. Hence the symbol  $(x_1, \dots, x_n)$  must satisfy the linear equation whose coefficients are given by the row of  $E_1$  corresponding to the vertex in question. If  $s_i$  and  $s_j$  are oppositely related to a vertex according to the matrix  $E_1$ , we must have  $x_i=1$  and  $x_j=1$  or  $x_i=-1$  and  $x_j=-1$  in the symbol for any directed polygon containing the sides  $\alpha_i$  and  $\alpha_j$ . Hence in this case also the linear equation given by the corresponding row of  $E_1$  must be satisfied. Finally, the equation given by a row of  $E_1$  corresponding to a point which is not a vertex of the polygon is satisfied because all the  $x$ 's corresponding to edges meeting at that point are zero. Hence *the symbol for a directed polygon must be a solution of the linear equations whose coefficients are the elements of the rows of the matrix  $E_1$* . These equations shall be denoted by  $(E_1)$ . In the case of the triangle they are

$$\begin{aligned} (6) \quad & x_3 + x_4 + x_5 + x_6 = 0, \\ & x_1 + x_2 - x_5 - x_6 = 0, \\ & -x_1 - x_2 - x_3 - x_4 = 0. \end{aligned}$$

By reasoning entirely analogous to that of § 186, it follows that any solution of  $(E_1)$  in integers represents one or more directed simple polygons. The situation here differs from that described in

the modulo 2 case, in that the same side may enter into more than one polygon and the same polygon may be counted any number of times in a set of polygons.

Since each column of the matrix  $E_1$  contains just one 1 and one  $-1$ , the sum of the left-hand members of the equations  $(E_1)$  vanishes identically. There can be no other linear homogeneous relation among the equations  $(E_1)$ , because the matrix  $E_1$  and the equations  $(E_1)$  when reduced modulo 2 are the same as  $H_1$  and  $(H_1)$ , and so any linear relation among the equations  $(E_1)$  would imply one among  $(H_1)$ .<sup>\*</sup> Hence the number of linearly independent equations of  $(E_1)$  is  $\alpha_0 - 1$ . The number of variables being  $\alpha_1$ , the number of linearly independent solutions is  $\alpha_1 - \alpha_0 + 1$ . In view of Theorem 74, this number is equal to  $\alpha_2$ .

It will be recalled that in the modulo 2 case one class of solutions of Equations  $(H_1)$  is given by the columns of the matrix  $H_2$ . These columns are the notation for the polygons bounding the convex regions  $a_1^1, \dots, a_{\alpha_2}^1$ . If each of these polygons be replaced by one of the two corresponding directed polygons, a set of solutions is determined for the equations  $(E_1)$ . These solutions are obtained directly from the matrix  $H_2$  by introducing minus signs so that the columns become solutions of  $(E_1)$ . This is possible in just two ways for each column, because each polygon bounding an  $a^2$  has two and only two sense-classes. A matrix so obtained shall be denoted by  $E_2$ . In the case of the triangle such a matrix is

$$E_2 \begin{pmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 \end{pmatrix}$$

It is evident on inspection that the rank of this matrix is equal to the number of columns. That is to say, unlike those of  $H_2$ , the

<sup>\*</sup> The coefficients of any linear homogeneous relation among the rows of  $E_1$  may be taken as integers having no common factor. Hence on reducing modulo 2 it would yield a linear relation among the rows of  $H_1$ . But as the only linear relation among the rows of  $H_1$  is that the sum of all the rows is zero, there is no linear relation among the rows of  $E_1$  not involving all the rows. There could not be two such relations among all the rows of  $E_1$ , because by combining them we could derive a relation involving a subset of the rows.

*columns of  $E_2$  are linearly independent.* The same proposition holds good for the matrix  $E_2$  in the general case. This can be proved as follows:

By the reasoning used above for the rows of  $E_1$  and  $H_1$ , it follows that any linear relation among the columns of  $E_2$  implies one among the columns of  $H_2$ . Since the only such relation among the columns of  $H_2$  involves all the columns, we need only investigate linear homogeneous relations among the columns of  $E_2$  in which all the coefficients are different from zero. If such a relation existed, two columns of  $E_2$  corresponding to regions having an edge in common would have numerically equal multipliers in the relation, else the elements corresponding to the common edge would not cancel. But since any two of the convex regions  $\alpha^2$  can be joined by a broken line consisting only of points of these regions and of the edges of their bounding polygons, it follows that all the coefficients in the relation would be numerically equal, i.e. they could all be taken as  $+1$  or  $-1$ .

Now the  $n$  lines  $l_1, \dots, l_n$  containing all the points and segments of our figure are not all concurrent; three of them, say  $l_1, l_2, l_3$ , form a triangle. Let us add together all the terms of the supposed relation corresponding to regions  $\alpha^2$  in one of the four triangular regions determined by  $l_1, l_2, l_3$ . The elements corresponding to edges interior to this triangular region must all cancel, because they cannot cancel against terms corresponding to regions  $\alpha^2$  exterior to the triangular region. The sum must represent an oriented polygon of which the edges are all on the boundary of the triangular region. This oriented polygon, by § 183, must be identical with the boundary of the triangular regions associated with one of its two sense-classes. If we operate similarly with the other three triangular regions determined by  $l_1, l_2, l_3$ , we obtain three other oriented polygons. But since the linear combination of the columns of  $E_2$  is supposed to vanish, each edge of the four triangular regions should appear once with one sense and once with the opposite sense, and this would imply that in the case of a triangle there would exist a linear homogeneous relation among the columns of  $E_2$ , contrary to the observation above. Hence in every case *the  $\alpha_2$  columns of  $E_2$  are linearly independent.*

Since there are only  $\alpha_2$  linearly independent solutions of the equations  $(E_1)$ , it follows that *all the solutions of  $(E_1)$  are linearly*

dependent on the columns of  $E_x$ . This is in sharp contrast with the property of the equations  $(H_1)$  stated at the end of § 186.

**196. Odd and even polygons in the projective plane.** Let us apply the results of the section above to the theory of odd and even polygons. Since any polygon is expressible in terms of the columns of  $E_x$ , an odd polygon must be so expressible. Let us write this expression in the form

$$(7) \quad \rho p = \sum_{i=1}^{i=a_x} y_i s_i^2,$$

where  $s_1^2, \dots, s_{a_x}^2$  represent the columns of  $E_x$ ,  $p$  is the symbol for the given oriented polygon, and  $\rho$  and  $y_1, \dots, y_{a_x}$  are integers which may be taken so as not to have a common factor.

Since the coefficients do not have 2 as a common factor, (7) does not vanish entirely when reduced modulo 2. But since an odd polygon is not expressible in terms of the columns of  $H_x$ ,  $\rho$  must contain the factor 2, and (7) must reduce, modulo 2, to an identity among the columns of  $H_x$ . The only such identity is the one involving all the columns of  $H_x$ . Hence the  $y_i$ 's are all odd. But in order that the edges not on the odd polygon  $p$  shall vanish, the  $y$ 's corresponding to  $s^2$ 's having an edge in common must be equal. Since any two points not on  $p$  can be joined by a broken line not meeting  $p$  (Theorem 76), it follows that all the  $y$ 's are equal. If they are all taken equal to  $\pm k$ , it is obvious that  $\rho = 2k$ . Hence we have the theorem:

**THEOREM 93.** *The symbol  $p$  for any odd polygon is expressible in the form*

$$(8) \quad 2p = \sum_{i=1}^{a_x} e_i s_i^2,$$

where each  $e_i$  is  $+1$  or  $-1$ .

This theorem may be verified in a special case by adding the columns of the matrix  $E_x$  given above for a triangle. The sum is  $(0, 0, 2, -2, 0, 0)$ , which represents a line counted twice. The number 2 is called the *coefficient of torsion* of the two-sided polygon (cf. Poincaré, Proceedings of the London Mathematical Society, Vol. XXXII (1900), p. 277. The systematic use of the matrices  $E_1, E_2$ , etc. is due to Poincaré).

Another form of statement for Theorem 93 is the following: *If the region bounded by an odd polygon  $p$  be decomposed into convex regions each bounded by an even polygon, each edge of  $p$  is on the boundary of two of these convex regions.*

An even polygon  $p$  is also expressible in the form (7). Aside from a common factor of all the coefficients, there is only one expression for  $p$  of the form (7), for if not, by eliminating  $p$  we could obtain a linear homogeneous relation among the columns of  $E_s$ .

Let  $R$  be one of the two regions bounded (Theorem 75) by  $p$ , which contains one of the convex regions  $a_i^2$  for which the corresponding  $y_i$  in (7) is not zero. Any two  $s^2$ 's corresponding to  $a^2$ 's having an edge in common must be multiplied by numerically equal  $y$ 's in (7) in order that the symbol for the common edge shall not appear in  $p$ . Since any two points of  $R$  can be joined by a broken line consisting entirely of points of  $R$ , this implies that the coefficients  $y_i$  corresponding to the  $a^2$ 's in  $R$  are all numerically equal to an integer  $k$ . From this it follows that the sum of the terms in the right-hand member of (7) which correspond to  $a^2$ 's in  $R$  is equal to  $p$ , because each edge of  $p$  is an edge of one and only one of the  $a^2$ 's in  $R$ . Since the equality just found is of the form (7), and (7) is unique, we have that  $p$  and  $y_1, \dots, y_n$  are all numerically equal to  $k$ . Obviously the factor  $k$  can be divided out of (7). Hence we have

**THEOREM 94.** *The symbol  $p$  for an even polygon is expressible in the form*

$$(9) \quad p = \sum_{i=1}^{a_2} e_i s_i^2,$$

where  $e_i$  is 0 or +1 or -1. The  $a_i^2$ 's such that the  $e_i$ 's with the same subscripts are not zero are the  $a_i^2$ 's in one of the regions  $R$  referred to in Theorem 75.

**DEFINITION.** By the *interior* (or *inside*) of an even polygon is meant that one of the two regions determined according to Theorem 94 which contains the  $a_i^2$ 's having the same subscripts as the non-zero  $e_i$ 's in (9). The other region is called the *exterior* of the polygon.

### EXERCISE

Identify the interior of a two-sided polygon as defined above with the interior as defined in § 187.

**197. One- and two-sided polygonal regions.** Let  $A_1, A_2, \dots, A_n$  be a polygon which is the boundary of a convex region  $R$  for which there is a convex region  $R'$  containing  $R$  and its boundary. If  $O$  and

$O'$  are any two points of  $R$ , then  $S(OA_1A_2) = S(O'A_1A_2)$  (cf. § 161) with respect to  $R'$  because  $O$  and  $O'$  are on the same side of the line  $A_1A_2$  in  $R'$ . Again,

$$S(OA_1A_2) = S(OA_2A_3) = \dots = S(OA_nA_1)$$

because  $A_1$  and  $A_2$  are on opposite sides of the line  $OA_2$ ,  $A_2$  and  $A_3$  are on opposite sides of the line  $OA_3$ , etc.

A sense-class in  $R$ , which we shall call *positive*, determines a positive sense-class in any convex region  $R'$  containing  $R$ , i.e. the sense-class containing the given sense-class of  $R$ . This, in view of the paragraph above, determines a unique sense-class on the polygon bounding  $R$ , by the rule that if  $S(OA_1A_2)$  is positive, where  $O$  is in  $R$ , then  $S(A_1A_2A_3)$  is positive on the boundary of  $R$ ; and if  $S(OA_2A_1)$  is positive, then  $S(A_2A_1A_n)$  is positive on the boundary of  $R$ . From § 161 it follows without difficulty that this determination is independent of the choice of the convex region  $R'$ .

Conversely, it is obvious that by this rule a sense-class on the boundary of  $R$  determines a definite sense-class in  $R$ .

**DEFINITION.** Let  $\alpha^2$  be any planar convex region which, with its boundary, is contained in a convex planar region  $R$ , and let  $a^1$  be any segment on the boundary of  $\alpha^2$ . Let  $s^1$  denote the oriented segment obtained by associating  $a^1$  with one of its sense-classes, and  $s^2$  denote the oriented region obtained by associating  $\alpha^2$  with one of its sense-classes. The sense-class of  $s^2$  is contained in a certain sense-class of  $R$  which may be denoted by  $S(OAB)$ , where  $O$  is in  $\alpha^2$  and  $A$  and  $B$  are on  $a^1$ . If  $S(AB)$  is the sense-class of  $s^1$ , then  $s^1$  and  $s^2$  are said to be *positively related*; and if  $S(AB)$  is not the sense-class of  $s^1$ , they are said to be *negatively related*.

As pointed out above, this definition is independent of the choice of  $R$ . Let  $R_1$  and  $R_2$  be two convex regions having no point in common and bounded by two polygons  $A_1A_2A_3 \dots A_m$  and  $A_1A_2B_3 \dots B_n$  respectively which have in common only the vertices  $A_1$  and  $A_2$  and the points of the edge  $A_1A_2$ . Suppose, also, that  $R_1$ ,  $R_2$  and their boundaries are contained in a convex region  $R$ . These conditions are satisfied if  $R_1$  and  $R_2$  are  $\alpha^2$ 's, and  $A_1A_2$  is an  $a^1$ , determined by a set of lines four of which are such that no three are concurrent.

The rule given above for determining positive sense on the boundaries of  $R_1$  and  $R_2$  requires that if  $S(OA_1A_2)$  is positive for  $O$  a point of  $R_1$ , then  $S(A_1A_2A_3)$  must be positive on the boundary

of  $R_1$ , where  $A$  is a point of the edge  $A_1A_2$ . If  $O'$  is any point of  $R_2$ , it is on the opposite side of the line  $A_1A_2$  from  $O$  in  $R$ . Hence  $S(O'A_2A_1)$  is positive, and hence  $S(A_2AA_1)$  must be the positive sense-class on the boundary of  $R$ .

Let  $R_1$  and  $R_2$  be two of the  $\alpha^2$ 's determined by a set of lines  $l_1, l_2, \dots, l_n$ ; let the boundary of  $R_1$  associated with the positive sense-class as determined in the last paragraph be denoted by  $(x_1, x_2, \dots, x_n)$  according to the notation of § 195; and let the boundary of  $R_2$  associated with the positive sense-class determined at the same time be  $(y_1, y_2, \dots, y_n)$ . The notation may be assigned so that  $x_1$  and  $y_1$  refer to the edge  $A_1A_2$  common to the boundaries of  $R_1$  and  $R_2$ . In this case, if  $x_1 = 1, y_1 = -1$ , and if  $x_1 = -1, y_1 = +1$ ; for the positive sense for the boundary of  $R_1$  is  $S(A_1AA_2)$  and for the boundary of  $R_2$  is  $S(A_2AA_1)$ . Hence the sum of the two symbols  $(x_1, x_2, \dots, x_n)$  and  $(y_1, y_2, \dots, y_n)$  is the symbol for the boundary of the region  $R'$  composed of  $R_1, R_2$  and the common edge  $A_1A_2$ , this boundary being associated with a sense-class  $S'$  which agrees with the positive sense-class on any edge of the boundary of  $R_1$  or  $R_2$  which is an edge of the boundary of  $R'$ .

By repeated use of these considerations it follows that if a set of  $\alpha^2$ 's with their boundaries constitute a convex region  $R$  and its boundary, the symbol  $(x_1, x_2, \dots, x_n)$  for the boundary of  $R$  associated with a sense-class which is designated as positive, is the sum of the symbols for the boundaries of the  $\alpha^2$ 's, each associated with its positive sense-class. In other words, the symbol for the boundary of  $R$  associated with its positive sense-class is the sum of a set of columns of  $H_2$ , each multiplied by  $+1$  or  $-1$  so that it shall be the symbol for the boundary of the corresponding  $\alpha^2$  associated with the sense-class which is positive relatively to the positive sense-class of  $R$ . By comparison with Theorem 94, it follows (as is obvious from other considerations also) that any polygon which is the boundary of a convex region is even.

The argument in the paragraph above applies without essential modification to any region bounded by a polygon and having a unique determination of sense according to § 168. Hence any polygon bounding a two-sided region is even.

Moreover the steps of the argument may be reversed as follows: If the symbol for any oriented polygon  $p$  be expressible in terms

of the columns of  $H_p$ , in the form (7), where the non-zero coefficients are  $e_{i_1}, e_{i_2}, \dots, e_{i_n}$ ,  $p$  is the boundary of the region  $R$  consisting of  $a_{i_1}^2, a_{i_2}^2, \dots, a_{i_n}^2$  and those points of their boundaries which are not on  $p$ . If  $R'$  is a convex region contained in  $R$ , and its positive sense-class be determined as agreeing with the positive sense-class of one of the regions  $a_{i_1}^2, a_{i_2}^2, \dots, a_{i_n}^2$ , it must agree with that of every  $a^2$  with which it has a point in common; for otherwise the symbols for the common edges of two of the  $a^2$ 's would not cancel in (7). If  $R''$  is any other convex region contained in  $R$ , and its positive sense-class is also determined by this rule, the positive sense-classes of  $R'$  and  $R''$  must, by definition, agree in any region common to  $R'$  and  $R''$ . Hence  $R$  is two-sided according to § 168. Thus we have by comparison with § 196

**THEOREM 95.** *The interior of an even polygon is a two-sided region.*

**198. One- and two-sided polyhedra.** Let the vertices of a polyhedron be denoted by  $a_1^0, a_2^0, \dots, a_{\alpha_0}^0$ , the edges by  $a_1^1, a_2^1, \dots, a_{\alpha_1}^1$  and the faces by  $a_1^2, a_2^2, \dots, a_{\alpha_2}^2$ . Assigning an arbitrary one of its sense-classes to each edge, there is determined a set of oriented segments  $s_1^1, s_2^1, \dots, s_{\alpha_1}^1$  and a matrix

$$E_1 = (\epsilon_{ij}^1),$$

in which  $i = 1, 2, \dots, \alpha_0$ ;  $j = 1, 2, \dots, \alpha_1$ ; and  $\epsilon_{ij}^1$  is  $+1, -1$ , or  $0$ , according as  $a_i^0$  is positively related to, negatively related to, or not an end of  $s_j^1$ .

Assigning an arbitrary one of its sense-classes to each face, there is determined a set of oriented planar convex regions  $s_1^2, s_2^2, \dots, s_{\alpha_2}^2$  and a matrix

$$E_2 = (\epsilon_{ij}^2),$$

in which  $i = 1, 2, \dots, \alpha_1$ ;  $j = 1, 2, \dots, \alpha_2$ ; and  $\epsilon_{ij}^2$  is  $+1, -1$ , or  $0$ , according as  $s_i^1$  is positively related to (cf. § 197), negatively related to, or not on the boundary of  $s_j^2$ . By the last section each column of  $E_2$  is the symbol  $(x_1, x_2, \dots, x_{\alpha_2})$ , in the sense explained in § 195, for an oriented polygon obtained by associating the polygon bounding one of the  $a^2$ 's with one of its sense-classes. Changing the sense-class assigned to any  $a^2$  to determine the corresponding  $s^2$  amounts to multiplying all elements of the corresponding column of  $E_2$  by  $-1$ .

For simplicity let us at first restrict attention to polyhedra in which each edge is on the boundaries of two and only two faces.

In this case there are just two non-zero elements in each row of  $E_2$ . Hence the sum of the columns of  $E_2$  will reduce to  $(0, 0, \dots, 0)$  if and only if the sense-classes have been assigned to the faces of the polyhedron in such a way that one of these elements is  $+1$  and the other  $-1$  in each row. This means that each  $s^1$  is positively related to one of the  $s^2$ 's on whose boundary it is and negatively related to the other. Thus the faces are related as are the  $\alpha^2$ 's which constitute a two-sided region bounded by an even polygon in the plane (§ 197).

DEFINITION. A polyhedron for which the sense-classes can be assigned to the edges and faces in such a way that each edge is positively related to one of the faces on whose boundary it is and negatively related to the other, is said to be *two-sided*, or *bilateral*; and one for which this assignment of sense-classes is not possible is said to be *one-sided*, or *unilateral*.

Changing the assignment of sense-classes on an edge amounts merely to multiplying the corresponding column of  $E_1$  and row of  $E_2$  by  $-1$ , and changing the assignment of sense-classes on a face amounts to the same operation on a column of  $E_2$ . Consequently the polyhedron is two-sided if there is a linear relation whose coefficients are  $1$ 's and  $-1$ 's among all the columns of  $E_2$ , and it is one-sided if there is no such relation. It is also obvious from these considerations that if a polyhedron satisfies the definition of two-sidedness (or of one-sidedness) for one assignment of sense-classes to its edges, it does so for all assignments. We therefore infer at once:

THEOREM 96. *A polyhedron is one- or two-sided according as the rank of  $E_2$  is  $\alpha_2$  or  $\alpha_2 - 1$ .*

By reference to § 195 we find

COROLLARY. *The projective plane is a one-sided polyhedron.*

In the case of any polyhedron in which each edge is on the boundary of only two faces, it is seen that the only possible linear relation among the columns of  $E_2$  reduces to one in which each coefficient is  $+1$  or  $-1$ , for any other relation would imply that a subset of the faces determines a polyhedron.

THEOREM 97. *A polyhedron bounding a convex region  $R$  which is contained within its boundary in a convex region  $R'$ , is two-sided.*

*Proof.* Let sense-classes be assigned to the edges in an arbitrary way, but let sense-classes be assigned to the faces according to the

following rule: Let a given sense-class  $S(PQRT)$  in  $R'$  be designated as positive. Let  $O$  be any point of  $R$  and  $A, B, C$ , three non-collinear points of a face of the polyhedron. The sense-class  $S(ABC)$  is assigned to this face if and only if  $S(OABC)$  is positive.

There is no difficulty in proving that if  $C$  and  $D$  are two points of an edge  $s_i^1$  of the polyhedron bounding  $R$ , and  $E$  and  $E'$  points of the two faces having this edge on their boundaries, then  $E$  and  $E'$  are on opposite sides of the plane  $OCD$ . Hence

$$S(OCDE) \neq S(OCDE').$$

Hence the sense-classes are assigned according to the rule above to the two faces having the edge  $s_i^1$  on their boundaries in such a way that  $s_i^1$  is positively related to one and negatively related to the other.

DEFINITION. By an *oriented polyhedron* is meant the set of oriented two-dimensional convex regions  $[s^2]$  obtained by associating each face of a two-sided polyhedron with a sense-class in such a way that if sense-classes are assigned arbitrarily to the edges to determine directed segments, each of these directed segments is positively related to one of the oriented two-dimensional convex regions on whose boundary it is and negatively related to the other. The  $s^2$ 's are called the *oriented faces* of the oriented polyhedron, and the  $s$ 's its *oriented edges*.

COROLLARY. *A given two-sided polyhedron determines two and only two oriented polyhedra according to the definition above.*

DEFINITION. Let  $a^3$  be a three-dimensional convex region which is contained with its boundary in a convex region  $R$ , and  $a^2$  a two-dimensional convex region on the boundary of  $a^3$ . Let  $s^3$  denote  $a^3$  associated with one of its sense-classes; and let  $s^2$  denote  $a^2$  associated with one of its sense-classes. The sense-class of  $s^3$  is contained in one of the sense-classes, say  $\bar{S}$ , of  $R$ . Let  $O$  be a point of  $a^3$ , and  $A, B, C$  three points of  $a^2$ , such that  $S(OABC)$  is  $\bar{S}$ . Then if  $S(ABC)$  is the sense-class associated with  $a^2$  to form  $s^2$ ,  $s^3$  and  $s^2$  are said to be *positively related*. Otherwise they are said to be *negatively related*.

By § 161 this definition is independent of any particular choice of the convex region  $R$  containing  $a^3$  and its boundary. From what has been proved above it follows that if each  $a^2$  on the boundary of an  $a^3$  is associated with a sense-class in such a way as to be positively related to the oriented region determined by  $a^3$  and one of its

sense-classes, this set of oriented two-dimensional convex regions is an oriented polyhedron.

The definitions made in this section are extended to polyhedra in which each edge is on an even number of faces (instead of only two, as we have been supposing) as follows:

**DEFINITION.** A polyhedron is said to be *two-sided* if sense-classes can be assigned to the edges and faces in such a way that each resulting oriented edge is positively related and negatively related to equal numbers of the resulting oriented faces.

### EXERCISES

1. An odd polyhedron is one-sided and an even polyhedron is two-sided.
2. Make a discussion of one- and two-sided polyhedral regions in space analogous to the discussion for the two-dimensional case in § 197.

**199. Orientation of space.** The matrices of § 195 can be generalized to the three-dimensional case. Let  $s_1^1, s_2^1, \dots, s_{\alpha_1}^1$  be the oriented segments obtained by associating each of the segments  $a_1^1, a_2^1, \dots, a_{\alpha_1}^1$  with an arbitrary one of its sense-classes. In the first matrix,

$$E_1 = (\epsilon_{ij}^1),$$

$i=1, 2, \dots, \alpha_0$ ;  $j=1, 2, \dots, \alpha_1$ ; and  $\epsilon_{ij}^1$  is  $+1$ ,  $-1$ , or  $0$  according as  $a_i^0$  is positively related to, negatively related to, or not an end of  $s_j^1$ .  $E_1$  can be formed from  $H_1$  by changing one  $1$  to a  $-1$  in each column. The choice of the  $-1$  in the  $j$ 'th column amounts to the choice of the sense-class on  $a_j^1$  which determines  $s_j^1$ . As an exercise, the reader should form  $E_1$  from the  $H_1$  given for a tetrahedron in § 152.

Sets of oriented segments  $s^1$  are represented as in § 195 by symbols of the form  $(x_1, x_2, \dots, x_{\alpha_1})$ , where the  $x$ 's are positive or negative integers. By the same argument as in § 195, if this symbol represents a set of oriented segments each of which is an edge of a polygon associated with that one of its sense-classes which agrees with a fixed sense-class of the polygon, it is a solution of the equations,

$$(E_1) \quad \sum_{j=1}^{\alpha_1} \epsilon_{ij}^1 x_j = 0, \quad (i=1, 2, \dots, \alpha_0)$$

and, conversely, any solution of these equations is the symbol for one or more such sets of oriented segments. Thus any solution of  $(E_1)$  may be regarded as representing one or more oriented polygons.

Let  $s_1^2, s_2^2, \dots, s_{\alpha_1}^2$  be the oriented two-dimensional convex regions obtained by associating each  $\alpha^2$  with an arbitrary one of its sense-classes. The oriented two-dimensional regions obtained by associating the  $s^2$ 's with the opposite sense-classes may be denoted by  $-s_1^2, -s_2^2, \dots, -s_{\alpha_1}^2$  respectively. In the second matrix,

$$E_2 = (\epsilon_{ij}^2),$$

$i = 1, 2, \dots, \alpha_1; j = 1, 2, \dots, \alpha_2$ ; and  $\epsilon_{ij}^2$  is 1, -1, or 0 according as  $s_i^2$  is positively related to, negatively related to, or not on the boundary of  $s_j^2$ .  $E_2$  can be formed from  $H_2$  by changing some of the 1's in each column of  $H_2$  to -1's in such a way that each column shall be a symbol  $(x_1, x_2, \dots, x_{\alpha_1})$  for a set of  $s$ 's whose sense-classes all agree with that of the oriented polygon determined by associating the boundary of  $s^2$  with one of its sense-classes. This is possible by the argument at the beginning of § 197, since each column of  $H_2$  is the symbol for the boundary of one and only one  $\alpha^2$ . As an exercise, the reader should form  $E_2$  from the  $H_2$  given for a tetrahedron in § 152.

A symbol of the form  $(x_1, x_2, \dots, x_{\alpha_1})$  in which each  $x$  is a positive or negative integer or zero may be taken to represent a set of oriented two-dimensional convex regions which includes  $s_i^2$  counted  $x_i$  times if  $x_i$  is positive,  $-s_i^2$  counted  $-x_i$  times if  $x_i$  is negative, and does not include  $s_i^2$  if  $x_i$  is zero. If this symbol represents an oriented polyhedron (§ 197), it is a solution of the equations

$$(E_2) \quad \sum_{j=1}^{\alpha_2} \epsilon_{ij}^2 x_j = 0, \quad (i = 1, 2, \dots, \alpha_1).$$

For consider the  $i$ th of these equations:

$$\epsilon_{i1}^2 x_1 + \epsilon_{i2}^2 x_2 + \dots + \epsilon_{i\alpha_2}^2 x_{\alpha_2} = 0.$$

If an oriented face of the oriented polyhedron is positively related to  $s_i^2$ , it contributes a term +1 to the left-hand member of this equation; for if  $s_k^2$  is this oriented face,  $x_k = 1$  and  $\epsilon_{ik}^2 = 1$ ; and if  $-s_k^2$  is this oriented face,  $x_k = -1$  and  $\epsilon_{ik}^2 = -1$ . An oriented face which is negatively related to  $s_i^2$  contributes a term -1 to the left-hand member of this equation; for if  $s_k^2$  is this oriented face,  $x_k = 1$  and  $\epsilon_{ik}^2 = -1$ ; and if  $-s_k^2$  is this oriented face,  $x_k = -1$  and  $\epsilon_{ik}^2 = 1$ . Hence there are as many terms equal to +1 as there are oriented faces positively related to  $s_i^2$ , and as many terms equal to -1 as there are

oriented faces negatively related to  $s_i^1$ . If neither  $s_k^2$  nor  $-s_k^2$  is in the oriented polyhedron, or if  $s_k^2$  does not have  $s_i^1$  on its boundary, the  $k$ th term of this equation is zero, for in the first case  $x_k = 0$  and in the second case  $\epsilon_k^2 = 0$ . Hence by the definition of an oriented polyhedron, each of the equations  $(E_2)$  is satisfied if  $(x_1, x_2, \dots, x_{\alpha_2})$  represents an oriented polyhedron. In particular (Theorem 97) the symbol for either oriented polyhedron determined by the boundary of an  $\alpha^3$  is a solution of  $(E_2)$ .

One-sided polyhedra do not give rise to solutions of  $(E_2)$ .

Let  $s_1^3$  and  $-s_1^3$ ,  $s_2^3$  and  $-s_2^3$ ,  $\dots$ ,  $s_{\alpha_3}^3$  and  $-s_{\alpha_3}^3$  be the pairs of oriented three-dimensional convex regions determined by  $\alpha_1^3, \alpha_2^3, \dots, \alpha_{\alpha_3}^3$  respectively according to the definition in § 197. In the third matrix,

$$E_3 = (\epsilon_{ij}^3),$$

$i = 1, 2, \dots, \alpha_3$ ;  $j = 1, 2, \dots, \alpha_3$ ; and  $\epsilon_{ij}^3$  is  $+1$ ,  $-1$ , or  $0$  according as  $s_i^3$  is positively related to, negatively related to, or not on the boundary of  $s_j^3$ . The matrix  $E_3$  can be formed from  $H_3$  by changing  $1$ 's to  $-1$ 's in the columns of  $H_3$  in such a way that the resulting columns are the symbols for oriented polyhedra and therefore solutions of  $(E_2)$ . This is possible by Theorem 96. As an exercise, the reader should form  $E_3$  from the  $H_3$  given for a tetrahedron in § 152.

The sum of the columns of  $E_1$  is  $(0, 0, \dots, 0)$  because each row of  $E_1$  contains one  $+1$  and one  $-1$ . There can be no other linear relation among the columns of  $E_1$ , because this would imply, on reducing modulo 2, more than one linear relation among the columns of  $H_1$ . Hence the rank of  $E_1$  is  $\alpha_1 - 1$ , and the number of solutions of  $E_1$  in a linearly independent set on which all the solutions are linearly dependent is  $\alpha_1 - \alpha_0 + 1$ .

Since the rank of  $H_2$  is  $\alpha_1 - \alpha_0$ , and since every homogeneous linear relation among the columns of  $E_2$  implies one among the columns of  $H_2$ , the rank of  $E_2$  is at least  $\alpha_1 - \alpha_0$ . It is, in fact, at least  $\alpha_1 - \alpha_0 + 1$  because, by Theorem 93, the symbols for a set of columns  $c_1, c_2, \dots, c_t$  which represent oriented polygons bounding all the  $s^2$ 's of a projective plane satisfy a relation of the form

$$(10) \quad e_1 c_1 + e_2 c_2 + \dots + e_t c_t = 2l,$$

where  $l$  is the symbol for a line in this plane and  $e_1, e_2, \dots, e_t$  are  $+1$  or  $-1$ . Reducing modulo 2, this gives rise to a homogeneous linear relation among the columns of  $H_2$ , which is not one of those

obtained by reducing the homogeneous linear relations among the columns of  $E_2$ .

Thus there are at least  $\alpha_1 - \alpha_0 + 1$  linearly independent columns of  $E_2$ . These are all solutions of  $(E_1)$ , and as there are not more than  $\alpha_1 - \alpha_0 + 1$  linearly independent solutions of  $(E_1)$ , there are not more than  $\alpha_1 - \alpha_0 + 1$  linearly independent columns of  $E_2$ . Hence the rank of  $E_2$  is  $\alpha_1 - \alpha_0 + 1$ , which by Theorem 85 is the same as  $\alpha_3 - \alpha_0 + 1$ .

In consequence, the symbol  $(x_1, x_2, \dots, x_{\alpha_1})$  for any oriented polygon is linearly expressible in terms of the symbols for oriented polygons which bound convex planar regions. It can easily be proved that in case of an odd polygon this expression takes the form (10) where, however, the polygons denoted by  $c_1, c_2, \dots, c_k$  are not necessarily all in the same plane.

Since the number of variables in the equations  $(E_2)$  is  $\alpha_2$  and the rank of  $E_2$  is  $\alpha_2 - \alpha_3 + 1$ , the number of solutions in a linearly independent set on which all solutions are linearly dependent is  $\alpha_3 - 1$ . The columns of  $E_3$  are all solutions of  $(E_2)$ . Hence the rank of  $E_3$  cannot be greater than  $\alpha_3 - 1$ . It cannot be less than  $\alpha_3 - 1$ , because, on reducing modulo 2, this would imply that the rank of  $H_3$  was less than  $\alpha_3 - 1$ . Hence the rank of  $E_3$  is  $\alpha_3 - 1$ . Since the symbol for any oriented polyhedron whose oriented faces are  $s^2$ 's or  $-s^2$ 's is a solution of  $(E_2)$ , it follows that it is expressible linearly in terms of the symbols for oriented polyhedra which bound convex three-dimensional regions.

Since the rank of  $E_3$  is  $\alpha_3 - 1$ , the set of equations

$$(E_3) \quad \sum_{j=1}^{\alpha_3} \epsilon_{ij}^3 x_j = 0 \quad (i=1, 2, \dots, \alpha_2)$$

must have one solution distinct from  $(0, 0, \dots, 0)$ . When reduced modulo (2) this solution must satisfy  $(H_3)$  and therefore, by § 190, reduce to  $(1, 1, \dots, 1)$ . Since each equation in the set  $(E_3)$  has only two coefficients different from zero, and these coefficients are  $\pm 1$ , it follows that all the  $x$ 's are numerically equal in a solution  $(x_1, x_2, \dots, x_{\alpha_3})$  of  $(E_3)$ . Since the equations are homogeneous, all the  $x$ 's may be taken to be  $+1$  or  $-1$ .

The  $i$ th of these equations is of the form

$$\epsilon_{i1}^3 x_{j_1} + \epsilon_{i2}^3 x_{j_2} = 0,$$

$\epsilon_{y_i}^s$  being  $+1$  or  $-1$  according as  $s_i^s$  is positively or negatively related to  $s_{j_1}^s$ , and  $\epsilon_{y_i}^s$  being  $+1$  or  $-1$  according as  $s_i^s$  is positively or negatively related to  $s_{j_2}^s$ . Hence, if the set of regions represented by a solution in which the  $x$ 's are  $\pm 1$  includes that one of  $s_{j_1}^s$  and  $-s_{j_1}^s$  to which  $s_i^s$  is positively related, it also includes that one of  $s_{j_2}^s$  and  $-s_{j_2}^s$  to which  $s_i^s$  is negatively related; and if it includes that one of  $s_{j_1}^s$  and  $-s_{j_1}^s$  to which  $s_i^s$  is negatively related, it also includes that one of  $s_{j_2}^s$  and  $-s_{j_2}^s$  to which  $s_i^s$  is positively related.

Hence the existence of a solution of  $(E_s)$  other than  $(0, 0, \dots, 0)$  implies the existence of a set of  $s^s$ 's and  $-s^s$ 's such that each  $s^s$  is positively related to one of them and negatively related to another. Since the notation  $s_j^s$  and  $-s_j^s$  may be interchanged by multiplying the  $j$ th column of  $E_s$  by  $-1$ , the notation may be so arranged that  $(1, 1, \dots, 1)$  is a solution of  $E_s$ . With the notation so arranged, each  $s^s$  is positively related to one  $s^s$  and negatively related to another. We thus have

**THEOREM 98.** *If each of the  $a^s$ 's determined by a set of planes  $\pi_1, \pi_2, \dots, \pi_n$  in a projective space is arbitrarily associated with one of its sense-classes to determine an oriented planar convex region  $s^s$ , each of the  $a^s$ 's can be associated with one of its sense-classes to determine a three-dimensional convex region  $s^s$  in such a way that each  $s^s$  is positively related to one  $s^s$  and negatively related to another.*

The set of  $s^s$ 's described in this theorem is a generalization of an oriented polyhedron as defined in § 198. If the definition of unilateral and bilateral polyhedra be generalized to any number of dimensions, it is a consequence of this theorem that the three-dimensional space is a bilateral polyhedron. In general, it can easily be verified, by generalizing the matrices  $E_1, E_2, E_3$  etc., that projective spaces of even dimensionality are unilateral polyhedra and projective spaces of odd dimensionality are bilateral polyhedra.

#### EXERCISE

An odd two-dimensional polyhedron in a three-dimensional space is one-sided and an even one is two-sided.

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